Hedging with Friction

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This talk is based on joint projects with

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- Ludovic Moreau (E&Y), Johannes Muhle-Karbe (CMU),
- Min Dai (NUS), Steve Kou (NUS), Chen Yang (ETH).

- In portfolio management problems, one generally derives desirable optimal solutions for idealized markets.
- Most of these solutions are too costly to be implementable.
- So it is of great interest to construct implementable portfolios that are close to a given optimal solution.
- In this talk, I will outline a solution with a certain friction.
- Then, as an example I apply this approach to the problem of tracking of a leveraged exchange traded fund.

- In the famous Merton utility problem, the optimal target is to keep a constant proportion, π_M, of the wealth in the stock.
- Assuming a geometric Brownian motion model for the stock price S_t, the optimal wealth process X_t^* solves,

$$dS_t = S_t\left[\mu dt + \sigma dW_t\right],$$
$$dX_t^* = X_t^*\ \left(rdt + \pi_M[(\mu - r)dt + \sigma dW_t]\right),$$

where W is Brownian motion, μ, r, π_M are constants.

- So the target portfolio position is

$$\xi_t = \frac{\text{Money invested in Stock}}{\text{Stock price}} = \frac{\pi_M\ X_t^*}{S_t}.$$

- Although π_M is constant, the optimal number of shares ξ_t that we need to have at a given time is a stochastic process which moves like Brownian motion.
- Typical friction is from bid-ask spread. Then, a constant proportion of $|\dot{\xi}_t|$ is lost to transaction costs. So any implementable process has to be absolutely continuous in time. This rules out the optimal Merton solution.
- There can also be fixed costs.
- A paper by Altorovici, Reppen, S. (SICON, 2017) considers the Merton problem with fixed and proportional costs. and also its asymptotics.

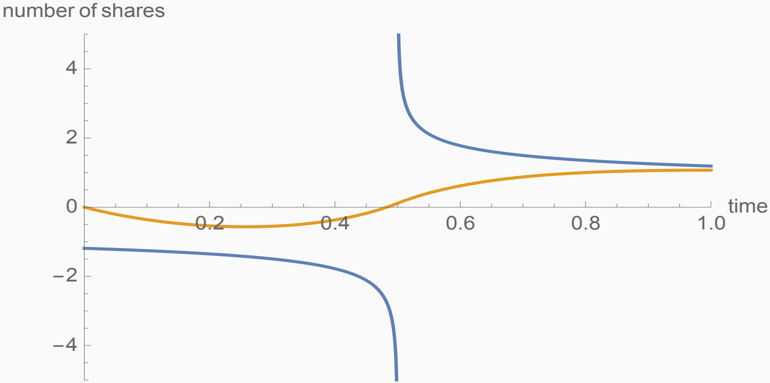
Fix an underlying probabilistic structure $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$.

Assume that we are given a target portfolio, $\{\xi_t\}_{t\in[0,T]}$.

Given an initial portfolio $Z_0 = z$, the problem is to minimize

$$Z \in \mathcal{A}_z \quad \to \quad J(Z) := \mathbb{E}\int_0^T \left[|\xi_t - Z_t|^2 + \lambda|\dot{Z}_t|^2\right] \, dt,$$

where $\lambda > 0$ is a given gearing factor (related to utility if there is one), the admissible class $\mathcal{A}_z$ is simply all differentiable and adapted processes starting at z, i.e., $Z_0 = z$.



Here the blue curve ξ is the target related to a discrete Asian option and the optimal solution is the orange curve Z^*.

There are two terms in the cost functional :

- Tracking Error : $\mathbb{E} \int_0^T |\xi_t - Z_t|^2 \; dt,$
- Tracking Effort : $\lambda \, \mathbb{E} \int_0^T |\dot{Z}_t|^2 \, dt.$

The second one is related to friction modelled as in Garleanu & Pedersen (JF, 2013). It is argued there that λ is related to Kyle's lambda. A similar model is used by Rogers & Singh (MF, 2010) and related to the price impact model of Almgren & Chriss.

The choice for quadratic structure makes the problem tractable. Indeed, this is a linear quadratic regulator type problem. However, ξ is given not through a differential equation but rather as a process.

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Theorem (Bank, Soner, Voss, MAFE 2016)

The optimal solution Z^* is given as the solution of

$$\dot{Z}_t^* = C(t,\lambda)\,(\hat{\xi}_t - Z_t^*), \qquad Z_0^* = x,$$

$$\hat{\xi}_t := \mathbb{E}\left[\int_t^T K(t,u)\,\xi_u\,du \mid \mathcal{F}_t\right],$$

where $C(t,\lambda) \approx 1/\sqrt{\lambda}$, the kernel $K(t,\cdot)$ are explicitly given and

$$K(t,u) > 0 \text{ and } \int_t^T K(t,u)du = 1.$$

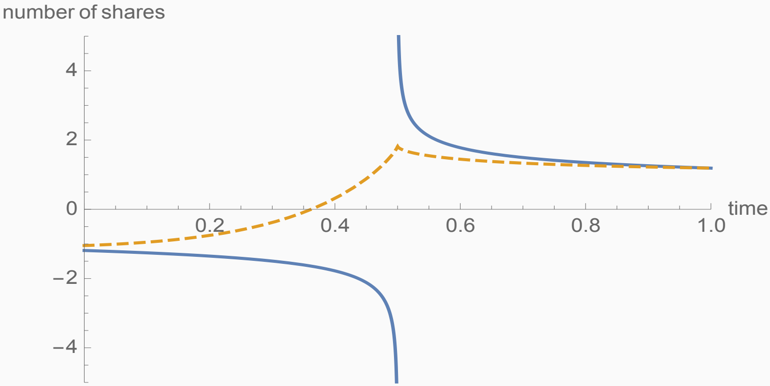
The optimal solution targets not ξ but rather $\hat{\xi}$ given by

$$\hat{\xi}_t := \mathbb{E}\left[\int_t^T K(t,u)\,\xi_u\,du \mid \mathcal{F}_t\right].$$

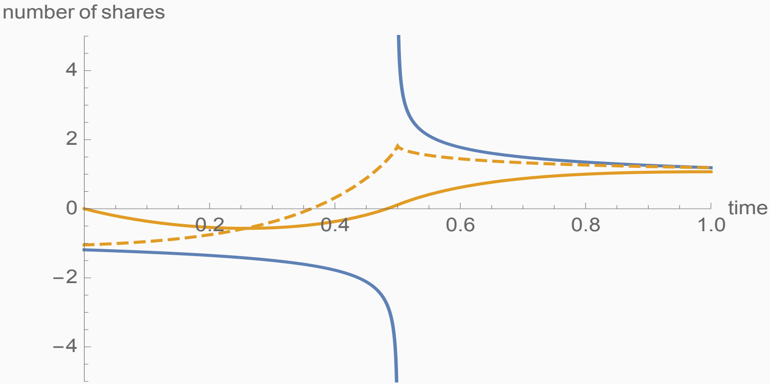
The modified target is the conditional expectation of the weighted future average of the original ξ.

The Kernel $K(t,\cdot)$ in fact depends on λ as well. The smaller the λ is the more concentrated it is around t. In fact,

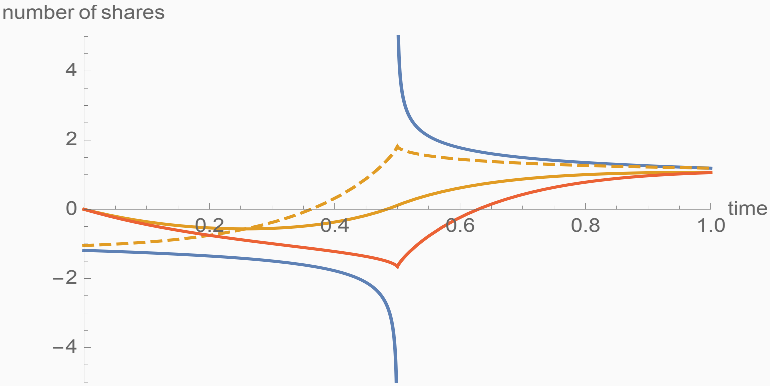
$$K(t,u)du \quad \rightarrow \quad \delta_t(du) \qquad \text{as } \lambda \text{ tends to } 0.$$



Blue is the target ξ, dashed orange is the modified target $\hat{\xi}$



The orange curve is the solution Z^*.



Red curve is the solution to myopic one, i.e, aiming at ξ and not $\hat{\xi}$.

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Garleanu & Pedersen quote Wayne Gretzky, "*A great hockey player skates to where the puck is going to be, not where it is.*"

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- An exchange-traded fund, is a marketable security that tracks an index, commodities, bonds, or baskets of assets like index funds.
- Unlike mutual funds, an ETF trades like a common stock on a stock exchange.
- ETFs experience price changes throughout the day as they are bought and sold.
- ETFs typically have higher daily liquidity and lower fees than mutual fund shares, making them an attractive alternative for individual investors.

- One of the most widely known and traded ETF tracks the S&P 500 Index, and is called the Spider (SPDR) with ticker SPX.
- IWM tracks the Russell 2000 and DIA tracks the Dow Jones.
- Sector ETFs exist that track individual industries such as oil companies, (POIH), energy companies (XLE), financial companies (XLF), and so on.
- Commodity ETFs include crude oil (USO), gold (GLD), silver (SLV), and natural gas (UNG) among others.
- There is even an ETF or ETFs.

- Some ETFs utilize leverage, to create inverse or leveraged ETFs.
- Inverse ETFs track the opposite return of that of the underlying assets for example the inverse gold ETF would gain 1% for every 1% drop in the price of the metal.
- Leveraged ETFs seek to gain a multiple return of that of the underlying. A 2x gold ETF would gain 2% for every 1% gain in the price of the metal.

We call them LETF and $\beta \in \{2, 3, -2, -3\}$ is the leverage factor.

We use the following notation throughout the talk :

- S_t is the net asset value NAV of the ETF, at time $t \geq 0$.
- X_t is the NAV of the LETF, at time $t \geq 0$.
- β is the leverage ratio for the LETF. Typically, $\beta = 2, 3, -2$.
- The daily return on S or X is

$$R_t^S := \frac{S_t - S_{t-1}}{S_{t-1}}, \quad R_t^X := \frac{X_t - X_{t-1}}{X_{t-1}}, \quad t = 1, 2, \cdots .$$

Since we assume that the ETF is perfectly tracked, we need to track β times S_t. So the goal is

$$\boxed{R_t^X = \beta R_t^S, \quad \forall\ t = 1, 2, \cdots .}$$

- The goal is $R_t^X = \beta R_t^S$, for all discrete time points $t = 1, 2, \ldots$.
- Then, $X_t \neq \beta S_t$. In fact, $dX_t / X_t \approx \beta dS_t / S_t$.
- Long term returns (more than daily) do not track due to convexity.
- So one cannot simply hold β shares of S all the time.
- The piece-wise constant hedge

$$\boxed{Z_t = \beta \, \frac{X_t}{S_t}, \quad t = 1, 2, \ldots.}$$

perfectly tracks.

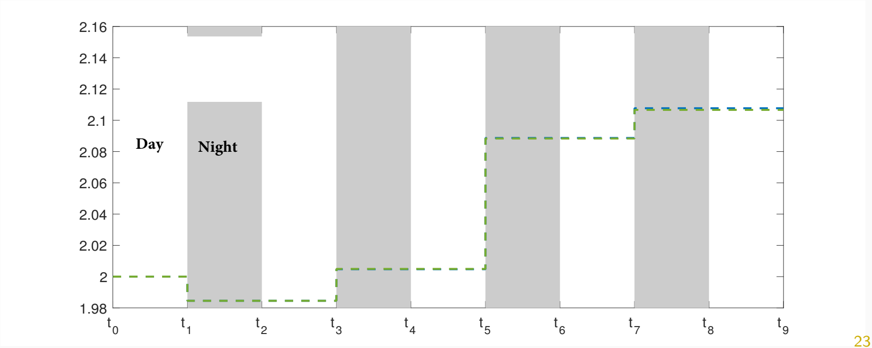
Consider the simple example with no friction and with $\beta = 2$.

At each time $t = 1, 2$, the portfolio position needs to move from Z_{t^-} to the target value $Z_t = \beta X_t / S_t$.

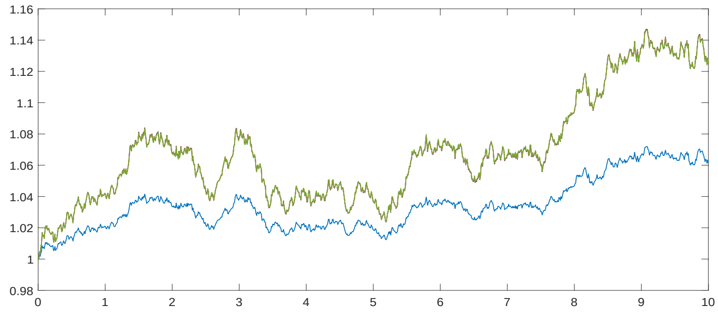
Daily Rebalancing Example with no friction and $\beta = 2$

Time	S	X	Z	Target $\beta X/S$	Rebalancing
0	1	1	2	2	–
1^-	0.9	0.8	2	$1.78 = 2 \times 0.8/0.9$	sell
2^-	0.99	0.96	1.78	$1.94 = 2 \times 0.96/0.99$	buy

The following is the desired portfolio position. They could adjusted during the day. When there is no friction, it is optimal to rebalance at the end of each day.



Below is a Matone Carlo simulation of ETF and LETF with $\beta = 2$.



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Tang and Xu (JFQA, 2013) studied the problem with data between 2006-2010. We recalculated using data between 2006-2016.

Daily NAV Return Deviations of LETFs

		(1x)NAV Return			(2x)NAV Return			
LETF Family	Statistics	Target	Actual	Actual-Target	Target	Actual	Actual-Target	Percent Slippage
SPX	Mean	0.0394	0.0394	0.0000	0.0788	0.0729	-0.0059	7.5%
	Std dev	1.2932	1.2893	0.0054	2.5864	2.5857	0.0227	
INDU	Mean	0.0406	0.0407	0.0000	0.0813	0.0751	-0.0062	7.6%
	Std dev	1.1886	1.1865	0.0031	2.3772	2.3750	0.1156	
NDX	Mean	0.0593	0.0591	-0.0002	0.1186	0.1119	-0.0066	5.6%
	Std dev	1.3856	1.3845	0.0228	2.7712	2.7675	0.0225	
MID	Mean	0.0478	0.0476	-0.0002	0.0955	0.0890	-0.0066	6.9%
	Std dev	1.4503	1.4459	0.0750	2.9007	2.9014	0.0309	

We would like to achieve the followings :

- provide a model that allows for slippage ;
- trading is subject to frictions ;
- a tractable model that can be computed ;
- add an over-night component ;
- calibrate if possible.

We assume that fast trade is penalized through its time derivative. In particular, we assume that portfolio position (i,e., the number of ETFs we hold at a given time) is absolutely continuous in time.

Then, state variables are :

- X_t is the NAV of the LETF ; $dX_t = Z_t dS_t$,
- S_t is the NAV of the ETF ; $dS_t = S_t\left[\mu dt + \sigma dW_t\right]$,
- Z_t is the number of ETFs in the portfolio ; $dZ_t = \theta_t dt$,

where W is Brownian motion, $\mu, \sigma > 0$ are given constants. The control variable is

$$\theta_t := \frac{d}{dt} Z_t.$$

Daily cost for the period $[t-1, t)$ is

$$C_t := C_t(\theta_\cdot) = \mathbb{E}\left[T_t + F_t\right],$$

where λ has to be chosen or better calibrated and

$$T_t := X_{t-1}^2 \left(R_t^X - \beta R_t^S\right)^2, \quad F_t := \lambda \int_{t-1}^{t} \theta_u^2 \, S_u^2 \, du.$$

Future costs to be added after a discounting at an appropriate rate $\rho > 0$. Then the total cost is given by

$$\begin{aligned} C(X_0, S_0, Z_0; \theta_\cdot) &:= \sum_{t=0}^{\infty} e^{-\rho t} C_{t+1} \\ &= C_1 + e^{-\rho} \mathbb{E}\left[C(X_1, S_1, Z_1; \theta_\cdot)\right]. \end{aligned}$$

To formulate the problem as standard control problem and be able to use dynamic programming, it is necessary that we start with any time $t \in [0,1]$ and calculate returns with respect to any $(\bar{x}, \bar{s})$:

$$T_t(X_1, S_1; \bar{x}, \bar{s}) := \bar{x}^2 \left(\frac{X_1 - \bar{x}}{\bar{x}} - \beta \frac{S_1 - \bar{s}}{\bar{s}} \right)^2 .$$

For $t \in [0,1]$ and $(X_t, S_t, Z_t) = (x, y, z)$, the value function is given by as the fixed point of

$$V(t, x, s, z; \bar{x}, \bar{s}) := \inf_{\theta_\cdot} \mathbb{E}[\lambda \int_t^1 \theta_u^2 S_u^2 du + T_t(X_1, S_1; \bar{x}, \bar{s}) + e^{-\rho} V(X_1, S_1, Z_1; X_1, S_1)].$$

Hard to solve but we have the following homothety

$$v(t,\alpha x,\alpha s,z;\alpha\bar{x},\alpha\bar{s}) = \alpha^2 v(t,x,s,z;\bar{x},\bar{s}), \quad \alpha > 0,$$

$$v(t,\eta x,s,\eta z;\eta\bar{x},\bar{s}) = \eta^2 v(t,x,s,z;\bar{x},\bar{s}) \quad \eta > 0.$$

Hence, with $\alpha = 1/\bar{s}$, $\eta = \bar{s}/\bar{x}$,

$$v(t,x,s,z;\bar{x},\bar{s}) = \bar{x}^2\ v(t,\frac{x}{\bar{x}},\frac{s}{\bar{s}},\frac{\bar{s}}{\bar{x}}\ z,;1,1) =: \bar{x}^2\ w(t,\frac{x}{\bar{x}},\frac{s}{\bar{s}},\frac{\bar{s}}{\bar{x}}\ z).$$

$$w(t,x,s,z) = \inf_{\theta.} \mathbb{E}[\lambda \int_t^1 \theta_u^2 S_u^2\ du + ((X_1-1)-\beta(S_1-1))^2 + e^{-\rho} v(0,X_1,S_1,Z_1;X_1,S_1)].$$

This a PDE in (1+3) variables and is a fixed point.
Computationally could be quite expensive. But the solution has the following almost quadratic form,

$$
\begin{aligned}
v(t,s,x,z) = a(t)x^2 &+ d(t)x \\
&+ (b_0(t) + b_1(t)z)xs \\
&+ (c_0(t) + c_1(t)z + c_2(t)z^2)s^2 \\
&+ (e_0(t) + e_1(t)z)s + f(t),
\end{aligned}
$$

where all ten coefficients solve an ODE which is linear in some components and quadratic in the others.

$$
\begin{aligned}
a'(t) &= \frac{1}{2\lambda} b_1(t)^2, \\
b_0'(t) &= \frac{1}{\lambda} b_1(t)\; c_1(t) - \mu b_0(t), \\
b_1'(t) &= \frac{2}{\lambda} b_1(t)\; c_2(t) - \mu b_1(t) - \mu a(t), \\
c_0'(t) &= -(\sigma^2 + 2\mu) c_0(t) + \frac{1}{2\lambda} c_1(t)^2, \\
c_1'(t) &= -(\sigma^2 + 2\mu)[c_1(t) + b_0(t)] + \frac{1}{\lambda} c_1(t) c_2(t), \\
c_2'(t) &= -(\sigma^2 + 2\mu)[c_2(t) + b_1(t)] - \sigma^2 a(t) + \frac{2}{\lambda} c_2(t)^2.
\end{aligned}
$$

$$
\begin{aligned}
d'(t) &= \frac{1}{\lambda} b_1(t) e_1(t), \\
e_0'(t) &= -\mu e_0(t) + \frac{1}{\lambda} c_1(t)\ e_1(t), \\
e_1'(t) &= \frac{2}{\lambda} e_1(t)\ c_2(t) - \mu e_1(t) - \mu d(t), \\
f'(t) &= \frac{1}{2\lambda} e_1(t)^2.
\end{aligned}
$$

For a given

$$\alpha := (a, b_0, \ldots, f) \in \mathbb{R}^{10},$$

we define

$$\begin{aligned} U(x,s,z;\alpha) := &\; a\, x^2 + d\, x \\ &+ (b_0 + b_1\, z)\, x\, s \\ &+ (c_0 + c_1\, z + c_2\, z^2)\, s^2 \\ &+ (e_0 + e_1\, z)\, s + f. \end{aligned}$$

We say $\alpha \in \mathcal{P}$ if $U \geq 0$.

$$\alpha(1) = \alpha^* + e^{-\rho}\mathcal{L}(\alpha(0)), \quad \text{where}$$

$$a^* = \frac{1}{2}, \; b_0^* = -\beta, \; c_0^* = \frac{\beta^2}{2},$$
$$e_0^* = \beta(1-\beta), \; d^* = \beta - 1, \; f^* = (\beta-1)^2,$$

and the linear map $\bar{\alpha} := \mathcal{L}(\alpha)$ is given by,

$$\bar{a} = a + b_0 + c_0 + d + e_0 + f,$$
$$\bar{b}_1 = b_1 + c_1 + e_1, \quad \bar{c}_2 = c_2,$$

and all the other components are zero. Note $\alpha^* \in \mathcal{P}$.

We solve the ODE with any final condition

$$\alpha(1) = \beta.$$

Since the ODE has quadratic nonlinearity, the existence of a solution is not guaranteed by the general theory. But it has short time existence.

Importantly, by control arguments and verification, if $\beta \in \mathcal{P}$ then $\alpha(t) \in \mathcal{P}$ for all $t \leq 1$. This means that there is no blow-up and the solution exists for all time.

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Theorem

There exists a unique solution $\alpha^*(t)$ to the ODE with final data $\alpha^*(1) = \alpha^* + e^{-\rho}\mathcal{L}(\alpha(0))$ for all $t \leq 1$. The value function is

$$v(t, x, s, z) = U(x, s, z; \alpha^*(t)), \quad \forall\ t \in [0, 1].$$

Moreover, the optimal control is given by,

$$\theta^*(t) = -\frac{1}{\lambda}\left[b_1(t)\frac{X_t}{S_t} + c_1(t)\frac{\bar{x}}{\bar{s}} + 2c_2(t)Z_t + e_1(t)\frac{\bar{x}}{S_t}\right].$$

So far we were able to achieve :

- A tractable model that allows for slippage ;
- An explicit solution through a fixed point of an ODE in $\mathbb{R}^{10}$.
- An explicit hedging strategy.
- Two parameters λ and ρ to be chosen or calibrated.
- Other parameters can be estimated from data easily.
- Can integrate over night period as done in Dai, Li, Liu & Wang (MS 2015). Numerics will be shown with this.

There are two *utility type parameters*; λ, ρ. We need to either estimate or calibrate them.

Choosing ρ.

Since the cost is quadratic and $dX_t \approx \beta[\mu dt + \sigma dt]X_t$,

$$\mathbb{E}[X_t^2] \approx \exp([2\beta\mu + \beta^2\sigma^2]t).$$

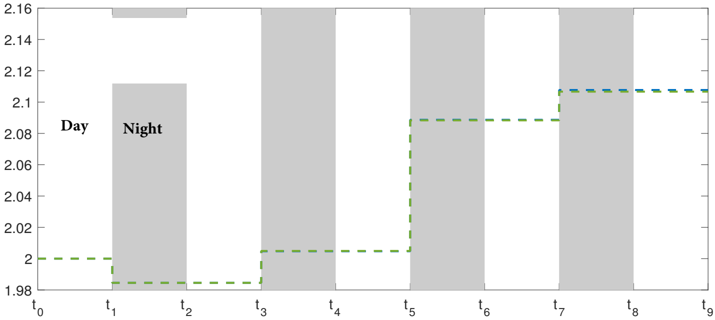
So ρ has to be at least larger than the growth rate of $\mathbb{E}[X_t^2]$. So

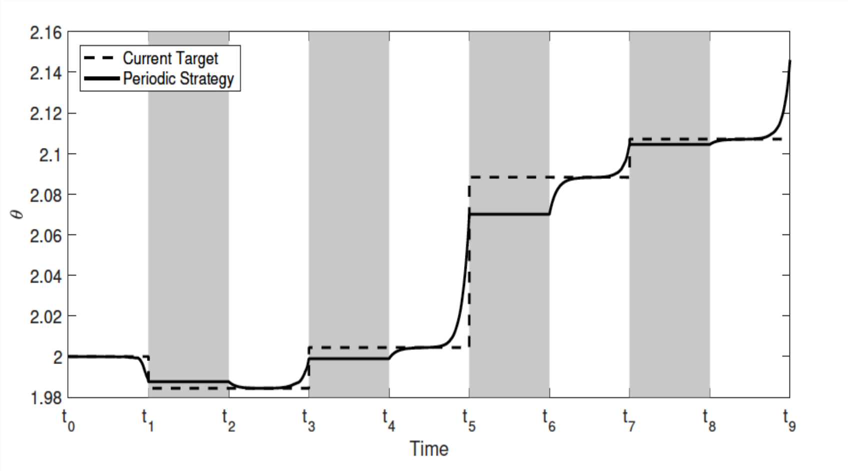
$$\rho > 2\beta\mu + \beta^2\sigma^2.$$

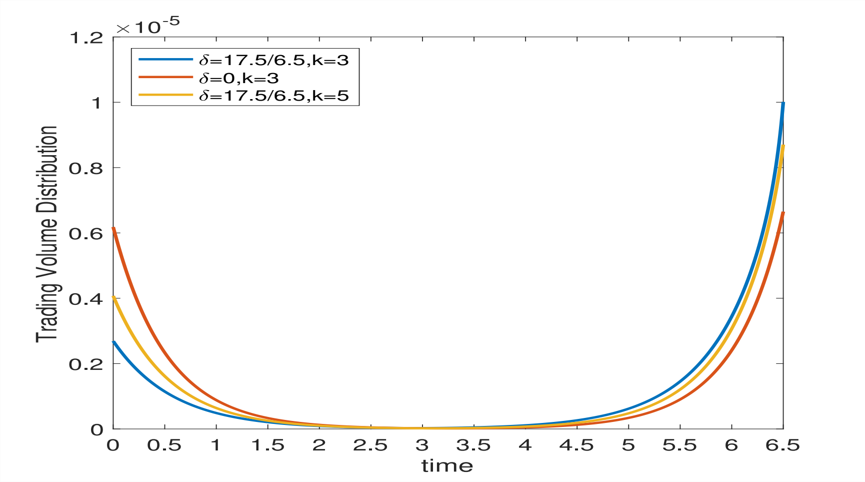
Choosing λ.

The smaller the λ, the smaller the slippage. So one can calibrate it to the amount tolerable or to the amount observed empirically.

We rebalance at the end of each day. This is a run from a Monte-Carlo simulation.




Current Target
Periodic Strategy
2.16
2.14
2.12
2.1
2.08
2.06
2.04
2.02
2
1.98
θ
t0 t1 t2 t3 t4 t5 t6 t7 t8 t9
Time


×10^-5
1.2
1
0.8
0.6
0.4
0.2
0
δ=17.5/6.5,k=3
δ=0,k=3
δ=17.5/6.5,k=5
Trading Volume Distribution
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
time

The optimal control problem at hand has a new structure.

It is continuously controlled but discretely monitored. This structure yields a fixed point result for the value function.

Similar structure was also central in the recent work by Keppo, Reppen, S. on optimal dividend payments. There, the dividend payments are made annually while other actions such as issuance can be done continuously.

- LETFs are new instruments and much to be understood.
- Control problems with a periodic discrete action or monitoring can be solved by a fixed point approach.
- This approach is quite tractable and continuous time approximation is not necessary.
- Asymptotics can also be used to simplify.

THANK YOU FOR YOUR ATTENTION.