On Ergodic Impulse Control with Constraint

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Statement of the Problem
(as in JL Menaldi's talk except for the cost and ergodic assumptions)

- The uncontrolled state is described by a Markov-Feller process x_t (values in E metric compact)
- impulse control $\nu = (\theta_i, \xi_i)_{i \geq 1}$, θ_i increasing sequence of stopping times, ξ_i E valued random variable
- constraint on impulse controls: $\theta_i > 0$ and θ_i is a jump time of the signal process y_t

$$y_{\tau_n} = 0,\ y_t = t - \tau_n \ \text{ for } \ \tau_n \leq t \leq \tau_{n+1},\ n \geq 1,$$

$T_n = \tau_{n+1} - \tau_n$, conditionally to x_t as IID random variables with intensity $\lambda(x, y)$
- $\xi_i \in \Gamma(x_{\theta_i})$, $\Gamma(x)$ closed set of E and $\forall \xi \in \Gamma(x)$, $\Gamma(\xi) \subset \Gamma(x)$

Statement of the Problem (2)

- running cost $f(x,y)$ and impulse cost $c(x,\xi)$, both positive bounded and continuous,

$$c(x,\xi) \geq c_0 > 0 \quad \text{and} \quad c(x,\xi) + c(\xi,\xi') \geq c(x,\xi')$$

- $Mg(x) \equiv \inf_{\xi \in \Gamma(x)}\{c(x,\xi) + g(x)\}$ is assumed to be continuous if g is continuous and there exists a measurable selector $\hat{\xi}(x,g)$.

$\mathcal{V}$ will denote the set of admissible controls

$$\mathcal{V} = \{(\theta_i, \xi_i),\ i \geq 1,\ \theta_1 > 0,\ y_{\theta_i} = 0\}$$

$\mathcal{V}_0$ the set of admissible controls satisfying the constraint, but $\theta_1 = 0$ is allowed

Statement of the Problem (3)

- Controlled process

The controlled process for a control ν is defined
on the product space Ω^∞, $\Omega = D(\mathbb{R}^+; E \times \mathbb{R}^+)$
by a probability P^ν_{xy},
and $(x^\nu_t, y^\nu_t) = (x^i_t, y^i_t)$ for $\theta_{i-1} \leq t < \theta_i$
evolves as the uncontrolled process between impulses instants.

Statement of the Problem (4)

- The average cost to be minimized:

$$J(x,y,\nu) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^{\nu}_{xy}\Big\{ \int_0^T f(x^\nu_s, y^\nu_s)\mathrm{d}s + \sum_i \mathbb{1}_{\theta_i \le T} c(x^{i-1}_{\theta_i}, \xi_i)\Big\}$$

$$\mu(x,y) = \inf\{J(x,y,\nu) : \nu \in \mathcal{V}\}$$

We will use an auxiliary problem

$$\tilde{J}(x,y,\nu) = \liminf_{n\to\infty} \frac{1}{\mathbb{E}^{\nu}_{xy}\tau_n}\mathbb{E}^{\nu}_{xy}\Big\{ \int_0^{\tau_n} f(x^\nu_s, y^\nu_s)\mathrm{d}s + \sum_i \mathbb{1}_{\theta_i \le \tau_n} c(x^{i-1}_{\theta_i}, \xi_i)\Big\}$$

$$\mu_0(x,y) = \inf\{J(x,y,\nu) : \nu \in \mathcal{V}_0\}$$

Additional Assumptions

$\lambda(x, y)$ is ≥ 0 bounded and continuous and

$$0 < a_1 \leq \mathbb{E}_{x0}(\tau_1) \leq a_2$$

Ergodicity assumption:

$$P(x, B) = \mathbb{E}_{x0} \mathbb{1}_B(x_{\tau_1}) \quad \forall B \in \mathcal{B}(E)$$

satisfies: there exists a positive measure m on E s.t. $0 < m(E) \leq 1$ and $P(x, B) \geq m(B)\ \forall B \in \mathcal{B}(E)$

Example: reflected diffusions and reflected diffusions with jumps for which the transition density satisfies

$$p(x, t, x') \geq k(\varepsilon) \quad \text{on} \quad E \times [\varepsilon, \infty[\times E$$

HJB Equation

Heuristic argument with the discounted problem

$$u_0^\alpha(x,0) = \min\left\{ Mu_0^\alpha(x,0), \mathbb{E}_{x0}\Big\{ \int_0^{\tau_1} \mathrm{e}^{-\alpha t} f \,\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} u_0^\alpha(x_{\tau_1},0) \Big\} \right\}$$

Let $m_\alpha = \inf u_0^\alpha(x,0)$, $w_0^\alpha = u_0^\alpha - m_\alpha$,

$$w_0^\alpha(x,0) = \min\left\{ Mw_0^\alpha(x,0), \mathbb{E}_{x0}\Big\{ \int_0^{\tau_1} \mathrm{e}^{-\alpha t}(f - \alpha m_\alpha)\mathrm{d}t + \mathrm{e}^{-\alpha\tau_1} w_0^\alpha(x_{\tau_1},0) \Big\} \right\}$$

Assuming $w_0^\alpha \to w_0$ a function, and $\alpha m_\alpha \to \mu_0$ a constant,

$$w_0(x,0) = \min\left\{ Mw_0(x,0), \mathbb{E}_{x0}\Big\{ \int_0^{\tau_1} (f - \mu_0)\mathrm{d}t + w_0(x_{\tau_1},0) \Big\} \right\}$$

One can also use heuristic argument on

$$u_0^T(t,x,y) = \inf \mathbb{E}_{x0}^\nu\Big\{ \int_0^{T-t} f\,\mathrm{d}t + \sum_i \mathbb{1}_{\theta_i \le T-t}\, c(x_{\theta_i}^{i-1}, \xi_i) \Big\}.$$

HJB Equation (2)

- For the auxiliary problem:
Find (w_0, μ_0), μ_0 constant, such that

$$w_0(x,0) = \min\left\{ Mw_0(x,0), \mathbb{E}_{x0}\left\{ \int_0^{\tau_1} [f - \mu_0]\mathrm{d}s + w_0(x_{\tau_1}, 0) \right\} \right\}$$

then $w_0(x,y)$, for $y > 0$, is given by

$$w_0(x,y) = \mathbb{E}_{xy}\left\{ \int_0^{\tau_1} [f - \mu_0]\mathrm{d}s + w_0(x_{\tau_1}, 0) \right\}$$

- For the initial problem: (w_0, μ_0) gives $w(x,y)$ as

$$w(x,y) = \mathbb{E}_{xy}\left\{ \int_0^{\tau_1} [f - \mu_0]\mathrm{d}s + w_0(x_{\tau_1}, 0) \right\}$$

Solution (μ_0, w_0)

- A discrete time HJB equation for $\big(\mu_0, w_0(x,0)\big)$: define

$$\ell(x) = \mathbb{E}_{x0}\Big\{\int_0^{\tau_1} f(x_s, y_s)\mathrm{d}s\Big\}, \qquad Pg(x) = \mathbb{E}_{x0}g(x_{\tau_1})$$
$$\tau(x) = \mathbb{E}_{x0}\tau_1, \qquad w_0(x) \equiv w_0(x,0),$$

then

$$w_0(x) = \min\Big\{\inf_{\xi\in\Gamma(x)}\big\{c(x,\xi) + w_0(\xi)\big\}, \ell(x) - \mu_0\tau(x) + Pw_0(x)\Big\}$$

is equivalent to the previous HJB equation

Proposition

There exists a solution (μ_0, w_0) in $\mathbb{R}^+ \times C(E)$ of the HJB equation.

Remark: If w_0 is solution, w_0 + constant is also solution. The uniqueness of μ_0 will come from the stochastic interpretation.

Arguments

The proof uses the following equivalent form of the HJB equation

$$w_0(x) = \inf_{\xi \in \Gamma(x) \cup \{x\}} \left\{ \ell(\xi) + \mathbb{1}_{\xi \neq x} c(x, \xi) - \mu_0 \tau(\xi) + P w_0(\xi) \right\} = R w_0$$

and the fact that

$$P(x, B) \geq \tau(x) \gamma(B) \quad \text{for a positive measure } \gamma \text{ on } E$$

satisfying $\gamma(E) > \frac{1-\beta}{\tau(x)}$, $0 < \beta < 1$.

Then R is a contraction on $C(E)$.

w_0 is the unique fixed point and

$$\mu_0 = \int_E w_0(x) \gamma(\mathrm{d}x)$$

Existence of an Optimal Control

Additional assumptions: the (uncontrolled) Markov process (x_t, y_t) has a unique invariant measure ζ and there exists a continuous function $h(x, y)$ s.t.

$$\mathbb{E}_{xy} h(x_\tau, y_\tau) = h(x, y) - \mathbb{E}_{xy}\Big\{ \int_0^\tau [f(x_t, y_t) - \bar{f}]\mathrm{d}t \Big\},$$

for any finite stopping time τ, with

$$\bar{f} = \int_{E\times\mathbb{R}^+} f(x, y)\mathrm{d}\zeta.$$

Remark: if $f(x, y) = f(x)$, then it is sufficient to assume that the Poisson equation for x_t, i.e. $-A_x h = f(x) - \bar{f}$ has a continuous solution.

Existence of an Optimal Control (2)

Theorem

With the additional assumption, we have

$$\mu_0 = \inf\left(\tilde{J}(x,0,\nu), \nu \in \mathcal{V}_0\right)$$

and there exists an optimal control $\hat{\nu}_0$ *of the auxiliary problem*
case 1. $\mu_0 = \bar{f}$*: then it is optimal to "do nothing"*
case 2. $\mu_0 < \bar{f}$*: one can rewrite the HJB equation*

$$\tilde{w}(x) = \min\left\{\tilde{\psi}(x), \tilde{\ell}(x) + P\tilde{w}(x)\right\}$$

with $\tilde{w} = w_0 - h(x,0)$, $\tilde{\psi} = Mw_0 - h(x,0)$, $\tilde{\ell}(x) = (\bar{f} - \mu_0)\mathbb{E}_{x0}\tau_1$. *This is the HJB equation of a discrete optimal stopping problem which has an optimal control* $\hat{\eta} = \inf\left\{n \geq 0 : \tilde{w}(x_n) = \tilde{\psi}(x_n)\right\}$*, i.e.,* $\hat{\eta} = \inf\left\{n \geq 0 : w_0(x_n) = Mw_0(x_n)\right\}$ *where* x_n *is the Markov chain* x_{τ_n}*. From this, we deduce an optimal control with* $\theta_1 = \tau_{\hat{\eta}}$*, and* $\theta_i = \tau_{\hat{\eta}_i}$ *with* $\hat{\eta}_i = \inf\left\{n \geq \hat{\eta}_i : w_0(x_n) = Mw_0(x_n)\right\}$.

Existence of an Optimal Control (3)

Corollary

$\mu_0 = \inf \left\{ \tilde{J}(x, y, \nu), v \in \mathcal{V} \right\}$ *and the optimal control* $\hat{\nu}$ *obtained by translation by* τ_1 *of the control* $\hat{\nu}_0$. □

The final result is given by

Theorem

$$\mu_0 = \inf\{J(x, y, \nu) : \nu \in \mathcal{V}\} = J(x, y, \hat{\nu})$$

Existence of an Optimal Control (4)

A first step is to prove:

Proposition

(μ_0, w_0) *being the solution previously obtained and recalling that*

$$w(x,y) = \mathbb{E}_{x0}\Big\{ \int_0^{\tau_1} [f - \mu_0]\mathrm{d}s + w_0(x_{\tau_1}, 0) \Big\}$$

(μ_0, w) *is solution of*

$$-A_{xy} w(x,y) + \lambda(x,y)[w(x,0) - Mw(x,0)]^+ = f - \mu_0.$$

where A_{xy} *is the (weak) infinitesimal generator of the uncontrolled process*

$$A_{xy}\varphi = A_x\varphi + \frac{\partial \varphi}{\partial y} + \lambda(x,y)\big[\varphi(x,0) - \varphi(x,y)\big].$$

Existence of an Optimal Control (5)

To prove the proposition, one first shows

$$w_0(x) = \min\{w(x,0), Mw(x,0)\}$$

which gives

$$w(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^{\tau_1} [f - \mu_0]\mathrm{d}t + w(x_{\tau_1},0) - [w(x_{\tau_1},0) - Mw]^+ \Big\}$$

from which we deduce that equation.□
Next, the proposition allows us to show that

$$M_T = \int_0^T [f(x_t,y_t) - \mu_0]\mathrm{d}s + w(x_T,y_T) \quad \text{is a submartingale.}$$

This gives $\mu_0 \leq J(x,y,\nu)$, $\forall\, \nu \in \mathcal{V}$, and, from the first expression of $w(x,y)$, on obtains $\mu_0 = J(x,y,\nu)$.

Extensions

E locally compact.

- If we assume $\Phi(t)C_0(E) \subset C_0(E)$ all other assumptions being unchanged, one can still obtain the results on the HJB equation.
- However, the "additional assumption" is no longer sufficient to get the result on the existence of an optimal cost.
- adding "$h(x, 0)$ bounded" would be sufficient (e.g., if $-A_{xy}h = f - \bar{f}$ has a bounded solution), but more general assumptions would require further work.
- When λ is independent of x and f independent of y, one can obtain an optimal control without an additional assumption.

Extensions (2)

- $P(x, B) \geq m(B)$ is also restrictive for E locally compact

Replacing by a "localized" condition like

$$P(x, B) \geq \alpha \mathbb{1}_K(x) m(B),$$

allows to obtains the results on the HJB equations and, if $h(x, 0)$ is bounded, the existence of an optimal control.

References (among others)

Besides the classical works on impulse control,
A. Bensoussan and J.L. Lions (book 1982), etc, . . . ,

• For discrete time control: A. Bensoussan (book, 2011),
O. Hernandez-Lerma and J. Lasserre (books 1996, 1999), . . .

• For average control, ergodicity: Bensoussan (book 1988),
M. Kurano (paper 1989),
F. Luque-Vasquez and O. Hernandez-Lerma (paper 1999),
L. Stettner (paper 1986), D. Gatarek and L. Stettner (paper 1990),
A. Arapostathis et al. (survey paper 1993),
M.G. Garroni and J.L. Menaldi (book 2002),
Several papers with J.L. Menaldi (1997, 2013, . . .),
Papers to appear with J.L. Menaldi, H. Jasso-Fuentes, T. Prieto-Rumeau.