Equilibrium Strategies for Time-Inconsistent Stochastic Switching Systems

Hongwei Mei
joint work with Jiongmin Yong

University of Kansas

May, 2018

Outline

Time-inconsistent Controlled diffusion

Consider a stochastic diffusion

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), & s \in [\tau, T], \\ X(\tau) = \xi, \qquad \alpha(\tau) = \iota, \end{cases}$$

and a cost function

$$J(\tau; \xi; u(\cdot)) = \mathbb{E}_\tau \Big[h(\tau; X(T), \alpha(T)) + \int_\tau^T g\big(\tau; s, X(s), u(s)\big) ds \Big].$$

which depends on its initial time τ.

A classical optimal control problem is to find an optimal control u^* such that

$$J(\tau; \xi; u^*(\cdot)) = \inf_{u \in \mathcal{U}} J(\tau; \xi; u(\cdot))$$

where $\mathcal{U}$ is some suitable control space.

Exponential discount and non-exponential discount

(1) Exponential discount.

$$g(\tau; s, x, u) = e^{\lambda(s-\tau)} g(x, u), \quad h(\tau; x) = e^{\lambda(T-\tau)} h(x).$$

then this is a time-consistent case; the optimal strategy at time τ is optimal in the future.

(2) Non-exponential discount.
Generally it is a time-inconsistent case. i.e. the optimal strategy $u(\tau; \cdot, \cdot)$ at time τ is not optimal in the future because of the non-exponential discount factor. Thus it's impossible to find an optimal control to optimize the all cost functional for any different initial time.

Equilibrium strategy

We try to find some equilibrium strategy $\bar{\Psi}(\cdot,\cdot)$ (independent of initial time) which satisfies:

Local optimality: for any $u(\cdot) \in \mathcal{U}([t, t+\varepsilon])$

$$\liminf_{\varepsilon \to 0} \frac{J\big(t; t, \xi; u(\cdot) \oplus \bar{\Psi}(\cdot,\cdot)\big|_{[t+\varepsilon, T]}\big) - J\big(t; t, \xi; \bar{\Psi}(\cdot\,,\cdot)\big|_{[t,T]}\big)}{\varepsilon} \geqslant 0.$$

where $\bar{X}(t) = \xi$.

Outline

1. Time-inconsistent optimal control
2. Controlled Regime switching Diffusion with Time-inconsistent cost
3. Approximate equilibrium strategy: Discretization
4. Local optimality

Regime switching Case

Consider a stochastic diffusion with regime switching

$$\begin{cases} dX(s) = b(s, X(s), \alpha(s), u(s))ds + \sigma(s, X(s), \alpha(s), u(s))dW(s), \ \ s \in [\tau, T], \\ X(\tau) = \xi, \qquad \alpha(\tau) = \iota, \end{cases}$$

where $\alpha(\cdot)$ is a finite state switching process with

$$\mathbb{P}\big(\alpha(s+\Delta s) = j \bigm| X(s) = x, \alpha(s) = i\big) = \begin{cases} q_{ij}(x)\Delta s + o(\Delta s), & \text{for } j \neq i, \\ 1 + q_{ii}(x)\Delta s + o(\Delta s), & \text{for } j = i, \end{cases}$$

Recursive cost functional

The initial triple $(s, \xi, \iota) \in \mathcal{D}[t, T]$, define

$$J(t; s, \xi, \iota; u(\cdot)) = Y(t; s) = \mathbb{E}_s\Big[h(t; X(T), \alpha(T)) + \int_s^T g\big(t; s, X(r), \alpha(r), Y(t; r), Z(t; r), \int_{\mathbb{R}} \Gamma(t; r, \theta)\pi(d\theta), u(r)\big)dr\Big],$$

Here, $(Y(t; \cdot), Z(t; \cdot), \Gamma(t; \cdot, \cdot))$ is the adapted solution of the following BSDE:

$$\begin{cases} dY(t; s) = -g\big(t, s, X(s), \alpha(s), Y(t; s), Z(t; s), \int_{\mathbb{R}} \Gamma(t; s, \theta)\pi(d\theta), u(s)\big)ds \\ \qquad +Z(t; s)dW(s) + \int_{\mathbb{R}} \Gamma(t; s-, \theta)\widetilde{N}(ds, d\theta), \qquad s \in [\tau, T], \\ Y(t; T) = h(t; X(T), \alpha(T)). \end{cases}$$

for non-switching diffusion, see Wei-Yong-Yu(2017)

Alternative representation

$$d\alpha(s) = \int_{\mathbb{R}} \mu(X(s), \alpha(s-), \theta) N(ds, d\theta), \qquad s \in [\tau, T].$$

Now, we take the map $\mu(\cdot\,,\cdot\,,\cdot)$ in the equation for $\alpha(\cdot)$ as follows:

$$\mu(x, i, \theta) = \sum_{j=1}^{m} (j - i) I_{\Delta_{ij}(x)}(\theta), \qquad (x, i, \theta) \in \mathbb{R}^n \times M \times \mathbb{R}.$$

or

$$\alpha(t) = i + \sum_{j=1}^{m} \int_{(\tau, t]} \big[j - \alpha(s-) \big] N(ds, \Delta_{\alpha(s-)j}(X(s))), \qquad t \in [\tau, T].$$

Another way to write α

Let $\beta_{ij} : \mathbb{R}^n \to \mathbb{R}$ $(1 \leqslant i, j \leqslant m)$ be deterministic functions satisfying the following:

$$
\begin{aligned}
-\beta_0 \leqslant \beta_{10}(x) &\equiv \beta_{11}(x) \leqslant \beta_{12}(x) \leqslant \cdots \leqslant \beta_{1m}(x) \\
&= \beta_{20}(x) \leqslant \beta_{21}(x) \equiv \beta_{22}(x) \leqslant \beta_{23}(x) \leqslant \cdots \leqslant \beta_{2m}(x) \\
&= \beta_{30}(x) \leqslant \cdots \\
&= \beta_{i0}(x) \leqslant \cdots \leqslant \beta_{i(i-1)}(x) \equiv \beta_{ii}(x) \leqslant \beta_{i(i+1)}(x) \leqslant \cdots \leqslant \beta_{im}(x) \\
&= \beta_{(i+1)0} \leqslant \cdots \\
&= \beta_{m0}(x) \leqslant \beta_{m1}(x) \leqslant \cdots \leqslant \beta_{m(m-1)}(x) \equiv \beta_{mm}(x) \leqslant \beta_0.
\end{aligned}
$$

where $\beta_0 > 0$ is a large fixed constant.

Regime switching process α

Define

$$\Delta_{ij}(x) = \left[\beta_{i(j-1)}(x), \beta_{ij}(x)\right), \qquad 1 \leqslant i, j \leqslant m, \tag{2.1}$$

and for any $\delta > 0$, denote

$$\Delta_{ij}^{\delta}(x) = \bigcup_{|y-x|\leqslant\delta} \Delta_{ij}(y), \qquad \Delta_{ij}^{-\delta}(x) = \bigcap_{|y-x|\leqslant\delta} \Delta_{ij}(y), \qquad x \in \mathbb{R}^n. \tag{2.2}$$

Note that for a given $x \in \mathbb{R}^n$, for some $i, j \in M$, the set $\Delta_{ij}(x)$ could be empty, in particular, it is always true that

$$\Delta_{ii} = \varnothing, \qquad \forall i \in M.$$

Also, it could be true, say, $\Delta_{14}(x) = [\beta_{13}(x), \beta_{14}(x)) = \varnothing$ if $\beta_{13}(x) = \beta_{14}(x)$, which will also lead to $\Delta_{14}^{-\delta}(x) = \varnothing$, for any $\delta > 0$.

Regime switching process α

Assumption **(H0)**: The Lévy measure $\pi(\cdot)$ is non-atomic on the Borel σ-field on $\mathbb{R}$. Moreover, the functions $\beta_{ij} : \mathbb{R}^n \to \mathbb{R}$ ($1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m$) are uniformly continuous . Further, there exists a $K > 0$ such that

$$\pi\big(\Delta_{ij}(x)\big) \leqslant K, \qquad \forall x \in \mathbb{R}^n,\ 1 \leqslant i,j \leqslant m.$$

and

$$\pi\big(\Delta_{ij}^{\delta}(x) \setminus \Delta_{ij}(x)\big) + \pi\big(\Delta_{ij}(x) \setminus \Delta_{ij}^{-\delta}(x)\big) \leqslant K\delta, \qquad \forall \delta > 0,\ 1 \leqslant i,j \leqslant m.$$

i.e.

$$q_{ij}(x) = \pi(\Delta_{ij}(x)).$$

Regime switching process α

Proposition

Let (H0) *hold. Suppose* $X(\cdot)$ *is an* $\mathbb{F}$*-adapted process with* $X(s) = x$ *satisfying*

$$\mathbb{E}\Big[\sup_{s\leqslant\tau\leqslant s+\Delta s}|X(\tau)-X(s)|^q\Big]\leqslant K|\Delta s|^p,$$

for some constants $q > 0$*,* $p > 1$ *and* $K > 0$*. Then the solution* $\alpha(\cdot)$ *satisfies*

$$\mathbb{P}\big(\alpha(s+\Delta s)=j\;\big|\;\alpha(s)=i, X(s)=x\big)=q_{ij}(x)\Delta s+o(\Delta s),\quad \textit{for } j\neq i.$$

Existence and Uniqueness of solution

Assumption **(H1)**:
The maps $b : [0, T] \times \mathbb{R}^n \times M \times U \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times M \times U \to \mathbb{R}^{n\times d}$ are continuous and there exists a constant $L > 0$ and a fixed $u_0 \in U$ such that for any $(s, i) \in [0, T] \times M \times U$, $(x_1, u_1), (x_2, u_2) \in \mathbb{R}^n \times U$,

$$\begin{cases} |b(s, x_1, i, u) - b(s, x_2, i, u)| + |\sigma(s, x_1, i, u) - \sigma(s, x_2, i, u)| \\ \qquad \leqslant L\big(|x_1 - x_2|\big), \\ |b(s, 0, i, u_0)| + |\sigma(s, 0, i, u_0)| \leqslant L(1 + |u_0|). \end{cases}$$

Existence

Lemma

Let (H0)–(H1) *hold. Then for any initial triple* $(\tau, \xi, \iota) \in \mathcal{D}^r$ *and* $u(\cdot) \in \mathcal{U}^p[\tau, T]$ *with* $p > 2$*, state equation admits a unique solution* $(X(\cdot), \alpha(\cdot))$*. Moreover, the following estimates hold:*

$$\mathbb{E}_\tau \Big[\sup_{\tau \leqslant s \leqslant T} |X(s)|^p \Big] \leqslant K \Big(1 + |\xi|^p + \mathbb{E}_\tau \int_\tau^T |u(s)|^p dr \Big),$$

$$\mathbb{E}_\tau \Big[\sup_{\tau \leqslant r \leqslant s} |X(r) - \xi|^p \Big] \leqslant K(s-\tau)^{\frac{p}{2}-1} \Big[(s-\tau)(1 + |\xi|^p) + \int_\tau^s |u(r)|^p dr \Big].$$

Uniqueness

Lemma

Further, if $(X_1(\cdot), \alpha_1(\cdot))$ *and* $(X_2(\cdot), \alpha_2(\cdot))$ *are solutions corresponding to the initial triples* $(\tau, \xi_1, \iota), (\tau, \xi_2, \iota) \in \mathcal{D}^p$*, then*

$$\Big(\mathbb{E}_\tau[I_{A_T^c}]\Big)^2 + \mathbb{E}_\tau\Big[\sup_{\tau\leqslant s\leqslant T} |X_1(s) - X_2(s)|^2 I_{A_s}\Big] \leqslant K|\xi_1 - \xi_2|^2,$$

where

$$A_s = \{\omega \in \Omega \mid \alpha_1(t,\omega) = \alpha_2(t,\omega), \quad \tau \leqslant t \leqslant s\}, \qquad s \in [\tau, T].$$

Outline

Approximate equilibrium strategy: Discretization

see Yong (2012)
let Π be a partition of $[0, T]$:

$$\Pi : \quad 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T.$$

Let N players involve in a multi-person differential game. Player k takes over the system at time t_{k-1}, controls the system on $[t_{k-1}, t_k)$ and hand it over to Player $(k+1)$ at t_k.
The rule is:
Every player find an time-consistent optimal control during his time of control, using his own way of discounting for the future cost, even though he will not control the system then.

Cycle N.

- Define $\Theta^N(t_N, \cdot\,, \cdot) = h(t_{N-1}, \cdot\,, \cdot)$.

- Solve HJB equation system on $[t_{N-1}, T]$ with terminal condition $\Theta^N(t_N, \cdot\,, \cdot)$ to get $V^N(\cdot\,, \cdot\,, \cdot)$.

- Define $\bar{\Psi}^\Pi(\cdot\,, \cdot\,, \cdot)$ on $[t_{N-1}, T]$, using $V^N(\cdot\,, \cdot\,, \cdot)$, *Cycle k.*

• Solve representation PDE on $[t_k, T]$ to get $\Theta^k(\cdot\,,\cdot\,,\cdot)$, using $\bar{\Psi}^\Pi(\cdot\,,\cdot\,,\cdot)$.

• Solve HJB equation system on $[t_{k-1}, t_k)$ with terminal condition $\Theta^k(t_k,\cdot\,,\cdot)$ to get $V^k(\cdot\,,\cdot\,,\cdot)$.

• Extend $\bar{\Psi}^\Pi(\cdot\,,\cdot\,,\cdot)$ to $[t_{k-1}, T]$, using $V^\Pi(\cdot\,,\cdot\,,\cdot)$ (a concatenation of $V^k(\cdot\,,\cdot\,,\cdot), \cdots, V^N(\cdot\,,\cdot\,,\cdot)$).

Discretized controlled diffusion

Having constructed $\bar{\Psi}^{\Pi}(\cdot\,,\cdot\,,\cdot)$ on $[0, T] \times \mathbb{R}^n \times M$, for given $(x, i) \in \mathbb{R}^n \times M$, we solve the following closed-loop system on $[0, T]$:

$$\begin{cases} d\bar{X}^{\Pi}(s) = b\big(s, \bar{X}^{\Pi}(s), \bar{\alpha}^{\Pi}(s), \bar{\Psi}^{\Pi}(s, \bar{X}^{\Pi}(s), \bar{\alpha}^{\Pi}(s))\big)ds \\ \qquad\qquad +\sigma\big(s, \bar{X}^{\Pi}(s), \bar{\alpha}^{\Pi}(s), \bar{\Psi}^{\Pi}(s, \bar{X}^{\Pi}(s), \bar{\alpha}^{\Pi}(s))\big)dW(s), \qquad s \in [0, \\ \bar{X}^{\Pi}(0) = x, \qquad \bar{\alpha}^{\Pi}(0) = i. \end{cases}$$

Discretized Cost functional

Then the following holds:

$$\begin{cases} \Theta^\Pi_s(\tau,s,x,i) + \bar{\mathcal{A}}^\Pi \Theta^\Pi(\tau,s,x,i) + g^\Pi\big(\tau,s,x,i,\Theta^\Pi(\tau,s,x,i), \\ \qquad \Theta^\Pi_x(\tau,s,x,i)\sigma(s,x,i,\bar{\Psi}^\Pi(s,x,i)), [Q(x)\Theta^\Pi(\tau,s,x,\cdot)]_i, \bar{\Psi}^\Pi(s,x,i)\big) = 0; \\ \qquad\qquad (s,x,i) \in [0,T] \times \mathbb{R}^n \times M, \\ \Theta^\Pi(\tau,T,x,i) = h(t^\Pi(\tau),x,i), \qquad (x,i) \in \mathbb{R}^n \times M, \end{cases} \tag{3.1}$$

$$V^\Pi(s,x,i) = \Theta^\Pi(t^\Pi(s),s,x,i), \qquad (s,x,i) \in [0,T] \times \mathbb{R}^n \times M, \tag{3.2}$$

and

$$\bar{\Psi}^\Pi(s,x,i) = \psi\big(t^\Pi(s); s,x,i,\Theta^\Pi(t^\Pi(s),x,\cdot),\Theta^\Pi_x(t^\Pi(s),x,i),\Theta^\Pi_{xx}(t^\Pi(s),x,i)\big), \\ (s,x,i) \in [0,T] \times \mathbb{R}^n \times M. \tag{3.3}$$

We call $\bar{\Psi}^\Pi(\cdot,\cdot,\cdot)$ an *approximate equilibrium strategy* associated with the partition Π.

Equilibrium strategy

Assumption **(H4)**: There exists a $\Theta(\tau, s, x, i)$ such that

$$\lim_{||\Pi|| \to 0} \Big(|\Theta^{\Pi}(\tau, s, x, i) - \Theta(\tau, s, x, i)| + |\Theta^{\Pi}_{x}(\tau, s, x, i) - \Theta_{x}(\tau, s, x, i)| + |\Theta^{\Pi}_{xx}(\tau, s, x, i) - \Theta_{xx}(\tau, s, x, i)| \Big) = 0,$$

uniformly for (τ, s, x) in any compact sets.

Taking $||\Pi|| \to 0$:

$$\bar{\Psi}(s,x,i) = \psi(s; s, x, i, \Theta(s,s,x,\cdot), \Theta_x(s,s,x,i), \Theta_{xx}(s,s,x,i)),$$

and $\Theta(\tau, t, x, i)$ satisfies the following non-linear PDE system, named *Equilibrium HJB equation*:

$$\begin{cases} \Theta_s(\tau,s,x,i) + \bar{\mathcal{A}}\Theta(\tau,s,x,i) + g\big(\tau,s,x,i,\Theta(\tau,s,x,i), \\ \qquad \Theta_x(\tau,s,x,i)\sigma(s,x,i,\bar{\Psi}(s,x,i)), [Q(x)\Theta(\tau,s,x,\cdot)]_i, \bar{\Psi}(s,x,i)\big) = 0; \\ \qquad\qquad (s,x,i) \in [0,T] \times \mathbb{R}^n \times M, \\ \Theta(\tau,T,x,i) = h(\tau,x,i), \qquad (x,i) \in \mathbb{R}^n \times M, \end{cases}$$

where

$$\bar{\mathcal{A}}\Theta(\tau,s,x,i) = \frac{1}{2}\mathrm{tr}\left[\sigma\big(s,x,i,\bar{\Psi}(s,x,i)\big)^\top \Theta_{xx}(\tau,s,x,i)\sigma\big(s,x,i,\bar{\Psi}(s,x,i)\big)\right] \\ + \Theta_x(\tau,s,x,i)b(s,x,i,\bar{\Psi}(s,x,i)) + \big[Q(x)\Theta(\tau,s,x,\cdot)\big)\big]$$

$$\begin{cases} \Theta_s(\tau, s, x, i) + \mathbb{H}(\tau, s, x, i, \Theta(\tau, s, x, \cdot), \Theta_x(\tau, s, x, i), \Theta_{xx}(\tau, s, x, i), \bar{\Psi}(s, x, i)) \\ \qquad\qquad\qquad\qquad\qquad\qquad (s, x, i) \in [0, T] \times \mathbb{R}^n \times M, \\ \Theta(\tau, T, x, i) = h(\tau, x, i), \qquad\qquad (x, i) \in \mathbb{R}^n \times M. \end{cases}$$

and

$$\bar{\Psi}(s, x, i) = \psi(s; s, x, i, \Theta(s, s, x, \cdot), \Theta_x(s, s, x, i), \Theta_{xx}(s, s, x, i)).$$

Outline

Local optimality

Equilibrium strategy

$$\bar{\Psi}(\cdot,\cdot,\cdot)$$

Perturbed strategy

$$[u(\cdot)\oplus\bar{\Psi}(\cdot,\cdot,\cdot)](s)=\begin{cases} u(s), & s\in[t,t+\varepsilon),\\ \bar{\Psi}(s,\cdot,\cdot), & s\in[t+\varepsilon,T],\end{cases}$$

ε-optimal strategy

$$\bar{\Psi}^{\varepsilon}(t;\cdot,\cdot,)|_{[t,t+\varepsilon]}\oplus\bar{\Psi}(\cdot,\cdot,\cdot)$$

It is easy to see that

$$
\begin{aligned}
J\big(t, \bar{X}(t), \bar{\alpha}(t); \bar{\Psi}^{\varepsilon}(\cdot\,,\cdot\,,\cdot) \oplus \bar{\Psi}(\cdot\,,\cdot\,,\cdot)\big|_{[t+\varepsilon, T]}\big) \\
\leqslant J\big(t, \bar{X}(t), \bar{\alpha}(t); u(\cdot) \oplus \bar{\Psi}(\cdot\,,\cdot\,,\cdot)\big|_{[t+\varepsilon, T]}\big), \\
\forall u(\cdot) \in \mathcal{U}[t, t+\varepsilon],
\end{aligned}
$$

It suffices to prove the local optimality by

$$
\lim_{\varepsilon \to 0} \frac{J\big(t, \bar{X}(t), \bar{\alpha}(t), \bar{\Psi}^{\varepsilon}(\cdot\,,\cdot\,,\cdot) \oplus \bar{\Psi}(\cdot\,,\cdot\,,\cdot)\big) - J\big(t, \bar{X}(t), \bar{\alpha}(t); \bar{\Psi}(\cdot\,,\cdot)\big)}{\varepsilon} = 0.
$$

Note that

$$
|\bar{\Psi}^{\varepsilon}(s, x, i) - \bar{\Psi}(s, x, i)| \leq K\varepsilon.
$$

Continuity w.r.t. the initial value

Lemma

Let (H0)–(H1) *hold. Let* $(X_1(\cdot), \alpha_1(\cdot))$ *and* $(X_2(\cdot), \alpha_2(\cdot))$ *be the solutions of the state equation on* $[t, T]$, *under the equilibrium strategy* $\bar{\Psi}(\cdot\,,\cdot\,,\cdot)$ *with initial triples* (t, x_1, i_0) *and* (t, x_2, i_0), *respectively. Let*

$$A_s := \{\omega \in \Omega \mid \alpha_1(r, \omega) = \alpha_2(r, \omega),\ r \in [t, s]\}.$$

Then for some constant $K > 0$, *the following holds:*

$$\mathbb{P}(A_T^c)^2 + \mathbb{E}\Big[\sup_{t \leqslant r \leqslant T} |X_1(r) - X_2(r)|^2 I_{A_T}\Big] \leqslant K|x_1 - x_2|^2. \tag{4.1}$$

Lemma

Let $(\bar{X}(\cdot), \bar{\alpha}(\cdot))$ *and* $(\bar{X}^{\varepsilon}(\cdot), \bar{\alpha}^{\varepsilon}(\cdot))$ *be defined as above with same initial* (x, i) *at time* t*. Let*

$$A_s^{\varepsilon} = \left\{ \bar{\alpha}(\tau) = \bar{\alpha}^{\varepsilon}(\tau) \;\middle|\; t \leqslant \tau \leqslant s \right\}.$$

Then for any $r \geqslant 1$*, there exists a constant* $K > 0$ *such that*

$$\mathbb{E}\Big[\sup_{t \leqslant s \leqslant t+\varepsilon} \left| \bar{X}(s) - \bar{X}^{\varepsilon}(s) \right|^r I_{A_s^{\varepsilon}} \Big] \leqslant K \varepsilon^{\frac{3r}{2}}, \qquad \forall \varepsilon > 0,$$

and for any $p > 2$*,*

$$\mathbb{P}\big[(A_{t+\varepsilon}^{\varepsilon})^c \big] + \mathbb{P}\big[(A_T^{\varepsilon})^c \big] \leqslant K \varepsilon^{1 + \frac{p-2}{2(p+1)}}, \qquad \forall \varepsilon > 0.$$

Theorem

For any $p > 2$,

$$|\bar{Y}(t;t) - \bar{Y}^{\varepsilon}(t;t)| \leqslant K\varepsilon^{1+\frac{p-2}{4(p+1)}}.$$

Thank you!