Time-Inconsistent Stochastic Optimal Control Problems

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May 8, 2018

Outline

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1. Introduction: Time-Consistency

Optimal Control Problem: Consider

$$\begin{cases} \dot X(s) = b(s, X(s), u(s)), & s \in [t, T], \\ X(t) = x, \end{cases}$$

with cost functional

$$J(t, x; u(\cdot)) = h(X(T)) + \int_t^T g(s, X(s), u(s))ds,$$

$$u(\cdot) \in \mathcal{U}[t, T] = \{ u : [t, T] \to U \mid u(\cdot) \text{ is measurable } \}.$$

Problem (C). For $(t, x) \in [0, T) \times \mathbb{R}^n$, find $\bar u(\cdot) \in \mathcal{U}[t, T]$ s.t.

$$J(t, x; \bar u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x; u(\cdot)) \equiv V(t, x).$$

Bellman Optimality Principle: For any $\tau \in [t, T]$,

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \Big[\int_t^\tau g(s, X(s), u(s))ds + V\big(\tau, X(\tau; t, x, u(\cdot))\big) \Big].$$

Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be optimal for $(t,x) \in [0, T) \times \mathbb{R}^n$.

$$\begin{aligned} V(t,x) &= J(t,x;\bar{u}(\cdot)) \\ &= \int_t^\tau g(s, \bar{X}(s), \bar{u}(s))ds + J\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot)); \bar{u}(\cdot)\big|_{[\tau,T]}\big) \\ &\geq \int_t^\tau g(s, \bar{X}(s), \bar{u}(s))ds + V\big(\tau, \bar{X}(\tau;t,x,\bar{u}(\cdot))\big) \\ &\geq \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \int_t^\tau g(s, X(s), u(s))ds + V\big(\tau, X(\tau; t, x, u(\cdot))\big) = V(t,x). \end{aligned}$$

Thus, all the equalities hold.

Consequently,

$$\begin{aligned} J\big(\tau, \bar{X}(\tau); \bar{u}(\cdot)\big|_{[\tau,T]}\big) &= V(\tau, \bar{X}(\tau)) \\ &= \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J\big(\tau, \bar{X}(\tau); u(\cdot)\big), \quad \text{a.s.} \end{aligned}$$

Hence, $\bar{u}(\cdot)\big|_{[\tau,T]} \in \mathcal{U}[\tau, T]$ is **optimal** for $\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot))\big)$.

This is called the **time-consistency** of Problem (C).

Stochastic Optimal Control Problem:

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ — filtered complete probability space,
$W(\cdot)$ — d-dim standard Brownian motion, $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0} \equiv \mathbb{F}^W$

$$\mathcal{U}[t,T] = \Big\{ u : [t,T]\times\Omega \to U \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-prog. meas.} \Big\}.$$

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \ \ s \in [t,T], \\ X(t) = x, \end{cases}$$

$$J(t,x;u(\cdot)) = \mathbb{E}\Big[\int_t^T g(s, X(s), u(s))ds + h(X(T))\Big].$$

Problem (S). For given $(t,x) \in [0,T) \times \mathbb{R}^n$, find a $\bar u(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t,x;\bar u(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)).$$

It is also **Time-consistent**.

A modified problem:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \quad s \in [t, T], \\ X(t) = x, \end{cases}$$

$$\widetilde{J}(t, x; u(\cdot)) = \mathbb{E}\Big[e^{-\lambda(T-t)}h(X(T)) + \int_t^T e^{-\lambda(s-t)}g(s, X(s), u(s))ds\Big].$$

$e^{-\lambda \cdot}$ — **exponential discount** λ — **discount rate**

$$e^{-\lambda t}\widetilde{J}(t, x; u(\cdot)) = \mathbb{E}\Big[e^{-\lambda T}h(X(T)) + \int_t^T e^{-\lambda s}g(s, X(s), u(s))ds\Big]$$

Minimizing $u(\cdot) \mapsto \widetilde{J}(t, x; u(\cdot))$ is equivalent to minimizing $u(\cdot) \mapsto e^{-\lambda t}\widetilde{J}(t, x; u(\cdot))$.

Remains time-consistent.

Recursive Utility/Cost Functional

Adam Smith (1759), "The Theory of Moral Sentiments"

Utility is not intertemporally sparable but rather that past and future experiences, jointly with current ones, provide current utility.

Roughly, in mathematical terms, one should have

$$U(t, X(t)) = f(U(t-r, X(t-r)), U(t+\tau, X(t+\tau)),$$

where $U(t, X)$ is the utility at (t, X).

* Epstein–Duffie (1992) introduced **stochastic differential utility**: The current utility $Y(t)$ of the payoff ξ at future time $T > t$ satisfies

$$Y(t) = \mathbb{E}_t\Big[\int_t^T g(s, Y(s))ds + \xi\Big].$$

Thus, current utility depends on the future utility. The above is equivalent to the following BSDE:

$$Y(t) = \xi - \int_t^T g(s, Y(s))ds - \int_t^T Z(s)dW(s), \qquad t \in [0, T].$$

A natural extension:

$$Y(t) = h(X(T)) + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s).$$

$(Y(\cdot), Z(\cdot))$ — **Recursive utility/disutility process**

A further extension:

$$Y(t) = h(X(T)) + \int_t^T g(s, X(s), u(s), Y(s), Z(s))ds - \int_t^T Z(s)dW(s),$$

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \\ X(t) = x. \end{cases}$$

Define **recursive cost functional**

$$\begin{aligned} J(t, x; u(\cdot)) &= Y(t) \\ &\equiv \mathbb{E}_t\Big[h(X(T)) + \int_t^T g(s, X(s), u(s), Y(s), Z(s))ds\Big]. \end{aligned}$$

If $g(s, x, u, y, z) = g(s, x, u)$, it becomes the **classical** one.

Further adding exponential discount:

$$J(t,x;u(\cdot)) = \mathbb{E}_t\Big[e^{-\lambda(T-t)}h(X(T)) + \int_t^T e^{-\lambda(s-t)}g(s,X(s),u(s),Y(s),Z(s))ds\Big]$$

If the above is called $Y(t)$ (for $t \in [0,T]$), then

$$Y(t) = h(X(T)) + \int_t^T \Big[\lambda Y(s) + g(s,X(s),u(s),Y(s),Z(s))ds - \int_t^T Z(s)dW(s).$$

- The corresponding problem is still **time-consistent**.
- Dynamic programming principle holds, HJB equation is valid, etc.

2. Time-Inconsistent Problems

In reality, problems are **hardly time-consistent**:

An optimal decision/policy made at time t, more than often, will not stay optimal, thereafter.

Main reason: When building the model, describing the utility/cost, etc., the following are used:

subjective Time-Preferences and

subjective Risk-Preferences.

- **Time-Preferences:**

Most people do not discount exponentially! Instead, they over discount on the utility of immediate future outcomes.

* What if a car in front not moving **2 seconds** after the light turned green? (Give a horn!)
* Plan to finish a job within next week (Will you finish it **Monday**? or **Friday**?)

* Shopping using credit cards (meet **immediate** satisfaction)

* Unintentionally pollute the environment due to over-development

....

Immediate utility weighs heavier!

Annual rate is $r = 10\% = 0.1$

Option (A): Get \$100 today (5/8/2018).

Option (B): Get \$105 ($> 100(1 + \frac{0.1}{12})$) on 6/8/2018.

Option (A′): Get \$110 ($= 100 \times 1.10$) on 5/8/2019.

Option (B′): Get \$115.50 ($> 110(1 + \frac{0.1}{12})$) on 6/30/2019.

For a **time-consistent** person,

$$(\mathrm{A}) \asymp (\mathrm{A}'), \qquad (\mathrm{B}) \asymp (\mathrm{B}'), \qquad \text{(no difference)}$$

$$(\mathrm{B}) \succ (\mathrm{A}), \qquad (\mathrm{B}') \succ (\mathrm{A}'), \qquad \text{(consistent preferences).}$$

However, for an **uncertainty-averse** person,

$$(\mathrm{A}) \succ (\mathrm{B}), \qquad (\mathrm{B}') \succ (\mathrm{A}'), \qquad \text{(inconsistent preferences).}$$

Magnifying the example:

Option (A): Get \$1M today (5/8/2018).

Option (B): Get \$1.05M ($> 1\text{M}(1+\frac{0.1}{12})$) on 6/8/2018.

Option (A′): Get \$1.1M ($= 1\text{M} \times 1.10$) on 5/8/2019.

Option (B′): Get \$1.155M($> 1.1\text{M}(1+\frac{0.1}{12})$) on 6/8/2019.

For an **uncertainty-averse** person,

$$(\text{A}) \succ (\text{B}), \qquad (\text{B}') \succ (\text{A}').$$

The feeling is stronger?

More rational in the farther future.

Exponential discounting: $\lambda_e(t) = e^{-rt}$, $\quad r > 0$ — discount rate

Hyperbolic discounting: $\lambda_h(t) = \frac{1}{1+kt}$ — a hyperbola

If let $k = e^r - 1$, i.e., $e^{-r} = \lambda_e(1) = \lambda_h(1) = \frac{1}{1+k}$, then

$$\lambda_e(t) = e^{-rt} = \frac{1}{(1+k)^t}, \qquad \lambda_h(t) = \frac{1}{1+kt}.$$

For $t \sim 0$, $t \mapsto \frac{1}{1+kt}$ decreases faster than $t \mapsto \frac{1}{(1+k)^t}$:

$$\lambda_h'(0) = -k < -\ln(1+k) = \lambda_e'(0),$$

Hyperbolic discounting actually appears in **people's behavior**.

* D. Hume (1739), “A Treatise of Human Nature”

“**Reason is, and ought only to be the slave of the passions.**”

People’s actions/behaviors are due to their **passions**.

Generalized Merton Problem

$$\begin{cases} dX(s) = \big[rX(s) + (\mu - r)u(s) - c(s)\big]ds + \sigma u(s)dW(s), \\ X(t) = x. \end{cases}$$

$$J(t,x;u(\cdot),c(\cdot)) = \mathbb{E}_t\Big[\int_t^T \nu(t,s)c(s)^\beta(s)ds + \rho(t)X(T)^\beta\Big],$$

with $\beta \in (0,1)$. Classical case:

$$\nu(t,s) = e^{-\delta(s-t)}, \quad \rho(t) = e^{-\delta(T-t)}, \qquad 0 \le t \le s \le T.$$

Problem. Find $(\bar{u}(\cdot), \bar{c}(\cdot))$ to maximize $J(t,x;u(\cdot),c(\cdot))$.

For given $t \in [0, T)$, optimal solution:

$$\begin{cases} \bar{u}^t(s) = \dfrac{(\mu - r)\bar{X}^t(s)}{\sigma^2(1-\beta)}, \\ \bar{c}^t(s) = \dfrac{\nu(t,s)^{\frac{1}{1-\beta}}\bar{X}^t(s)}{e^{\frac{\lambda}{1-\beta}(T-s)}\rho(t)^{\frac{1}{1-\beta}} + \int_s^T e^{\frac{\lambda}{1-\beta}(\tau-s)}\nu(t,\tau)^{\frac{1}{\beta}}d\tau} \end{cases}$$

$$\lambda = \frac{[2r\sigma^2(1-\beta) + (\mu-r)^2]\beta}{2\sigma^2(1-\beta)}$$

It is **time-inconsistent**.

* Palacious–Huerta (2003), survey on history
* Strotz (1956), Pollak (1968), Laibson (1997), ...
* Finn E. Kydland and Edward C. Prescott, (1977)
  (2004 Nobel Prize winners)
  (classical **optimal control theory** not working)
* Ekeland–Lazrak (2008)
* Yong (2011, 2012) (Multi-person **differential games**)
* Wei–Yong–Yu (2017) (**recursive cost functional** case)
* Karnam–Ma–Zhang (2017)
* Mei–Yong (2017) (with regime-switching)

- **Risk-Preferences:**

Consider two investments whose returns are: R_1 and R_2 with

$$\mathbb{P}(R_1 = 100) = \frac{1}{2}, \qquad \mathbb{P}(R_1 = -50) = \frac{1}{2},$$
$$\mathbb{P}(R_2 = 150) = \frac{1}{3}, \qquad \mathbb{P}(R_2 = -60) = \frac{2}{3}.$$

Which one you prefer?

$$\mathbb{E}R_1 = \frac{1}{2}100 + \frac{1}{2}(-50) = 25,$$
$$\mathbb{E}R_2 = \frac{1}{3}150 + \frac{2}{3}(-60) = 10.$$

So R_1 seems to be better.

* St. Petersburg Paradox: (posed by Nicolas Bernoulli in 1713)

$$\mathbb{P}(X = 2^n) = \frac{1}{2^n}, \qquad n \geq 1,$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} 2^n \mathbb{P}(X = 2^n) = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \infty.$$

Question: How much are you willing to pay to play the game?

How about $10,000? Or $1,000? Or ???

In 1738, Daniel Bernoulli (a cousin of Nicolas) introduced **expected utility**: $\mathbb{E}[u(X)]$. With $u(x) = \sqrt{x}$, one has

$$\mathbb{E}\sqrt{X} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = 1 + \sqrt{2}.$$

* 1944, von Neumann–Morgenstern: Introduced “rationality” axioms: Completeness, Transitivity, Independence, Continuity.

Standard stochastic optimal control theory is based on the expected utility theory.

- Decision-making based on expected utility theory is **time-consistent**.
- In classical expected utility theory, the probability is **objective**.
- It is controversial whether a probability should be **objective**.
- Early relevant works: Ramsey (1926), de Finetti (1937)

Allais Paradox (1953). Let X_i be a payoff

Option 1. $\mathbb{P}(X_1 = 100) = 100\%$

Option 2. $\mathbb{P}(X_2 = 100) = 89\%$, $\mathbb{P}(X_2 = 0) = 1\%$,
$\mathbb{P}(X_2 = 500) = 10\%$

Option 3. $\mathbb{P}(X_3 = 0) = 89\%,\ \mathbb{P}(X_3 = 100) = 11\%$

Option 4. $\mathbb{P}(X_4 = 0) = 90\%,\ \mathbb{P}(X_4 = 500) = 10\%$

Most people have the following preferences:

$$X_2 \prec X_1, \qquad X_3 \prec X_4.$$

If there exists a utility function $u : \mathbb{R} \to \mathbb{R}^+$ such that

$$X \prec Y \quad \Longleftrightarrow \quad \mathbb{E}[u(X)] < \mathbb{E}[u(Y)],$$

then

$$\begin{aligned} X_2 \prec X_1 \quad \Rightarrow \quad & \mathbb{E}[u(X_2)] = 0.89u(100) + 0.1u(500) + 0.01u(0) \\ & < \mathbb{E}[u(X_1)] = u(100) \\ X_3 \prec X_4 \quad \Rightarrow \quad & \mathbb{E}[u(X_3)] = 0.89u(0) + 0.11u(100) \\ & < \mathbb{E}[u(X_4)] = 0.9u(0) + 0.1u(500), \end{aligned}$$

Thus,

$$0.11u(100) > 0.1u(500) + 0.01u(0),$$

$$0.11u(100) < 0.01u(0) + 0.1u(500).$$

Relevant Literature:

* Subjective expected utility theory (Savage 1954)

* Mean-variance preference (Markowitz 1952)
  leading to nonlinear appearance of conditional expectation

* Choquet integral (1953)
  leading to Choquet expected utility theory

* Prospect Theory (Kahneman–Tversky 1979)
  (Kahneman won 2002 Nobel Prize)

* Distorted probability (Wang–Young–Panjer 1997)
  widely used in insurance/actuarial science

* BSDEs, g-expectation (Peng 1997)
  leading to time-consistent nonlinear expectation

* BSVIEs (Yong 2006, 2008)
  leading to time-inconsistent dynamic risk measure

Recent Relevant Literatures:

* Björk–Murgoci (2008), Björk–Murgoci–Zhou (2013)

* Hu–Jin–Zhou (2012, 2015)

* Yong (2014, 2015)

* Björk–Khapko–Murgoci (2016)

* Hu–Huang–Li (2017)

Only the case that $\mathbb{E}_t[X(\cdot)]$ and/or $\mathbb{E}_t[u(\cdot)]$ nonlinearly appear.

- **A Summary:**

Time-Preferences: (Exponential/General) Discounting.

Risk-Preferences: (Subjective/Objective) Expected Utility.

Exponential discounting + **objective** expected utility/disutility leads to **time-consistency**.

Otherwise, the problem will be **time-inconsistent**.

3. Equilibrium Strategies

Time-consistent solution:

Instead of finding an optimal solution
(which is **time-inconsistent**),

find an equilibrium strategy
(which is **time-consistent**).

Sacrifice some immediate satisfaction,

save some for the future

(retirement plan, controlling economy growth speed, ...)

A General Formulation:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \quad s \in [t, T], \\ X(t) = x, \end{cases}$$

$$\begin{cases} dY(s) = -g(t, s, X(s), u(s), Y(s), Z(s))ds + Z(s)dW(s), \ \ s \in [t, T], \\ Y(T) = h(t, X(T)). \end{cases}$$

Let $(Y(\cdot), Z(\cdot)) \equiv (Y(\cdot\,; t, x, u(\cdot)), Z(\cdot\,; t, x, u(\cdot))$. Then define

$$\begin{aligned} & J(t, x; u(\cdot)) = Y(t; t, x, u(\cdot)) \\ & = \mathbb{E}_t \Big[h(t, X(T)) + \int_t^T g(t, s, X(s), u(s), Y(s), Z(s))ds \Big]. \end{aligned}$$

Problem (N). For $(t, x) \in [0, T) \times \mathbb{R}^n$, find $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ s.t.

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x; u(\cdot)).$$

This problem is **time-inconsistent**.

Comparison:

Recursive cost functional with **exponential** discounting:

$$J(t,x;u(\cdot)) = \mathbb{E}_t\Big[e^{-\lambda(T-t)}h(X(T)) + \int_t^T e^{-\lambda(s-t)}g(s,X(s),u(s),Y(s),Z(s))ds\Big]$$

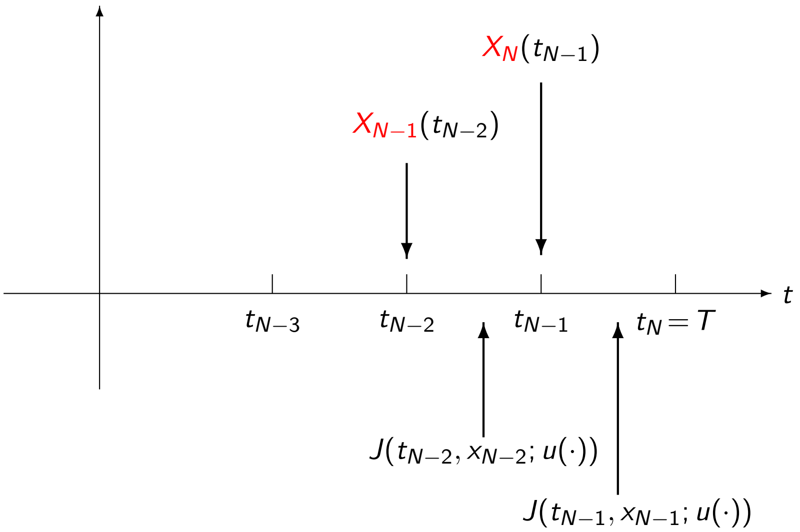
Recursive cost functional with **general** discounting:

$$J(t,x;u(\cdot)) = Y(t;t,x,u(\cdot)) = \mathbb{E}_t\Big[h(t,X(T)) + \int_t^T g(t,s,X(s),u(s),Y(s),Z(s))ds\Big].$$

• For a time-inconsistent problem, one should seek an equilibrium strategy which is time-consistent and locally optimal (in some sense).

• The idea behind: Sacrifice some near future utility to meet some farther needs (such as retirement fund).

• Mathematically, this can be achieved via a multi-person differential games.



$X_N(t_{N-1})$
$X_{N-1}(t_{N-2})$
t
t_{N-3}
t_{N-2}
t_{N-1}
$t_N = T$
$J(t_{N-2}, x_{N-2}; u(\cdot))$
$J(t_{N-1}, x_{N-1}; u(\cdot))$

Idea of Seeking Equilibrium Strategies.

• Partition the interval $[0, T]$:

$$[0, T] = \bigcup_{k=1}^{N} [t_{k-1}, t_k], \qquad \Pi : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N.$$

• Solve an optimal control problem on $[t_{N-1}, t_N]$, with cost functional:

$$J_N(u) = \mathbb{E}_{t_{N-1}}\Big[h(t_{N-1}, X(T)) + \int_{t_{N-1}}^{t_N} g(t_{N-1}, s, X(s), u(s), Y(s), Z(s))ds\Big],$$

obtaining optimal 4-tuple $(X_N(\cdot), u_N(\cdot), Y_N(\cdot), Z_N(\cdot))$, depending on the initial pair (t_{N-1}, x_{N-1}).

• Solve an optimal control problem on $[t_{N-2}, t_{N-1}]$ with a **sophisticated** cost functional:

$$J_{N-1}(u) = \mathbb{E}\Big[h(t_{N-2}, X_N(T)) + \int_{t_{N-1}}^{t_N} g(t_{N-2}, s, X_N(s), u_N(s), Y(s), Z(s))ds + \int_{t_{N-2}}^{t_{N-1}} g(t_{N-2}, s, X(s), u(s), Y(s), Z(s))ds\Big].$$

• By induction to get an **approximate equilibrium strategy**, depending on Π.

• Let $\|\Pi\| \to 0$ to get a limit.

Definition. $\Psi : [0,T] \times \mathbb{R}^n \to U$ is called a *time-consistent equilibrium strategy* if for any $x \in \mathbb{R}^n$,

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0,T], \\ \bar{X}(0) = x \end{cases}$$

admits a unique solution $\bar{X}(\cdot)$. For some $\Psi^{\Pi} : [0,T] \times \mathbb{R}^n \to U$,

$$\lim_{\|\Pi\| \to 0} d\Big(\Psi^{\Pi}(t,x), \Psi(t,x)\Big) = 0,$$

uniformly for (t,x) in any compact sets, where
$\Pi : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, and (**local optimality**)

$$J_k\big(t_{k-1}, X^{\Pi}(t_{k-1}); \Psi^{\Pi}(\cdot)\big|_{[t_{k-1},T]}\big)$$
$$\le J^k\big(t_{k-1}, X^{\Pi}(t_{k-1}); u^k(\cdot) \oplus \Psi^{\Pi}(\cdot)\big|_{[t_k,T]}\big), \quad \forall u^k(\cdot) \in \mathcal{U}[t_{k-1}, t_k],$$

$J_k(\cdot)$ — **sophisticated** cost functional.

$$
\begin{cases}
dX^{\Pi}(s) = b(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))ds \\
\qquad\qquad +\sigma(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))dW(s), \quad s \in [0, T], \\
X^{\Pi}(0) = x
\end{cases}
$$

$$
[u^k(\cdot) \oplus \Psi^{\Pi}(\cdot)\big|_{[t_k, T]}](s) = \begin{cases}
u^k(s), & s \in [t_{k-1}, t_k), \\
\Psi^{\Pi}(s, X^k(s)), & s \in [t_k, T],
\end{cases}
$$

$$
\begin{cases}
dX^k(s) = b(s, X^k(s), u^k(s))ds \\
\qquad\qquad +\sigma(s, X^k(s), u^k(s))dW(s), \quad s \in [t_{k-1}, t_k), \\
dX^k(s) = b(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))ds \\
\qquad\qquad +\sigma(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))dW(s), \quad s \in [t_k, T], \\
X^k(t_{k-1}) = X^{\Pi}(t_{k-1}).
\end{cases}
$$

Equilibrium control:

$$\bar{u}(s) = \Psi(s, \bar{X}(s)), \qquad s \in [0, T].$$

Equilibrium state process $\bar{X}(\cdot)$, satisfying:

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0, T], \\ \bar{X}(0) = x \end{cases}$$

Equilibrium value function:

$$V(t, \bar{X}(t)) = J(t, \bar{X}(t); \bar{u}(\cdot)).$$

Let $D[0,T]=\{(\tau,t)\mid 0\le\tau\le t\le T\}$. Define

$$a(t,x,u)=\frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^T,\quad \forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U,$$

$$\mathbb{H}(\tau,t,x,u,\theta,p,P)=\operatorname{tr}\left[a(t,x,u)P\right]+\langle\, b(t,x,u),p\,\rangle$$
$$+g(\tau,t,x,u,\theta,p^\top\sigma(t,s,u)),$$
$$\forall(\tau,t,x,u,p,P)\in D[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{S}^n,$$

Let $\psi:\mathcal{D}(\psi)\subseteq D[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n\to U$ such that

$$\mathbb{H}(\tau,t,x,\psi(\tau,t,x,\theta,p,P),\theta,p,P)=\inf_{u\in U}\mathbb{H}(\tau,t,x,u,\theta,p,P)>-\infty,$$
$$\forall(\tau,t,x,p,P)\in\mathcal{D}(\psi).$$

In **classical** case, it just needs

$$H(t,x,p,P)=\inf_{u\in U}\mathbb{H}(t,x,u,p,P)>-\infty,$$
$$\forall(t,x,p,P)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^n.$$

Equilibrium HJB equation:

$$\begin{cases} \Theta_t(\tau,t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)\Theta_{xx}(\tau,t,x)\big] \\ +\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle \\ +g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x)),\Theta(\tau,t,x), \\ \quad \Theta_x(\tau,t,x)^\top\sigma(t,x,\psi(t,t,x,\Theta(t,t,x),\Theta_x(t,t,x),\Theta_{xx}(t,t,x)))\big)=0, \\ \qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n, \\ \Theta(\tau,T,x)=h(\tau,x), \qquad (\tau,x)\in[0,T]\times\mathbb{R}^n. \end{cases}$$

Classical HJB Equation:

$$\begin{cases} \Theta_t(t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)\Theta_{xx}(t,x)\big] \\ +\langle\, b\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big),\Theta_x(t,x)\,\rangle \\ +g\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)=0, \qquad (t,x)\in[0,T]\times\mathbb{R}^n, \\ \Theta(T,x)=h(x), \quad x\in\mathbb{R}^n. \end{cases}$$

Comparison:

Classical HJB equation:

$$\begin{cases} \Theta_t(t,x)+H\big(t,x,\Theta_x(t,x),\Theta_{xx}(t,x)\big)=0, \quad (t,x)\in[0,T]\times\mathbb{R}^n, \\ \Theta(T,x)=h(x), \quad x\in\mathbb{R}^n. \end{cases}$$

Equilibrium HJB equation:

$$\begin{cases} \Theta_t(\tau,t,x)+H\big(\tau,t,x,\Theta_x(\tau,t,x),\Theta_{xx}(\tau,t,x), \\ \qquad\qquad \Theta(t,t,x),\Theta_x(t,t,x),\Theta_{xx}(t,t,x)\big)=0, \\ \qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n, \\ \Theta(\tau,T,x)=h(\tau,x), \qquad (\tau,x)\in[0,T]\times\mathbb{R}^n. \end{cases}$$

Equilibrium value function:

$$V(t,x)=\Theta(t,t,x), \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^n.$$

It satisfies

$$V(t,\bar X(t;x))=J\big(t,\bar X(t;x);\Psi(\cdot)\big|_{[t,T]}\big), \qquad (t,x)\in[0,T]\times\mathbb{R}^n.$$

Equilibrium strategy:

$$\Psi(t,x)=\psi(t,t,x,V_x(t,x),V_{xx}(t,x)), \qquad (t,x)\in[0,T]\times\mathbb{R}^n.$$

Theorem. *Under proper conditions, the equilibrium HJB equation admits a unique classical solution* $\Theta(\cdot\,,\cdot\,,\cdot)$. *Hence, an equilibrium strategy* $\Psi(\cdot\,,\cdot)$ *exists.*

Equilibrium strategy $\Psi(\cdot\,,\cdot)$ has the following properties:

- **Time-consistent:** $t \mapsto \Psi(t, \bar{X}(t))$.
- **Local approximately optimality:**

For any $t \in [0, T)$, any $\varepsilon > 0$, and any $u(\cdot) \in \mathcal{U}[t, t+\varepsilon)$, let

$$\big[u(\cdot) \oplus \Psi(\cdot\,,\cdot)\big](s,x) = \begin{cases} u(s), & (s,x) \in [t, t+\varepsilon) \times \mathbb{R}^n, \\ \Psi(s,x), & (s,x) \in [t+\varepsilon, T] \times \mathbb{R}^n. \end{cases}$$

$$\varliminf_{\varepsilon \to 0} \frac{J\big(t, \bar{X}(t); u(\cdot) \oplus \Psi(\cdot\,,\cdot)\big|_{[t+\varepsilon, T]}\big) - J\big(t, \bar{X}(t); \Psi(\cdot\,,\cdot)\big|_{[t,T]}\big)}{\varepsilon} \geq 0.$$

$$J\big(t, \bar{X}(t); \bar{\Psi}(\cdot\,, \bar{X}(\cdot))\big) \leq J\big(t, x; u(\cdot) \oplus \Psi(\cdot\,, \bar{X}(\cdot))\big) + o(\varepsilon).$$

(Perturbed on $[t, t+\varepsilon)$.)

Return to Generalized Merton Problem

$$\begin{cases} dX(s) = \big[rX(s) + (\mu - r)u(s) - c(s)\big]ds + \sigma u(s)dW(s), \\ X(t) = x. \end{cases}$$

$$J(t,x;u(\cdot),c(\cdot)) = \mathbb{E}_t\Big[\int_t^T \nu(t,s)c(s)^\beta(s)ds + \rho(t)X(T)^\beta\Big],$$

with $\beta \in (0,1)$. Classical case:

$$\nu(t,s) = e^{-\delta(s-t)}, \quad \rho(t) = e^{-\delta(T-t)}, \qquad 0 \le t \le s \le T.$$

Problem. Find $(\bar{u}(\cdot), \bar{c}(\cdot))$ to maximize $J(t,x;u(\cdot),c(\cdot))$.

Time-consistent equilibrium strategy:

$$\Psi(t,x) = \varphi(t)x^{\beta},$$

with $\varphi(\cdot)$ satisfying integral equation:

$$\varphi(t) = e^{\lambda(T-t)-\beta\int_t^T \left(\frac{\nu(s',s')}{\varphi(s')}\right)^{\frac{1}{1-\beta}} ds'} \rho(t)$$
$$+\int_t^T e^{\lambda(s-t)-\beta\int_t^s \left(\frac{\nu(s',s')}{\varphi(s')}\right)^{\frac{1}{1-\beta}} ds'} \left(\frac{\nu(s,s)}{\varphi(s)}\right)^{\frac{\beta}{1-\beta}} \nu(t,s)ds,$$
$$t \in [0,T].$$

4. Further Study

1. The well-posedness of the equilibrium HJB equation for the case $\sigma(t, x, u)$ is **not independent** of u.
2. The case that ψ is **not unique**, has **discontinuity**, etc.
3. The case that $\sigma(t, x, u)$ is **degenerate**, viscosity solution?
4. **Random** coefficient case (non-degenerate/degenerate cases).
5. The case involving **conditional expectation**.
6. **Infinite horizon** problems.

Thank You!