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Based on joint papers with M. Robin (2016, 2017)

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IMA, Minneapolis, MN
May 7–11, 2018
Wed May 09, 2018, 10:00-10:50
Optimal Stopping Time & Impulse Control Problem

Control $\theta = $ stop evolution of a Markov-Feller $x_t$ on a Polish space
Optimal Problem $= $ maximize/minimize the reward/cost functional

$$ J_x(\theta, \psi) = \mathbb{E}_x \left\{ \int_0^{\theta} e^{-\alpha t} f(x_t) dt + e^{-\alpha \theta} \psi(x_\theta) \right\}, \quad (1) $$

$\theta$ stopping time, $f, \psi \geq 0$ (running & terminal), $\alpha > 0$ (discount).
Optimal cost:

$$ \nu(x) = \inf \left\{ J_x(\theta, \psi) : \theta \text{ stopping time} \right\}. $$

Cost functional (with intervention)

$$ J_x(\nu) = \mathbb{E}_x \left\{ \int_0^\infty e^{-\alpha t} f(x^\nu_t) dt + \sum_{i=0}^{\infty} e^{-\alpha \vartheta_i} c(x^\nu_{\vartheta_i}, \xi_i) \right\}, \quad (2) $$

for any $\nu$ impulse-switching control, and the optimal cost function

$$ \nu(x) = \inf_{\nu} J_x(\nu). $$
VI & QVI - Stopping & Impulse - Constraint

The dynamic programming gives a HJB equation (VI or QVI) of the form

$$\min \{ Av - \alpha v + f, \psi - v \} = 0 \text{ or } \min \{ Av - \alpha v + f, Mv - v \} = 0,$$

with $A = \text{infinitesimal generator of Markov-Feller process } \{x_t, t \in [0, \infty]\}$ (with states in $E$, Polish space), and $Mu(x) = \inf \xi \{ c(x, \xi) + u(\xi) \}$. This QVI can be solved by using the sequence of variational inequalities

$$\min \{ Av^n - \alpha v^n + f, Mv^{n-1} - v^n \} = 0,$$

where $v^n$ is the optimal cost for the problem with stopping cost $Mv^{n-1}$.

Bensoussan and Lions (book 1978), Robin (thesis 1978), Menaldi (thesis 1980), etc, etc, Peskir and Shiryaev (book 2006), etc, etc, etc. (extremely long list, with variations)

**Constraint: ‘Wait for a signal’**. For instance, the system evolves according to a Wiener process $w_t$ (with drift $b$ and diffusion $\sigma$) and the signal is the jumps of a Poisson process $N_t$, independent of the Wiener process. Dupuis and Wang (2002) [also, Lempa (2012), Liang and Wei (2014)] studied this case for a geometric Brownian process.
1.- Introduction

**Constraint: Wait for a Signal**

The dynamic programming equation (or HJB equation) takes the form

\[ Au - \alpha u = \lambda [u - \psi]^+, \quad \text{with } f = 0, \]

where the verification theorem is based on an explicit solution of the HJB equation and on the solution of a discrete time problem.

**Objective:** To generalize the above problem in two directions:
- to replace the 1-d Wiener process with a Markov-Feller process,
- to replace the Poisson process by a more general counting process.

The simplest model \( \{\tau_n : n \geq 1\} \) are the jumps of a Poisson process, i.e., the time between two consecutive jumps \( \{T_1 = \tau_1, \ T_2 = \tau_2 - \tau_1, \ T_3 = \tau_3 - \tau_2, \ldots\} \) is necessarily an IID sequence, exponentially distributed. **However,** we focus first on \( \{T_n : n \geq 1\} \) being a sequence of IID random variables with common law \( \pi_0 \) (not exponential in general, i.e., not memoryless).
Signal as a Process

Let us introduce an homogeneous Markov process \( \{y_t : t \geq 0\} \), \( y_t \in [0, \infty[ \), independent of \( \{x_t : t \geq 0\} \) ‘time elapsed since the last signal’. So that almost surely, the cad-lag paths \( t \mapsto y_t \) are piecewise differentiable \( \dot{y}_t = 1 \), with jumps only back to zero, and expected infinitesimal generator

\[
A_1 \varphi(y) = \partial_y \varphi(t) + \lambda(y)[\varphi(0) - \varphi(y)], \quad \forall y \geq 0, \quad (3)
\]

where the intensity \( \lambda(y) \geq 0 \) is a Borel measurable function.

Thus, the signals are defined as functionals on \( \{y_t : t \geq 0\} \), namely,

\[
\tau_0 = 0, \quad \tau_n = \inf \left\{ t > \tau_{n-1} : y_t = 0 \right\}, \quad n \geq 1. \quad (4)
\]

The couple \( \{(x_t, y_t) : t \geq 0\} \) is an homogeneous Markov process in continuous time, and a signal arrives at a random instant \( \tau \) if and only if \( y_\tau = 0 \) (and \( \tau > 0 \)).
Two Optimal Costs - Stopping Problem

- $C_b = C_b(E \times [0, \infty[)$ continuous & bounded functions on $E \times [0, \infty[$,
- $\{(x_t, y_t) : t \geq 0\}$ Markov-Feller process, i.e., for any $\varphi(x, y)$ in $C_b$,

$$(x, y, t) \mapsto \mathbb{E}_{x, y}\{\varphi(x_t, y_t)\} \text{ is continuous, and}$$

$$\alpha > 0 \text{ and } f, \psi \geq 0, \in C_b(E \times [0, \infty[),$$

- The cost function is

$${J}_{x, y}(\theta, \psi) = \mathbb{E}_{x, y}\left\{ \int_0^\theta e^{-\alpha t} f(x_t, y_t) dt + e^{-\alpha \theta} \psi(x_\theta, y_\theta) \right\},$$

which yields an optimal cost

$$v(x, y) = \inf \left\{ {J}_{x, y}(\theta, \psi) : \theta > 0, y_\theta = 0 \right\},$$

i.e., $\theta$ is any admissible stopping time, and an auxiliary optimal cost is defined as

$$v_0(x, y) = \inf \left\{ {J}_{x, y}(\theta, \psi) : y_\theta = 0 \right\},$$

which is an homogeneous Markov model.
If \( \tau = \inf \{ t > 0 : y_t = 0 \} \) then HJB equations

\[
    u_0(x, 0) = \min \left\{ \psi, \mathbb{E}_{x,y} \left\{ \int_0^\tau e^{-\alpha t} f(x_t, y_t)dt + e^{-\alpha \tau} u_0(x_\tau, y_\tau) \right\} \right\},
\]

(9)

and

\[
    u(x, y) = \mathbb{E}_{x,y} \left\{ \int_0^\tau e^{-\alpha t} f(x_t, y_t)dt + e^{-\alpha \tau} \min \{ \psi, u \}(x_\tau, y_\tau) \right\},
\]

(10)

are the corresponding HJB equations, and both problems are related by the equation

\[
    u(x, y) = \mathbb{E}_{x,y} \left\{ \int_0^\tau e^{-\alpha t} f(x_t, y_t)dt + e^{-\alpha \tau} u_0(x_\tau, y_\tau) \right\}.
\]

(11)

Note that \( y_\tau = 0 \) and \( u_0(x, y) = u(x, y) \) for any \( y > 0 \).
Solving Optimal Stopping Problems

**Theorem**

Assume (5) & (6). Then VI (9) & (10) have each a unique solution in \( C_b(E \times [0, \infty]) \), which are the optimal costs (7) & (8). Moreover, the first exit time from the continuation region \([u < \psi]\) is optimal (discrete s.t.)

\[
\hat{\theta} = \inf \left\{ t > 0 : u(x_t, y_t) \geq \psi(x_t, y_t), \ y_t = 0 \right\}, \\
\hat{\theta}_0 = \inf \left\{ t \geq 0 : u_0(x_t, y_t) = \psi(x_t, y_t), \ y_t = 0 \right\}
\]  

are optimal, namely, \( u(x, y) = J_{x,y}(\hat{\theta}, \psi) \) and \( u_0(x, y) = J_{x,y}(\hat{\theta}_0, \psi) \).

Furthermore, the relation (11) holds.

Also \( u_0 = \min\{u, \psi\} \), which may \( \notin D(A_{x,y}) \subset C_b(E \times [0, \infty]) \). However, the optimal cost \( u \) given by (7) belongs to \( D(A_{x,y}) \) and for any \((x, y)\) in \( E \times [0, \infty]\),

\[
(HJB) \quad -A_{x,y}u(x, y) + \alpha u(x, y) + \lambda(x, y)[u(x, 0) - \psi(x, y)]^+ = f(x, y). \quad (13)
\]
Impulse Control Setting - Evolution

If $x_t$ represents the state of the ‘uncontrolled’ system then a sequence
\[ \nu = \{(\vartheta_i, \xi_i) : i \geq 1\} \]
of stopping times \( \{\vartheta_1 \leq \vartheta_2 \leq \cdots \rightarrow \infty\} \) and
\( E \)-valued random variables \( \{\xi_1, \xi_2, \ldots\} \) is called an impulse control.

The controlled system \( \{x'_\nu : t \geq 0\} \) behaves like \( x_t \) for \( t \) within \([\vartheta_{i-1}, \vartheta_i[\), \( i \geq 1 \), and \( x'_\vartheta_i = \xi_i \), in short, instantaneously moves from \( x'_{\vartheta_{i-1}} \) to \( \xi_i \).

A cost-per-impulse
\[ c : E \times \Gamma \rightarrow [c_0, \infty[ , \quad c \text{ is continuous and } c_0 > 0 , \quad (14) \]
with a closed subset \( \Gamma \) of \( E \) (set where to jump / admissible jumps).

• \( \{(x_t, y_t) : t \geq 0\} \) is a time-homogeneous Markov-Feller process, with
infinitesimal generator, \( x \in E, y \geq 0, \)
\[ A_{x,y} \varphi(x, y) = A_x \varphi(x, y) + \partial_y \varphi(x, y) + \lambda(x, y)[\varphi(x, 0) - \varphi(x, y)] , \quad (15) \]
i.e., now the intensity \( \lambda \) depends also on \( x \).
Cost per Interventions / Impulse costs

- First: ‘admissible’ impulse control $\nu = \{(\vartheta_i, \xi_i) : i \geq 1\}$ if $y_{\vartheta_i} = 0$, $\forall i \geq 1$, and $\vartheta_1 > 0$. If $\vartheta_1 = 0$ is allowed then $\nu$ is ‘zero-admissible’.

- 1st decision: choose $\mathbb{F}$-st $\vartheta_1$, a $\mathcal{F}_{\vartheta_1}$-meas. rv $\xi_1 : \Omega \rightarrow \Gamma \subset E$, cost up to $\vartheta_1$

$$J_{x,y}(\nu|\vartheta_1) = \mathbb{E}_{x,y}^{\nu|\vartheta_0} \left\{ \int_{\vartheta_0}^{\vartheta_1} f(x_t, y_t)e^{-\alpha t} \, dt + e^{-\alpha \vartheta_1} c(x_{\vartheta_1}, \xi_1) \right\}, \quad \vartheta_0 = 0,$$

- 2nd decision: choose a $\mathbb{F}$-st $\vartheta_2$, a $\mathcal{F}_{\vartheta_2}$-meas. rv $\xi_2 : \Omega \rightarrow \Gamma \subset E$, cost up to $\vartheta_2$

$$J_{x,y}(\nu|\vartheta_2) = J_{x,y}(\nu|\vartheta_1) + \mathbb{E}_{x,y}^{\nu|\vartheta_1} \left\{ \int_{\vartheta_1}^{\vartheta_2} f(x_t, y_t)e^{-\alpha t} \, dt + e^{-\alpha \vartheta_2} c(x_{\vartheta_2}, \xi_2) \right\} \quad \text{(16)}$$

- Iterating, cost upto $\vartheta_{k+1}$ ($\vartheta_{k+1} < \infty$ means ‘intervention’)

$$J_{x,y}(\nu|\vartheta_{k+1}) = J_{x,y}(\nu|\vartheta_k) + \mathbb{E}_{x,y}^{\nu|\vartheta_k} \left\{ \int_{\vartheta_k}^{\vartheta_{k+1}} f(x_t, y_t)e^{-\alpha t} \, dt + e^{-\alpha \vartheta_{k+1}} c(x_{\vartheta_{k+1}}, \xi_{k+1}) \right\}.$$
Optimal Impulse Costs

- Total cost: \( J_{x,y}(\nu) = \lim_k J_{x,y}(\nu|\vartheta_{k+1}) \). However, it is convenient to construct (in an \( \infty \times \) copy-space) a probability \( P_{x,y}^\nu \) such that

\[
J_{x,y}(\nu) = \mathbb{E}_{x,y}^\nu \left\{ \int_0^\infty e^{-\alpha t} f(x_t^\nu, y_t^\nu) dt + \sum_{i=1}^\infty e^{-\alpha \vartheta_i} c(x_{i-1,\vartheta_i,\xi_i}) \right\}, \quad (17)
\]

- \( \mathcal{V} \) (or \( \mathcal{V}_0 \)) = admissible (or zero-admissible) impulse controls, all relative to the initial condition \((x_0, y_0) = (x, y)\). Optimal cost:

\[
\nu(x, y) = \inf \{ J_{x,y}(\nu) : \nu \in \mathcal{V} \}, \quad \forall (x, y) \in E \times [0, \infty[, \quad (18)
\]

and also, the ‘time-homogeneous’ impulse control, with an optimal cost:

\[
\nu_0(x, y) = \inf \{ J_{x,y}(\nu) : \nu \in \mathcal{V}_0 \}, \quad \forall (x, y) \in E \times [0, \infty[, \quad (19)
\]

which is actually needed only for \( y = 0 \).
Dynamic Programming - State Model - Time Homogeneous

Two models: constraint ‘wait to intervene until a signal arrive’:

(a) $v(x, y)$ translated as ‘intervene only when $y = 0$ and $t > 0$’;

(b) (Markov/stationary) $v_0(x, y)$ translated as ‘intervene only when $y = 0$’.

(1) *Time homogeneous?* $v(x, y) \mapsto v(x, y, s)$, $v(x, y, 0) = v(x, y)$ and for $s > 0$,

$$J_{x, y, s}(v) = \mathbb{E}_{x, y}^v \left\{ \int_0^\infty e^{-\alpha(t-s)}f(x_{s+t}^\nu)dt + \sum_i e^{-\alpha(\vartheta_i-s)}c(x_{s+\vartheta_i}^{i-1,\nu}, \xi_i) \right\},$$

$$v(x, y, s) = \inf \{ J_{x, y, s}(\nu) : \nu \text{ any admissible impulse control} \},$$

where now ‘admissible’ means ‘intervene only when $y = 0$ and $s \neq 0$’.

(2) *Multiple impulses?* Are excluded from the optimal decision if

$$c(x, \xi_1) + c(\xi_1, \xi_2) \geq c(x, \xi_2), \quad \forall x \in E, \ \xi_1, \xi_2 \in \Gamma. \quad (>?) \quad (20)$$

and

$$v(x, y) \geq v_0(x, y), \quad \forall x \in E, \ y \geq 0, \ \text{with} = \ \text{if} \ \ y > 0,$$

$$v_0(x, 0) = \min \left\{ v(x, 0), \inf_{\xi \in \Gamma} \{ v(\xi, 0) + c(x, \xi) \} \right\}, \quad \forall x \in E. \quad (21)$$

hold true.
Dynamic Programming - Weak DP/HJB Equation

The relations

\[ u(x, y) = \mathbb{E}_{x, y} \left\{ \int_0^{\tau_1} e^{-\alpha t} f(x_t) \, dt + e^{-\alpha \tau_1} u_0(x_{\tau_1}, 0) \right\}. \]  

(22)

An equation with \( u \) alone

\[ u(x, y) = \mathbb{E}_{x, y} \left\{ \int_0^{\tau_1} e^{-\alpha t} f(x_t) \, dt + e^{-\alpha \tau_1} \min \left\{ u(x_{\tau_1}, 0), \inf_{\xi_1 \in \Gamma} \left\{ u(\xi_1, 0) + c(x_{\tau_1}, \xi_1) \right\} \right\} \right\}, \]

or equivalently

\[ u(x, y) = \left( R(\min \{u(\cdot, 0), (Mu(\cdot, 0))\}) \right)(x, y), \quad \forall x, y, \]  

(23)

Also

\[ u_0(x, 0) = \min \left\{ u(x, 0), (Mu(\cdot, 0))(x) \right\} = \]

\[ = \min \left\{ (Mu_0(\cdot, 0))(x), \mathbb{E}_{x, 0} \left\{ \int_0^{\tau_1} e^{-\alpha t} f(x_t) \, dt + e^{-\alpha \tau_1} u_0(x_{\tau_1}, 0) \right\} \right\}. \]
HJB Equation - Existence and Uniqueness

Either $-Au^0 + \alpha u^0 = f$ or

$$u^0(x) = \mathbb{E}_x \int_0^\infty e^{-\alpha t} f(x_t) dt, \quad \forall x \in E,$$

where $v, v_0 \in C(u^0) = \{ \varphi \in C(E) : 0 \leq \varphi(x) \leq u^0(x), \forall x \in E \}$

HJB equation with $u_0(x) = u_0(x, 0)$, either $u_0 = \min \{ Mu_0, Ru_0 \}$ or

$$u_0(x) = \min \left\{ \inf_{\xi \in \Gamma} \{ u_0(\xi) + c(x, \xi) \}, \mathbb{E}_{x,0} \left\{ \int_0^{\tau_1} e^{-\alpha t} f(x_t) dt + e^{-\alpha \tau_1} u_0(x_{\tau_1}) \right\} \right\}$$

Consider the scheme $u^n_0 = \min \{ Mu_0^{n-1}, Ru_0^n \}$, with $u^0_0 = u^0$, $n \geq 1$. This equation has a unique solution in $C(E)$, and $u^n_0 = v^n_0(\cdot, 0)$ with

$$v^n_0(x, y) = \inf_{\theta} \mathbb{E}_{x,y} \left\{ \int_0^\theta e^{-\alpha t} f(x_t) dt + e^{-\alpha \theta} Mv_0^{n-1}(x_{\theta}, 0) \right\},$$

where the minimization is over all zero-admissible stopping times $\theta$, i.e., $\theta = \tau_\eta$ for some discrete stopping time $\eta \geq 0$. 
Theorem

Under the previous assumptions, the decreasing sequence \( \{ u_0^n : n \geq 1 \} \) converges to \( u_0 \), and there are two constants \( C > 0 \) and \( \rho \) in \( ]0, 1[ \) such that

\[
0 \leq u_0^n(x) - u_0(x) \leq C \rho^n, \quad \forall x \in E, \ n \geq 1.
\]

(28)

Moreover, the HJB equation (26) has one and only one solution \( u_0 \) within the interval \( C(u^0) \), and the representation

\[
u_0(x) = \inf_{\theta} \mathbb{E}_{x,0} \left\{ \int_0^\theta e^{-\alpha t} f(x_t)dt + e^{-\alpha \theta} M u_0(x_\theta) \right\},
\]

(29)

holds true, where the minimization is over all zero-admissible stopping times \( \theta \), i.e., \( \theta = \tau_\eta \) for some discrete stopping time \( \eta \geq 0 \).
Existence and Uniqueness (Thm 2: \( u \))

**Theorem**

*Under previous assumptions, \( u^n \to u \), and \( \exists C > 0 \) and \( \rho \) in \( ]0, 1[ \) such that*

\[
0 \leq u^n(x) - u(x) \leq C \rho^n, \quad \forall x \in E, \ n \geq 1. \tag{30}
\]

*Moreover, the HJB equation (23) has one and only one solution \( u \) within the interval \( C(u^0) \), and*

\[
u(x) = \inf_{\theta} \mathbb{E}_{x,0} \left\{ \int_0^\theta e^{-\alpha t} f(x_t) dt + e^{-\alpha \theta} Mu(x_\theta) \right\}, \tag{31}\]

\( \theta \) admissible, i.e., \( \theta = \tau_\eta \) for some discrete stopping time \( \eta \geq 1 \).

**Furthermore**, \( u^n \) and \( u \) belong to \( D(A_{x,y}) \subset C_b(E \times [0, \infty[) \) and

\[
-A_{x,y} u(x, y) + \alpha u(x, y) + \lambda(x, y) \left[ u(x, 0) - Mu(\cdot, 0)(x) \right]^+ = f(x) \quad \tag{32}
\]

\[
-A_{x,y} u^n(x, y) + \alpha u(x, y) + \lambda \left[ u^n(x, 0) - Mu^{n-1}(\cdot, 0)(x) \right]^+ = f(x) \quad \tag{33}
\]
Solution: Optimal Cost and Feedback (Thm 3 & 4)

**Theorem**

*Under the previous assumptions, the unique solution of the HJB equation (23) is the optimal cost (18), i.e.,*

\[
u(x, y) = \inf \{ J_{x,y}(\nu) : \nu \text{ any admissible impulse control} \}, \tag{34}
\]

*for every \((x, y)\) in \(E \times [0, \infty[\).*

**Theorem**

*Under previous assumptions, the first exit time of the continuation region \([u < Mu]\) provides an optimal admissible impulse control.*
Extensions - Other Impulse Control Problems

- Instead of \( \Gamma \), may use ‘variable’ \( \Gamma(x) \),

\[
\Gamma : E \mapsto 2^E, \quad \text{with} \ \Gamma(x) \ \text{closed} \ \forall x \in E.
\]

Impulse intervention operator

\[
M \varphi(x) = \inf \{ \varphi(x) + c(x, \xi) : \xi \in \Gamma(x) \}, \quad \forall x \in E,
\]

should have the properties

\[
\begin{cases} 
(a) & M \text{ maps continuously } C_b(E) \text{ into itself,} \\
(b) & \text{there exists a Borel measurable minimizer for } M,
\end{cases}
\]

where a minimizer means a Borel function \( \hat{\xi} : E \times C_b(E) \to E \) satisfying

\[
\hat{\xi}(x, \varphi) \in \Gamma(x), \quad M \varphi(x) = \varphi(x) + c(x, \hat{\xi}(x, \varphi)), \quad \forall x, \varphi.
\]

Sometimes, the space \( C_b(E) \) is replaced by another suitable space.
Unbounded & Other Signals

- Initially $E$ compact, but $E = \mathbb{R}^d$, or in general, $E$ locally compact
- $f$ with polynomial growth, in $C_p(\mathbb{R}^d)$, $C_p(H)$, $H = \text{Hilbert}$
- Signal with a sequence $\{T_1, T_2, \ldots\}$ of independent identically distributed (IID conditionally to $\{x_t : t \geq 0\}$), or pure jump Markov processes, semi Markov processes, piecewise-deterministic Markov processes and diffusion processes with jumps. Essentially, it suffices to add a new variable to the initial process $\{x_t : t \geq 0\}$ to be reduced to the current situation.
- Instead of assuming $y_t$ in $\mathbb{R}^+$, we can consider the case $y_t$ in $[0, y_*]$, for some finite $y_* > 0$, and therefore $\pi([0, y_*]) = 1$
- In the case of IID variables independent of $\{x_t : t \geq 0\}$, we can consider a ‘general’ distribution $\pi_0$, without a density.
Hybrid Abstract Model

- a lot of references, many models, . . .
- a book in preparation with Héctor (Jasso-Fuentes) & Maurice (Robin)
- time $t$ in $[0, \infty[$, state variable $(x, n)$ in $S \subset \mathbb{R}^d \times \mathbb{R}^m$
- trajectories piecewise continuous in $x$, piecewise constant in $n$
- the continuous-type component $x$ may include some discrete evolution (i.e., taking discrete values) needed to trigger a discrete transition (or event), produced when $x$ hits the set-interface $D_n = \{x : (x, n) \in D\}$
- $n$ is sometime called ‘mode/regime’, example of evolution equation:

$$
(\dot{x}(t), \dot{n}(t)) = (g(x(t), n(t_i), v(t)), 0), \quad \text{for} \quad t \in [t_i, t_{i+1}[,
$$

$$
(x(t_i), n(t_i)) = (X(x(t_i), n(t_i), k_i), N(x(t_i), n(t_i), k_i)), \quad i = 0, 1, \ldots
$$

$$
t_{i+1} \equiv T(x(\cdot), n(t_i), t_i, D) := \inf \{t \geq t_i : (x(t), n(t)) \in D\}, \quad \text{if} \ t_i < \infty.
$$

with initial condition $(x(0), n(0)) = (x_0, n_0) \in S$, $t_0 = 0$
Theorem

Under the assumptions

\[(X, N) : D \times K \rightarrow S \setminus D, \text{ uniformly continuous.}\]

\[
\begin{align*}
g : S \times V &\rightarrow \mathbb{R}^d, \text{ continuous} \\
|g(x, n, v)| &\leq C(1 + |x|), \ \forall x, n, v \\
|g(x, n, v) - g(x', n, v)| &\leq M|x - x'|, \ \forall x, x', n, v \\
\inf_{k \in K} \inf_{(x, n), (\xi, \eta) \in D} \{|\xi - X(x, n, k)| + |\eta - N(x, n, k)|\} &\geq c > 0
\end{align*}
\]

the trajectories are well defined.

- **Stochastic** Dynamics, SDEs, SPDEs, general Markov-Feller processes,
  ...

7. Hybrid Control

Automaton Case
The data is as follows:

- **state-space** $S \subset \mathbb{R}^d \times \mathbb{R}^m$, open or closed,
- **minimal set-interface** $D^\wedge \subset S$, closed,
- **maximal set-interface** $D^\vee \subset S$, closed, $D^\wedge \subset D^\vee$,
- **control-spaces** $V \times K \subset \mathbb{R}^p \times \mathbb{R}^q$, compact,
- **set-constraints** $R_V \subset S \times V$ and $R_K \subset S \times K$, closed.

The dynamics follows the rule:

- (1) a mandatory jump or switching is applied on $D^\wedge$,
- (2) a continuous evolution takes place on $S \setminus D^\vee$,
- (3) the controller chooses either (1) or (2) on $D^\vee \setminus D^\wedge$. 
Stopping/Impulse/Switching with Constraint

Particular cases: \((x, y) \in S = E \times [0, \infty], D^\wedge = \emptyset, D^\vee = E \times \{0\}\), i.e., the control is allowed only when \(y = 0\) (the ‘signal’ has arrived).

In our Optimal Stopping Time Problem (with constraint) we have two costs, \(u\) and \(u_0\) related by

\[
u(x, y) = \mathbb{E}_{x,y} \left\{ e^{-\alpha \tau_1} u_0(x_{\tau_1}, 0) \right\}, \quad \forall (x, y) \in E \times [0, \infty]. \tag{36}\]

The cost \(u_0\) corresponds to the hybrid problem ‘control only when \(y = 0\)’, but the cost that we are interested in is \(u\).

Current Works (besides with M. Robin): in hybrid control

• (with Héctor Jasso-Fuentes) Relaxation and linear programs in hybrid control problems, . . .
• (with Héctor Jasso-Fuentes and Tomás Prieto-Rumeau) Discrete-Time Hybrid Control in Borel Spaces, . . .

THANKS!