Large deviation control and an optimal investment model

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based on a joint work with S.J. Sheu
• Large deviation principle for the empirical mean of a Bernoulli sequence 

\[ X_1, X_2, \ldots, X_n, \ldots, \] : a Bernoulli sequence,

\[ P(X_i = 1) = p, \quad P(X_i = 0) = 1 - p, \quad 0 < p < 1 \]

\[ S_n = \sum_{i=1}^{n} X_i, \]

\[ \implies P\left( \frac{S_n}{n} \leq \kappa \right) \sim e^{-nI_0(\kappa)}, \quad \text{as } n \to \infty \]

\[ I_0(\kappa) = \sup_{\theta < 0} \{ \theta \kappa - \rho(\theta) \} : \text{ rate function} \]

\[ \rho(\theta) = \frac{1}{n} \log E[e^{\theta S_n}] = \frac{1}{n} \log (1 - p + pe^{\theta})^n = \log (1 - p + pe^{\theta}) \]

Since \( \rho(\theta) \) is strictly convex, \( \rho'(\theta) \) is increasing and

\[ I_0(\kappa) = \infty, \quad \text{for } \kappa < \rho'(-\infty) = 0, \]

\[ I_0(\kappa) = 0, \quad \text{for } \kappa \geq \rho'(0-) = p \]
• Evaluation of $P\left(\frac{S_n}{n} \leq \kappa\right)$ is meaningful for $\rho'(\infty) < \kappa < \rho'(0-)$. Let us call the interval $(\rho'(-\infty), \rho'(0-)) = (0, p)$ "effective domain" of the rate function $I_0(\kappa)$.

Note that

$$\rho'(0-) = p = \lim_{n \to \infty} \frac{S_n}{n} \quad "\text{the law of large numbers}"

• "The law of large numbers" rules in the right end point of the "effective domain"

• The left end point 0 is determined by definition of $S_n \geq 0$
Large deviation control

Consider minimizing the probability that the growth rate of a semi-martingale functional $Y_T(h)$ falls below $\kappa$, which may have the following asymptotic behavior in certain circumstances

\[
\inf_{h \in \mathcal{H}_x(T)} P\left( \frac{1}{T} Y_T(h) \leq \kappa \right) \sim e^{-TI(\kappa)}, \quad T \to \infty
\]

$\kappa$: a given target growth rate,

\[
Y_T(h) = \int_0^T g(X_s, h_s) ds + \int_0^T \varphi(X_s, h_s) dW_s,
\]

$X_s$: a diffusion process, $h_s$: a control process,

We are concerned with

- the rate function $I(\kappa)$
- the ”effective domain” of $\kappa$ where $0 < I(\kappa) < \infty$
- (asymptotically) optimal strategies
- The objective of large deviation control is to control arising probabilities of "rare events".
- Evaluation of the minimizing probability for $\kappa$ such that $I(\kappa)$ does not vanish nor diverge is meaningful and we are concerned with the effective domain.
  
  ● What is the rate function $I(\kappa)$ in the current case?
  
  ● What rules in the end points of the "effective domain"?
  
  ● The strategies which realize this asymptotics of the minimizing probability are said asymptotically optimal. How to construct them?

- Large deviation control problems may provide new aspects in stochastic control theory. Indeed, further probabilistic arguments than usual are required to evaluate such asymptotic behavior of the minimizing probability.
Motivation arising from math. finance:

Riskless asset:
\[ dS^0(t) = r(X_t)S^0(t)dt, \quad S^0(0) = s^0. \]

Risky assets:
\[ \begin{align*}
\text{d}S^i(t) &= S^i(t)\{\alpha^i(X_t)dt + \sum_{k=1}^{N} \sigma^i_k(X_t)dW^k_t\}, \\
S^i(0) &= s^i, \quad i = 1, \ldots, m, \quad N = n + m + 1
\end{align*} \]

Factors:
\[ \begin{align*}
\text{d}X_t &= \beta(X_t)dt + \lambda(X_t)dW_t, \quad X(0) = x \in \mathbb{R}^n \\
\beta : \mathbb{R}^n &\mapsto \mathbb{R}^n, \quad \lambda : \mathbb{R}^n &\mapsto \mathbb{R}^n \otimes \mathbb{R}^N,
\end{align*} \]

Benchmark:
\[ \begin{align*}
\frac{dL_t}{L_t} &= \gamma(X_t)dt + \xi(X_t)^*dW_t, \quad L_0 = l_0, \quad \xi : \mathbb{R}^n &\mapsto \mathbb{R}^N
\end{align*} \]
Total wealth:

\[ V_t = \sum_{i=0}^{m} N^i_t S^i_t \]

\( N^i_t \): Number of the shares

\( h^i_t = \frac{N^i_t S^i_t}{V_t} \): Portfolio proportion \( i = 0, 1, 2, \ldots, m. \)

\( h_t := (h^1_t, \ldots, h^m_t) \)

Under the self financing condition \( V_t \) satisfies

\[
\frac{dV_t}{V_t} = \{ r(X_t) + h(t)^* \hat{\alpha}(X_t) \} dt + h(t)^* \sigma(X_t) dW_t,
\]

and we are concerned with **downside risk minimization**:

\[ \inf_{h} P \left( \frac{1}{T} Y_T \leq \kappa \right) := \inf_{h} P \left( \frac{1}{T} \log \frac{V_T(h)}{L_T} \leq \kappa \right) \sim e^{-TI(\kappa)}, \text{ as } T \to \infty \]
\[ \log V_T = \log V_0 + \int_0^T \{ -\frac{1}{2} h_s^* \sigma^*(X_s) h_s + h_s^* \tilde{\alpha}(X_s) + r(X_s) \} dt + \int_0^T h_s^* \sigma(X_s) dW_s, \]

\[ \tilde{\alpha}(x) = \alpha(x) - r(x) 1, \quad 1 = (1, 1, \ldots, 1)^* \]

and

\[ \log L_T = \log l_0 + \int_0^T \{ \gamma(X_s) - \frac{1}{2} \xi^* \xi(X_s) \} ds + \int_0^T \xi(X_s)^* dW_s \]

we have

\[ Y_T(h) := \log \frac{V_T}{L_T} = \log \frac{V_0}{l_0} + \int_0^T g(X_s, h_s) ds + \int_0^T (h_s^* \sigma(X_s) - \xi^*(X_s)) dW_s \]

\[ g(x, h) = -\frac{1}{2} h^* \sigma^*(x) h + h^* \tilde{\alpha}(x) + \tilde{g}(x), \quad \tilde{g}(x) = r(x) - \gamma(x) + \frac{1}{2} \xi^* \xi(x) \]
• Previous results (without stochastic benchmark):

\[(LC) \quad J_0(\kappa) := \lim_{T \to \infty} \frac{1}{T} \inf_{h \in \mathcal{H}(T)} \log P\left(\frac{1}{T} \log V_T(h) \leq \kappa\right)\]

\[(DR) \quad J_0(\kappa) = -\sup_{\theta < 0} \{\theta \kappa - \chi_0(\theta)\}, \quad \chi_0'(-\infty) < \kappa < \chi_0'(0-)\]

\[(RS) \quad \chi_0(\theta) := \lim_{T \to \infty} \frac{1}{T} \inf_{h} \log E[e^{\theta \log V_T(h)}], \quad \theta < 0.\]

- Duality relationship (DR) between downside risk minimization (LC) and risk-sensitive portfolio optimization over large time (RS) is shown for several models.
- Establishing (DR) and analyzing (RS) give asymp. optimal strategies of (LC):
  1. pick up a given constant \(\kappa\) in (LC)
  2. take constant \(\theta(\kappa)\) which attains the supremum in (DR) with this \(\kappa\)
  3. select an asymptotically optimal strategy of (RS) with \(\theta = \theta(\kappa)\)

C.f.
Hata - N. - Sheu (2010) AAP; N.(2011) QF (Linear Gaussian models)
Hata (2011) APFM (CIR models); N. (2012) AAP (General factor models)
Watanabe (2013) SPA (Hidden Markov models)
Downside risk minimization and asymptotic arbitrage

Note that
\[ \inf_h P\left( \frac{1}{T} \log V_T(h) \leq \kappa \right) = 1 - \sup_h P\left( \frac{1}{T} \log V_T(h) > \kappa \right) \]

and that we are looking at the asymptotic behavior for \( \kappa \) such that \( 0 < I(\kappa) < \infty \),

\[ \inf_h P\left( \frac{1}{T} \log V_T(h) \leq \kappa \right) \sim e^{-TI(\kappa)}, \quad T \to \infty \]

which is close to 0, and hence \( \sup_h P\left( \frac{1}{T} \log V_T(h) > \kappa \right) \) is close to 1. Thus, it implies the relationship between downside risk minimization and "asymptotic arbitrage" discussed by Föllmer-Schachermayer ’08.
• New aspects in this talk

We are going to talk about the cases with a stochastic benchmark $L_T$

$$\inf_{h \in H_{x}(T)} P\left( \frac{1}{T} \log \frac{V_T(h)}{L_T} \leq \kappa \right) \sim e^{-TI(\kappa)}, \quad T \to \infty.$$ 

We shall see the duality relationship between (LC) and (RS) over large time holds in a similar context as before. On the other hand, distinct difference appears in the "effective domain" at $\chi'(-\infty)$.

• Financial meaning of benchmark $L_T$:


  (Linear Gaussian models, Finite time horizon)

  ▶ The concept of the benchmark is ubiquitous in the investment management industry, where 'benchmarks' are preset portfolio or indexes used to compare the performance of an actual portfolio or fund.

  ▶ The investment committee (for insurance company, pension funds and mutual funds) or the end investor sets the benchmark and mandates the investment manager to produce at least the same level of returns as the benchmark or to outperform it.

  ▶ What is actually used as a benchmark?

Examples: S&P500 (Standard and Poors 500), MSCI World index, Barclays Capital Aggregate Bond Index, Nikkei 225, Nomura BPI ... etc.
Remarks on **Upside chance maximization**:

\[
J_u(\kappa) = \sup_h \lim_{T \to \infty} \frac{1}{T} \log P\left( \frac{1}{T} \log V_T(h) \geq \kappa \right)
\]

\[
\chi_+(\theta) := \sup_{h \in A} \lim_{T \to \infty} \frac{1}{T} \log E[e^{\theta \log V_T(h)}], \quad 0 < \theta < 1
\]

The arguments are rather similar to the proof of the large deviation principle without control parameter. Indeed, "Gärtner-Ellis" theorem holds. Namely, If \( \chi_+(\theta) \) is differentiable with respect to \( \theta < \theta^* \) and \( \lim_{\theta \to \theta^*} \chi'_+(\theta) = \infty \) for some \( 0 < \theta^* \leq 1 \), then, for \( \chi'_+(0+) < \kappa < \infty \), one can obtain duality relationship:

\[
J_u(\kappa) = - \inf_{k \in [\kappa, \infty)} \sup_{\theta \in [0, \theta^*)} \{ \theta k - \chi_+(\theta) \} = - \sup_{\theta \in [0, \theta^*)} \{ \theta \kappa - \chi_+(\theta) \}.
\]

Serious difficulties lie in specifying the constant \( \theta^* \) except the case of one dimension.


For some \( 0 < \theta < 1 \), \( \chi_+(\theta) = \infty \), and if \( 0 < \theta < 1 \) is small, then \( \chi_+(\theta) < \infty \) and verification holds for risk-sensitive portfolio optimization.
1. Problems setting

\{W_t\}: \text{N - dim. Brownian motion process on } (\Omega, \mathcal{F}, P; \mathcal{F}_t) \\

(n- \text{ dim.}) \text{ State process} \quad dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^n \\
\beta: \mathbb{R}^n \mapsto \mathbb{R}^n, \quad \lambda: \mathbb{R}^n \mapsto \mathbb{R}^n \otimes \mathbb{R}^N

Controlled functional with an (m - dim.) control parameter \(h\): 

\[ Y_T(h) = Y_0 + \int_0^T g(X_s, h_s)ds + \int_0^T \{\sigma(X_s)^* h_s - \xi(X_s)\}^* dW_s, \]

\[ g(x, h) := -\frac{1}{2} h^* \sigma^*(x) h + h^* \tilde{\alpha}(x) + \tilde{g}(x) \]

\[ \sigma: \mathbb{R}^n \mapsto \mathbb{R}^m \otimes \mathbb{R}^N, \; \tilde{\alpha}: \mathbb{R}^n \mapsto \mathbb{R}^m, \; \tilde{g}: \mathbb{R}^n \mapsto \mathbb{R}^1 \; \xi: \mathbb{R}^n \mapsto \mathbb{R}^N \]

We are going to study

(1.1) \[ J(\kappa) := \lim_{T \to \infty} \inf_{h \in \mathcal{H}(T)} \frac{1}{T} \log P\left(\frac{1}{T} Y_T(h) \leq \kappa\right), \]

\[ \mathcal{H}(T) = \{h(t); R^m\text{- valued, progr. m'ble s.t. } E[\int_0^T |h(s)|^2 ds] < \infty\} \]
Risk sensitive control problem:

(1.2) \[ \hat{\chi}(\theta) = \lim_{T \to \infty} \frac{1}{T} \inf_{h \in \mathcal{A}(T)} \log E[e^{\theta Y_T(h)}], \quad \theta < 0, \]

where the set \( \mathcal{A}_T \) of admissible strategies is defined as

\[ \mathcal{A}_T = \{ h_s \in \mathcal{H}_T; E[e^{M_{T}^{v,h} - \frac{1}{2}(\langle M_{T}^{v,h} \rangle)}] = 1 \}, \]

\[ M_{t}^{v,h} := \int_{0}^{t} [\nabla v(s, X_s) \lambda(X_s) + \theta \{ h_s^* \sigma(X_s) - \xi(X_s)^* \}] dW_s, \]

and \( v(t, x) \) is the solution to the HJB equation for

\[ \hat{v}(t, x) + \theta y_0 = \inf_{h \in \mathcal{A}(T)} \log E[e^{\theta Y_T(h)}] \]

- We establish the duality relationship between (1.1) and (1.2), and construct asymptotically optimal strategies. We then examine the effective domain of \( I(\kappa) \).
2. Duality theorem

\[ Y_t(h) = y_0 + \int_0^t g(X_s, h_s)ds + \int_0^t \{h^*_s\sigma(X_s) - \xi(X_s)^*\}dW_s. \]

Thus,

\[
e^{\theta Y_T(h)} = e^{\theta y_0} e^{\theta \int_0^T g(X_s, h_s)ds} + e^{\theta \int_0^T \{h^*_s\sigma(X_s) - \xi(X_s)^*\}dW_s}
\]

(2.1)

where

\[
M^h_T = \int_0^T \{h^*_s\sigma(X_s) - \xi(X_s)^*\}dW_s, \quad \eta(x, h) = g(x, h) + \frac{\theta}{2} |\sigma(x)^* h - \xi(x)|^2.
\]

If \( E[e^{\theta M^h_T - \frac{\theta^2}{2} \langle M^h \rangle_T}] = 1 \), then the value function can be written as

\[
\tilde{J}(t, x) = \theta y_0 + \inf_{h \in \mathcal{A}(T)} \log E^h[e^{\theta \int_0^{T-t} \eta(X_s, h_s)ds}] \equiv \theta y_0 + \tilde{v}(t, x)
\]

(2.2)
Since

\[ P^h(A) = E[e^{\theta M^h - \frac{\theta^2}{2} \langle M^h \rangle_t} : A] \]

defines a probability measure and

\[ W^h_t := W_t - \theta \int_0^t \{ \sigma^*(X_s) h_s - \xi(X_s) \} ds \]

is a Brownian motion process under \( P^h \) and

SDE:

(2.3) \quad dX_t = \{ \beta(X_t) + \theta \lambda(X_t)(\sigma^*(X_t)h_t - \xi(X_t)) \} dt + \lambda(X_t)dW^h_t, \quad X_0 = x

is regarded as the controlled dynamics.

**The HJB equation** for \( \tilde{v}(t, x) \) is seen to be

\[
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 v] + \frac{1}{2} (Dv)^* \lambda \lambda^* Dv \\
+ \inf_h \{ [\beta + \theta \lambda(\sigma^* h - \xi)] Dv + \theta \eta(x, h) \} = 0,
\end{aligned}
\]

(2.4)

\[ v(T, x) = 0, \]
or,

\[
(2.4)'
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 v] + \beta^* D v + \frac{1}{2} (D v)^* \lambda N_\theta^{-1} \lambda^* D v + U_\theta &= 0, \\
v(T, x) &= 0,
\end{aligned}
\]

where

\[
\beta_\theta = \beta + \theta \lambda N_\theta^{-1} (\sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} - \xi), \quad N_\theta^{-1} = I + \frac{\theta}{1 - \theta} \sigma^* (\sigma \sigma^*)^{-1} \sigma
\]

\[
U_\theta = \frac{\theta^2}{2} \{\sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} - \xi\} N_\theta^{-1} \{\sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} - \xi\} + \theta \frac{1}{2} \sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} + \tilde{g}(x)
\]

\[
\therefore \quad \frac{1}{1 - \theta} I \leq N_\theta^{-1} \leq I
\]

Also written as

\[
(2.4)''
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 v] + \beta^* D v \\
+ \frac{1}{2} \{\lambda^* D v + \theta (\sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} - \xi)\} N_\theta^{-1} \{\lambda^* D v + \theta (\sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} - \xi)\} \\
+ \theta \frac{1}{2} \hat{\alpha}^* (\sigma \sigma^*)^{-1} \hat{\alpha} + \tilde{g}(x)) = 0
\end{aligned}
\]

\[
v(T, x) = 0,
\]
Assumptions

(2.5) \( \lambda, \beta, \sigma, \hat{\alpha}, \hat{g}, \) and \( \xi \) are globally Lipschitz and smooth,

(2.6) \( \hat{g} \) is bounded below and \( \xi \) is bounded,

(2.7) \[
\begin{align*}
\{ & c_1|q|^2 \leq q^*\lambda\lambda^*(x)q \leq c_2|q|^2, \ c_1, c_2 > 0, \ q \in \mathbb{R}^n, \\
& c_1|\zeta|^2 \leq \zeta^*\sigma\sigma^*(x)\zeta \leq c_2|\zeta|^2, \ \zeta \in \mathbb{R}^m.
\end{align*}
\]

Proposition 1 Under assumptions (2.5) - (2.7), H-J-B equation (2.4) has a solution such that

\[
v(t,x) \leq K_0,
\]

\[
\frac{\partial v}{\partial t} \geq -C, \quad C > 0
\]

\[
|\nabla v|^2 + k(\frac{\partial v}{\partial t} + C) \leq C'(|\nabla(\lambda N_\theta^{-1}\lambda^*)|^2_{2r} + |\lambda N_\theta^{-1}\lambda^*|^2_{2r} + |\nabla(\lambda\lambda^*)|^2_{2r} + |\nabla\beta^*_\theta|_{2r}
\]

\[
+|\beta^*_\theta|^2_{2r} + |U_\theta|_{2r} + |\nabla U_\theta|_{2r} + 1), \quad x \in B_r, \ t \in [0, T),
\]

where \( |f|_r = \sup_{x \in B_r} |f(x)|, \ k > 0 \) is a sufficiently large positive constant, \( C' \) a positive constant depending on \( c_2, \frac{c_1}{1-\theta}, k \) and \( n \) but not on \( T, r \geq r_0, \) and \( C \) is the upper bound of \( U_\theta. \)
Remarks.

- c.f. N. '96 SICON, Hata-N.-Sheu '18 Preprint, Bensoussan-Frehse-N. '98, for the proof of Proposition 1.

- According to the above gradient estimate, we see that

\[ |\nabla v(t, x)| \leq M_1 (1 + |x|) \]

for some positive constant \( M_1 \) under our assumptions (2.5) - (2.7).

- Note that the solution \( v \) to HJB equation (2.4) such that \( |v(t, x)| \leq M(1 + |x|^2) \), \( M > 0 \) is unique (c.f. F. Da Lio and O. Ley '06 SICON ) and the set \( \mathcal{A}_T \) of admissible strategies in the verification theorem is defined by the solution specified in this sense.
We can see that the infimum in (2.4) is attained by

\[(2.8) \quad \hat{h}(t, x) := \frac{1}{1 - \theta} (\sigma \sigma^*)^{-1} \{\sigma \lambda^* \nabla v(t, x) + \tilde{\alpha}(x) - \theta \sigma \xi(x)\} \]

and the optimal strategy is constructed from this function. Indeed, we have the following verification theorem.

**Proposition 2** Let \(v(t, x; T)\) be a solution to H-J-B equation (2.4). Then, under assumptions (2.5)-(2.7), the following verification holds

\[\inf_{h \in \mathcal{A}_T} \log E[e^{\theta Y_T(h)}] = \log E[e^{\theta Y_T(\hat{h})}] = v(0, x; T) + \theta y_0,\]

where, \(\hat{h}_t = \hat{h}(t, X_t)\), and the set \(\mathcal{A}_T\) of admissible strategies is defined as

\[\mathcal{A}_T = \{h_s \in \mathcal{H}_T; E[e^{M_{t, h}^{v, h} - \frac{1}{2}M_{t, h}^{v, h}}} = 1\},\]

\[M_{t, v, h} := \int_{0}^{t} \left[\nabla v(s, X_s)^* \lambda(X_s) + \theta \{h_s^* \sigma(X_s) - \xi(X_s)^*\}\right] dW_s,\]

**Remark.** c.f. N. '15 in WS volume in the case without benchmark.

- Progressively measurable process \(h_t\) such that \(|h_t| \leq M(1 + |X_t|), t \leq T\) is seen to be \(h_t \in \mathcal{A}_T\)
Ergodic type HJB equation

(2.9) \( \chi(\theta) = \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 \bar{v}] + \frac{1}{2} (D \bar{v})^* \lambda \lambda^* D \bar{v} + \inf_h \{[\beta + \theta \lambda(\sigma^* h - \xi)]^* D \bar{v} + \theta \eta(x, h)\}, \)

(2.9)’ \( \chi(\theta) = \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 \bar{v}] + \beta^*_\theta D \bar{v} + \frac{1}{2} (D \bar{v})^* \lambda N^{-1}_\theta \lambda^* D \bar{v} + U_\theta. \)

Proposition 3  (i) Besides assumptions (2.5) -(2.7) we assume

(2.10) \( \tilde{\alpha}^*(\sigma^*)^{-1}\tilde{\alpha} \to \infty, \quad |x| \to \infty. \)

Then, there exists a solution \( (\chi(\theta), \bar{v}) \) to (2.9) such that \( \bar{v} \) is bounded above. Further, such a solution is unique up to additive constants with respect to \( \bar{v} \).

(ii) We further assume that there exists a positive constant \( c > 0 \) such that

(2.11) \( \lim_{|x| \to \infty} \frac{H(x, cx)}{1 + |x|^2} < 0, \quad H(x, p) := \beta^*_\theta p + \frac{1}{2} p^* \lambda N^{-1}_\theta \lambda^* p + U_\theta. \)

Then,

\( v(0, x; T) - (\bar{v}(x) + \chi(\theta)T) \to c_\infty, \quad T \to \infty, \)

uniformly on each compact set, where \( c_\infty \) is a constant.
(iii) If we assume that
\begin{equation}
\hat{\alpha}^*(\sigma \sigma^*)^{-1}\hat{\alpha} \geq c_\alpha |x|^2 - c'_\alpha \quad \exists c_\alpha, c'_\alpha > 0
\end{equation}
then, (2.11) holds and moreover we have
\[ \bar{v}(x) \leq -c_v |x|^2 + c'_v, \quad \exists c_v, c'_v > 0 \]

Remarks.

- cf. Bensoussan-Frehse '92; N. '96 SICON, '12 AAP, as for (i)
- cf. Ichihara-Sheu '13 SIMA, and N. '15 in WS volume, for (ii)
- cf. N. '12 AAP for (iii)

Further, we have the following estimates for the solution \( \bar{v} \) of EHJB
\[ -c_v |x|^2 - c'_v \leq \bar{v}(x) \leq -c_v |x|^2 + c'_v, \quad \exists c_v, c'_v > 0, \ c_v, c'_v > 0 \]
under (2.5) - (2.7) and (2.11)' since we have the gradient estimate
\[ |\nabla \bar{v}|^2 \leq M_1 ( |\nabla (\lambda \theta \lambda^*)|^2_{2r} + |\lambda \theta^{-1} \lambda^*|^2_{2r} + |\nabla (\lambda \lambda^*)|^2_{2r} + |\nabla \beta_\theta|^2_{2r} + |\beta_\theta|^2_{2r} + |U_\theta|^2_{2r} + |\nabla U_\theta|^2_{2r} + 1), \quad x \in B_r, \]
The growth rate of $U_\theta(x)$ plays a crucial role in studying the ergodic type equation and its leading term is $\frac{1}{2} \hat{\alpha}^* (\sigma \sigma^*)^{-1} \hat{\alpha}$

$$U_\theta = \frac{\theta^2}{2} \left\{ \sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} - \xi \right\} N_\theta^{-1} \left\{ \sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha} - \xi \right\} + \theta \left( \frac{1}{2} \hat{\alpha}^* (\sigma \sigma^*)^{-1} \hat{\alpha} + \hat{g}(x) \right)$$
Convexity

Rewrite (2.4)' as follows

\[ \frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 v] + \beta^* \nabla v \]
\[ + \frac{1}{2} \{ \lambda^* \nabla v + \theta (\sigma^* (\sigma \sigma^*)^{-1} \tilde{\alpha} - \xi) \} * N_\theta^{-1} \{ \lambda^* \nabla v + \theta (\sigma^* (\sigma \sigma^*)^{-1} \tilde{\alpha} - \xi) \} \]
\[ + \frac{\theta}{2} \tilde{\alpha}^* (\sigma \sigma^*)^{-1} \tilde{\alpha} + \theta \tilde{g}(x) = 0, \]
\[ v(T, x) = 0 \]

It is regarded as the HJB equation

\[ \left\{ \begin{align*}
\frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 v] + \beta^* \nabla v &+ \sup_{z \in \mathbb{R}^n} \left[ -\frac{1}{2} z^* N_\theta z + z^* \{ \lambda^* \nabla v + \theta (\sigma^* (\sigma \sigma^*)^{-1} \tilde{\alpha} - \xi) \} \right] \]
\[ + \frac{\theta}{2} \tilde{\alpha}^* (\sigma \sigma^*)^{-1} \tilde{\alpha} + \theta \tilde{g}(x) = 0 \]
\[ v(T, x) = 0 \]
\[ N_\theta \equiv I - \theta \sigma^* (\sigma \sigma^*)^{-1} \sigma \]
It corresponds to the stochastic control problem having the value function

\[ v(t, x; T) = \sup_{Z \in \mathcal{Z}_T} E[\int_0^{T-t} \Phi(Y_s, Z_s)ds] \]

with

\[ \Phi(x, z) = -\frac{1}{2} z^{*} \mathbf{N} \theta z + \theta z^{*} (\sigma^{*} (\sigma^{*})^{-1} \hat{\alpha} - \xi) + \frac{\theta}{2} \hat{\alpha}^{*} (\sigma^{*})^{-1} \hat{\alpha} + \theta \hat{g}(x) \]

and the control dynamics \( Y_t \) governed by

\[ dY_t = \lambda(Y_t)dW_t + \{ \beta(Y_t) + \lambda(Y_t)Z_t \}dt, \quad Y_0 = x \]

\[ \mathcal{Z}_T = \{ z_t; z_t \text{ is progr. m'ble s.t.} \quad E[\int_0^{T} |Z_s|^2 ds] < \infty \} \]

**Lemma 1** The solution of HJB equation (2.4)' is convex with respect to \( \theta \).

Indeed, since \( \Phi(x, z) \) is a linear function of \( \theta \), \( \hat{v}(t, x; T) \) is convex. Thus, solution \( v(t, x) \) to the HJB equation turns out to be convex.

- \( \chi(\theta) = \lim_{T \to \infty} \frac{1}{T} v(0, x; T) \) is also convex. (cf. Proposition 3 (ii))
• Solution \((\chi(\theta), \bar{v})\) to EHJB and its derivative

\[
\chi(\theta) = \frac{1}{2} \text{tr}[\lambda \lambda^* (x) D^2 \bar{v}] + \frac{1}{2} (D \bar{v})^* \lambda \lambda^* (x) D \bar{v} + \inf_{h \in \mathbb{R}^n} \left[ \{\beta(x) + \theta \lambda(x)(\sigma^*(x)h - \xi(x))\}^* D \bar{v} + \theta \eta(x, h) \right],
\]

\[
\chi(\theta) = \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 \bar{v}] + \beta^*_\theta D \bar{v} + \frac{1}{2} (D \bar{v})^* \lambda N_\theta^{-1} \lambda^* D \bar{v} + U_\theta
\]

\[
\chi(\theta) = \bar{L} \bar{v} - \frac{1}{2} (D \bar{v})^* \lambda N_\theta^{-1} \lambda^* D \bar{v} + U_\theta
\]

\[
\bar{L} \phi := \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 \phi] + \beta^*_\theta D \phi + (D \bar{v})^* \lambda N_\theta^{-1} \lambda^* D \phi : \text{ergodic}
\]

\[-\bar{v}(x) \to \infty, \quad -\bar{L} \bar{v}(x) \to -\infty, \ \text{as} \ |x| \to \infty
\]

Equation of the formal derivatives of EHJB:

\[(*) \quad \chi'(\theta) = \bar{L} u + \left( \frac{\partial \beta_\theta}{\partial \theta} \right)^* D \bar{v} + \frac{1}{2} (D \bar{v})^* \lambda \left( \frac{\partial N_\theta^{-1}}{\partial \theta} \right) \lambda^* D \bar{v} + \frac{\partial U_\theta}{\partial \theta}
\]
Poisson equation

\[
\rho(\theta) = \overline{Lu} + f(x)
\]

\((P)\)

\[
\begin{align*}
    u &\in W^{2,p}_{loc}, & \sup_{B_{R_0}^c} \frac{|u(x)|}{|\tilde{v}(x)|} &< \infty, & R_0 &> 0,
\end{align*}
\]

under

\[
\sup_{B_{R_0}^c} \frac{|f(x)|}{L \tilde{v}(x)} < \infty,
\]

where \(R_0\) is taken as

\[
\overline{L\tilde{v}} > 1, \quad \overline{L\tilde{v}} + \frac{1}{2(1 - \theta)\tilde{v}}(D\tilde{v})^* \lambda \lambda^* D\tilde{v} > 0, \quad x \in B_{R_0}^c
\]

(cf. Bensoussan '88)

If

\[
f(x) = (\frac{\partial \beta_\theta}{\partial \theta})^* D\tilde{v} + \frac{1}{2}(D\tilde{v})^* \lambda \frac{\partial N_{\theta}^{-1}}{\partial \theta} \lambda^* D\tilde{v} + \frac{\partial U_\theta}{\partial \theta},
\]

then can we see that

\[
\rho(\theta) = \chi'(\theta)
\]
**Theorem 1** i) Assume (2.5)-(2.7) and (2.11)'. Then, for \( \kappa \) such that 
\[ \chi'(-\infty) < \kappa < \chi'(0^-) \], we have

\[
\lim_{T \to \infty} \frac{1}{T} \inf_{h.} \log P \left( \frac{1}{T} Y_T(h) \leq \kappa \right) = -\sup_{\theta < 0} \{ \theta \kappa - \chi(\theta) \}.
\]

Further, take \( \theta(\kappa) \) which attains the supremum in the right hand side and set 
\( \hat{h}_t^{(T)} = \hat{h}(X_t, \nabla v(t, X_t)) \) with

\[
\hat{h}(x, p) = \frac{1}{1 - \theta \sigma \sigma^* - 1} \{ \sigma \lambda^* \nabla v(t, x) + \hat{\alpha}(x) - \theta \sigma \xi(x) \},
\]

where \( \theta = \theta(\kappa) \) and \( v(t, x) \) is the solution to the HJB equation (2.4) with \( \theta = \theta(\kappa) \).

Then,

\[
\lim_{T \to \infty} \frac{1}{T} \log P \left( \frac{1}{T} Y_T(\hat{h}_t^{(T)}) \leq \kappa \right) = -\sup_{\theta < 0} \{ \theta \kappa - \chi(\theta) \}.
\]
ii) If the solution $\bar{v}$ to equation (2.9) with $\theta = \theta(\kappa)$ satisfies

\[(\nabla \bar{v})^* \lambda \sigma^*(\sigma \sigma^*)^{-1} \sigma \lambda^* \nabla \bar{v} - (\bar{\alpha} - \theta \sigma \xi)^*(\sigma \sigma^*)^{-1} (\bar{\alpha} - \theta \sigma \xi) \to -\infty, \quad |x| \to \infty,\]

then, by setting $\bar{h}_t = \bar{h}(X_t, \nabla \bar{v}(X_t))$ with

$$\bar{h}(x, p) = \frac{1}{1 - \theta} (\sigma \sigma^*)^{-1} \{ \sigma \lambda^* (x) p + \bar{\alpha}(x) - \theta \sigma \xi(x) \},$$

we have

$$\lim_{T \to \infty} \frac{1}{T} \log P\left( \frac{1}{T} Y_T(\bar{h}) \leq \kappa \right) = \lim_{T \to \infty} \inf_h \frac{1}{T} \log P\left( \frac{1}{T} Y_T(h) \leq \kappa \right)$$

$$= \inf_h \lim_{T \to \infty} \frac{1}{T} \log P\left( \frac{1}{T} Y_T(h) \leq \kappa \right)$$

$$= -\sup_{\theta < 0} \{ \theta \kappa - \chi(\theta) \}.$$ 

**Remark**

If (*) holds with $\theta = \theta(\kappa)$, then

- \[
\lim_{T \to \infty} \inf_h \frac{1}{T} \log E[e^{\theta Y_T(h)}] = \inf_h \lim_{T \to \infty} \frac{1}{T} \log E[e^{\theta Y_T(h)}]
\]
Rough sketch of the proof of the **Lower estimates**

Equivalent HJB-Isaacs equation:

\[
\chi(\theta) = \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 \bar{v}] + \beta^* \nabla \bar{v} \\
+ \sup_{\nu \in \mathbb{R}^n} \inf_{h \in \mathbb{R}^n} \left\{ \nu^* \lambda^* \nabla \bar{v} + \theta g(x, h) + \theta(h^* \sigma - \xi^*) \nu - \frac{1}{2} |\nu|^2 \right\}
\]

\[
\tilde{h}(x, \nu) = (\sigma \sigma^*)^{-1}(x)(\hat{\alpha}(x) + \sigma(x) \nu), \quad \tilde{\nu}(x) = N_{\nu}^{-1}\{\lambda^* \nabla \bar{v} + \theta(\sigma^*(\sigma \sigma^*)^{-1}\hat{\alpha} - \xi)\}
\]

- \[
\chi(\theta) - \theta \chi'(\theta) = \overline{L}(\bar{v} - \theta \bar{v}') - \frac{1}{2} \tilde{\nu}^* \tilde{\nu},
\]

- \[
\chi'(\theta) = \overline{L} \bar{v}' + g(x, \tilde{h}) + (\tilde{h}^* \sigma - \xi^*) \tilde{\nu}, \quad \tilde{h} = \tilde{h}(x, \tilde{\nu})
\]
\[ \frac{d\tilde{P}}{dP} \bigg|_{\mathcal{F}_T} := e^{\int_0^T \tilde{v}_s dW_s - \frac{1}{2} \int_0^T |\tilde{v}_s|^2 ds}, \quad \tilde{v}_s = \tilde{v}(X_s), \quad \tilde{W}_t = W_t - \int_0^t \tilde{v}_s ds. \]

\( (X_t, \tilde{P}) : \mathcal{L} - \text{diffusion process} \)

\[ dX_t = \lambda(X_t) d\tilde{W}_t + \{\beta(X_t) + \lambda(X_t)\tilde{v}(X_t)\}dt, \quad X_0 = x \]

- \[ Y_T(h) = y_0 + \int_0^T g(X_s, h_s) ds + \int_0^T \{h_s^* \sigma(X_s) - \xi^*(X_s)\} dW_s \]

\[ = y_0 + \int_0^T \{g(X_s, h_s) + (h_s^* \sigma(X_s) - \xi^*(X_s)) \tilde{v}_s\} ds \]

\[ + \int_0^T \{h_s^* \sigma(X_s) - \xi^*(X_s)\} d\tilde{W}_s \]
Noting that $\sigma \sigma^* \tilde{h} = \tilde{\alpha} + \sigma \tilde{\nu}$, 
$
\tilde{h}(x, \nu) = (\sigma \sigma^*)^{-1}(x)(\tilde{\alpha}(x) + \sigma(x)\nu),
$

g(x, h) + (h^* \sigma(x) - \xi^*(x))\tilde{\nu}
$

= g(x, \tilde{h}) + (\tilde{h}^* \sigma(x) - \xi^*(x))\tilde{\nu} - \frac{1}{2}(h - \tilde{h})^* \sigma \sigma^*(h - \tilde{h})
$

= \chi'(\theta) - \overline{L}\overline{\nu}' - \frac{1}{2}(h - \tilde{h})^* \sigma \sigma^*(h - \tilde{h})

\begin{align*}
\nY_T(h) = y_0 + \int_0^T \{h_s^* \sigma(X_s) - \xi^*(X_s)\} d\tilde{W}_s + \int_0^T \{\chi'(\theta) - \overline{L}\overline{\nu}'(X_s)\} ds \\
- \frac{1}{2} \int_0^T (h_s - \tilde{h}_s)^* \sigma \sigma^*(X_s) (h - \tilde{h}) ds
\end{align*}

\begin{align*}
\nY_T(\tilde{h}) = y_0 + \int_0^T \{\tilde{h}_s^* \sigma(X_s) - \xi^*(X_s)\} d\tilde{W}_s + \int_0^T \{\chi'(\theta) - \overline{L}\overline{\nu}'(X_s)\} ds \\
= y_0 + \int_0^T \{\tilde{h}_s^* \sigma(X_s) - \xi^*(X_s)\} d\tilde{W}_s + \int_0^T \{g(X_s, \tilde{h}_s) + (\tilde{h}_s \sigma(X_s) - \xi(X_s)^*)\tilde{\nu}_s\} ds
\end{align*}
\[ \tilde{M}_t := \int_0^t \tilde{\nu}_s^* d\tilde{W}_s, \]

- \[
\frac{1}{2T} \langle \tilde{M} \rangle_T = \frac{1}{2T} \int_0^T |\tilde{\nu}_s|^2 ds \rightarrow -\left(\chi(\theta) - \theta \chi'(\theta)\right) \text{ a.s.}
\]

- \[
\frac{1}{T} \int_0^T \{g(X_s, \tilde{h}_s) + (\tilde{h}_s \sigma(X_s) - \xi(X_s)^*) \tilde{\nu}_s\} ds \rightarrow \chi'(\theta) \text{ a.s.}
\]

- \[
Y_T(h) - Y_T(\tilde{h}) = \int_0^T \{ (h_s - \tilde{h}_s)^* \sigma(X_s) - \xi^*(X_s) \} d\tilde{W}_s - \frac{1}{2} \int_0^T (h_s - \tilde{h}_s)^* \sigma \sigma^* (X_s) (h - \tilde{h}_s) \} ds
\]
\[ \chi'(-\infty) < \kappa < \chi'(0-) \quad \chi'(-\infty) < \chi'(\theta) + \epsilon < \kappa < \chi'(\theta) + 2\epsilon < \chi'(0-) \]

\[
P(Y_T(h) \leq \kappa T) = \mathbb{E}[e^{-\tilde{M}_T - \frac{1}{2}\langle \tilde{M} \rangle_T} \, Y_T \leq \kappa T] \\
\geq e^{(\chi(\theta) - \theta \chi'(\theta) - 2\epsilon)T} \, \tilde{P}(A_1 \cap A_2 \cap A_3) \\
\geq e^{(\chi(\theta) - \theta \chi'(\theta) - 2\epsilon)T} \{ 1 - \tilde{P}(A_1^c) - \tilde{P}(A_2^c) - \tilde{P}(A_3^c) \} \\
\]

\[
\chi(\theta) - \theta \chi'(\theta) \sim \chi(\theta) - \theta \kappa \geq \inf_{\theta < 0} \{ \chi(\theta) - \theta \kappa \}
\]

\[
A_1 = \{-\tilde{M}_T \geq -\epsilon T\}, \quad A_1^c = \{\tilde{M}_T > \epsilon T\}
\]

\[
A_2 = \{-\frac{1}{2}\langle \tilde{M} \rangle_T \geq (\chi(\theta) - \theta \chi'(\theta) - \epsilon)T\}, \quad A_2^c = \{\frac{1}{2}\langle \tilde{M} \rangle_T + (\chi(\theta) - \theta \chi'(\theta))T > \epsilon T\}
\]

\[
A_3 = \{Y_T(h) \leq \kappa T\},
\]

\[
A_3^c \subset \{Y_T(h) > (\chi'(\theta) + \epsilon)T\} \subset \{Y_T(h) - Y_T(\tilde{h}) > \frac{1}{2}\epsilon T\} \cup \{Y_T(\tilde{h}) - \chi'(\theta)T > \frac{1}{2}\epsilon T\}
\]
3. Linear Gaussian case

\[ \sigma(x) = \Sigma, \quad \beta(x) = Bx + b, \quad \lambda(x) = \Lambda, \]

\[ \xi(x) = \Xi, \quad \alpha(x) = \tilde{A}^* x + \tilde{a}, \quad g(x) = g_1^* x + g_0 \]

\[ \frac{\partial v}{\partial t} + \frac{1}{2} \text{tr}[\Lambda\Lambda^* D^2 v] + \beta_\theta(x)^* D v + \frac{1}{2} (Dv)^* \Lambda N_\theta^{-1} \Lambda^* Dv + U_\theta(x) = 0, \quad v(T, x) = 0 \]

\[ \beta_\theta(x) = Bx + b + \theta \Lambda N_\theta^{-1} (\Sigma^*(\Sigma \Sigma^*)^{-1} \alpha(x) + \Xi), \]

\[ U_\theta(x) = \frac{\theta^2}{2} (\Sigma^*(\Sigma \Sigma^*)^{-1} \alpha(x) + \Xi) N_\theta^{-1} (\Sigma^*(\Sigma \Sigma^*)^{-1} \alpha(x) + \Xi) + \frac{\theta}{2} \alpha^*(\Sigma \Sigma^*)^{-1} \alpha + \theta \tilde{g}(x), \]

\[ N_\theta^{-1} = I + \frac{\theta}{1 - \theta} \Sigma^*(\Sigma \Sigma^*)^{-1} \Sigma, \]

\[ \chi(\theta) = \frac{1}{2} \text{tr}[\Lambda\Lambda^* D^2 \bar{v}] + \beta_\theta(x)^* D \bar{v} + \frac{1}{2} (D\bar{v})^* \Lambda N_\theta^{-1} \Lambda^* D\bar{v} + U_\theta(x), \]
Explicit representation:  

\[
v(t, x) = \frac{1}{2} x^* P(t) x + q(t)^* x + l(t),
\]

\[
\dot{P} + K_1^* P + P K_1 + P \Lambda N_\theta^{-1} \Lambda^* P + \frac{\theta}{1 - \theta} \hat{A}^* (\Sigma \Sigma^*)^{-1} \hat{A} = 0, \quad P(T) = 0
\]

\[
K_1 := B + \theta \Lambda N_\theta^{-1} \Sigma^* (\Sigma \Sigma^*)^{-1} \hat{A}
\]

\[
\dot{q}(t) + (K_1 + \Lambda N_\theta^{-1} \Lambda^* P(t))^* q(t) + P(t) \{ b + \theta \Lambda N_\theta^{-1} (\Sigma^* (\Sigma \Sigma^*)^{-1} \hat{a} - \Xi) \}
\]

\[
\frac{\theta}{1 - \theta} \hat{A}^* (\Sigma \Sigma^*)^{-1} (\hat{a} - \theta \Sigma \Xi) + \theta g_1 = 0, \quad q(T) = 0
\]

\[
i + \frac{1}{2} \text{tr}[\Lambda \Lambda^* P] + \{ b + \theta \Lambda N_\theta^{-1} (\Sigma^* (\Sigma \Sigma^*)^{-1} \hat{a} - \Xi) \} q + \frac{1}{2} q^* \Lambda N_\theta^{-1} \Lambda^* q
\]

\[
\quad + \frac{\theta^2}{2} \{ \Sigma^* (\Sigma \Sigma^*)^{-1} \hat{a} - \Xi \} N_\theta^{-1} \{ \Sigma^* (\Sigma \Sigma^*)^{-1} \hat{a} - \Xi \} + \frac{\theta}{2} \hat{a} (\Sigma \Sigma^*)^{-1} \hat{a} + \theta g_0 = 0,
\]

\[
l(T; T) = 0
\]
Explicit representation for the stationary solution: \[ \tilde{v}(x) = \frac{1}{2} x^* \bar{P} x + \bar{q}^* x \]

- \[ K_1^* \bar{P} + \bar{P} K_1 + \bar{P} \land N_\theta^{-1} \land^* \bar{P} + \frac{\theta}{1 - \theta} \bar{A}^* (\Sigma \Sigma^*)^{-1} \bar{A} = 0 \]

- \[(K_1 + \land N_\theta^{-1} \land^* \bar{P})^* \bar{q} + \bar{P} \{ b + \theta \land N_\theta^{-1} (\Sigma^* (\Sigma \Sigma^*)^{-1} \bar{a} - \Xi) \} \]
  \[+ \frac{\theta}{1 - \theta} \bar{A}^* (\Sigma \Sigma^*)^{-1} (\bar{a} - \theta \Sigma \Xi) + \theta g_1 = 0. \]

- \[ \chi(\theta) = \frac{1}{2} \text{tr}[\land \land^* \bar{P}] + \{ b + \theta \land N_\theta^{-1} (\Sigma^* (\Sigma \Sigma^*)^{-1} \bar{a} - \Xi) \} \bar{q} + \frac{1}{2} \bar{q}^* \land N_\theta^{-1} \land^* \bar{q} \]
  \[+ \frac{\theta^2}{2} \{ \Sigma^* (\Sigma \Sigma^*)^{-1} \bar{a} - \Xi \}^* N_\theta^{-1} \{ \Sigma^* (\Sigma \Sigma^*)^{-1} \bar{a} - \Xi \} + \frac{\theta}{2} \bar{a} (\Sigma \Sigma^*)^{-1} \bar{a} + \theta g_0 \]
• Under one of the following conditions:

\( G := B - \Lambda \Sigma^*(\Sigma \Sigma^*)^{-1} \tilde{A} \) is stable

(B) \( B \) is stable

(\Lambda A) \( \Lambda \Lambda^*, \) and \( \tilde{A}^* \tilde{A} \) are positive definite matrices

we have the solution \( \tilde{P} \) such that

\[
\lim_{T \to \infty} P(t; T) = \tilde{P}, \quad \text{and} \quad K_1 + \Lambda N_\theta^{-1} \Lambda^* \tilde{P} \quad \text{is stable.}
\]

Further, we have

\[
\lim_{T \to \infty} q(t; T) = \bar{q}, \quad \text{and} \quad \lim_{T \to \infty} \frac{l(0; T)}{T} = \chi(\theta),
\]
To study "effective domain" \((\chi'(-\infty), \chi'(0-))\), consider scaling of the above HJB equation.

\[
\tilde{v}(x) := \frac{v(x)}{\theta} = \frac{1}{2\theta} x^* \bar{P} x + \frac{1}{\theta} \bar{q}^* x =: \frac{1}{2} x^* \bar{P} x + \bar{q}^* x
\]

As \(\theta \to 0\), we consider

\[
\frac{\chi(\theta)}{\theta} = \frac{1}{2} \text{tr}[\lambda \lambda^* D^2 \tilde{v}] + \beta^*_\theta D \tilde{v} + \frac{\theta}{2} (D \tilde{v})^* \lambda N^{-1}_\theta \lambda^* D \tilde{v} + \frac{U_\theta}{\theta}.
\]

As \(\theta \to -\infty\), we consider

\[
\frac{\chi(\theta)}{\theta^2} = \frac{1}{2\theta} \text{tr}[\lambda \lambda^* D^2 \tilde{v}] + \frac{1}{\theta} \beta^*_\theta D \tilde{v} + \frac{1}{2} (D \tilde{v})^* \lambda N^{-1}_\theta \lambda^* D \tilde{v} + \frac{U_\theta}{\theta^2}.
\]
As $\theta \to 0$

We have an explicit representation for $\tilde{v}$: $\tilde{v}(x) = \frac{1}{2}x^*\bar{P}x + \tilde{q}^*x$

- $K_1^*\bar{P} + \bar{P}K_1 + \theta\bar{P}\Lambda N_{\theta}^{-1}\Lambda^*\bar{P} + \frac{1}{1-\theta}\tilde{A}^*(\Sigma\Sigma^*)^{-1}\tilde{A} = 0$

- $(K_1 + \Lambda N_{\theta}^{-1}\Lambda^*\bar{P})^*\tilde{q} + \bar{P}\{b + \theta\Lambda N_{\theta}^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\tilde{a} - \Xi)\}^*\tilde{q} + \frac{\theta}{2}\tilde{q}^*\Lambda N_{\theta}^{-1}\Lambda^*\tilde{q}

- $\tilde{\chi}(\theta) = \frac{\chi(\theta)}{\theta}$

- $\tilde{\chi}(\theta) = \frac{1}{2}\text{tr}[\Lambda^*\bar{P}] + \{b + \theta\Lambda N_{\theta}^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\tilde{a} - \Xi)\}^*\tilde{q} + \frac{\theta}{2}\tilde{q}^*\Lambda N_{\theta}^{-1}\Lambda^*\tilde{q} + \frac{\theta}{2}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\tilde{a} - \Xi\}^*N_{\theta}^{-1}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\tilde{a} - \Xi\} + \frac{1}{2}\tilde{a}(\Sigma\Sigma^*)^{-1}\tilde{a} + g_0$
If $B$ is stable, then
\[ 0 \geq \tilde{P} \geq \theta \int_0^\infty e^{sB} \tilde{A}^* (\Sigma \Sigma^*)^{-1} \tilde{A} e^{sB} ds \]
and $\tilde{P}(0-) = 0$ follows. Further, $\tilde{q}(0-) = 0$ since
\[ \lim_{\theta \to 0} \tilde{K}_1 \equiv \lim_{\theta \to 0} \{B + \theta \Lambda N_\theta^{-1} \Sigma^* (\Sigma \Sigma^*)^{-1} \tilde{A}\} = B, \]
and we have
\begin{itemize}
  \item $B^* \tilde{P}(0-) + \tilde{P}(0-) B + \tilde{A}^* (\Sigma \Sigma^*)^{-1} \tilde{A} = 0,$
  \item $B^* \tilde{q}(0-) + \tilde{P}(0-) b + \tilde{A}^* (\Sigma \Sigma^*)^{-1} \tilde{a} + g_1 = 0$
\end{itemize}
Explicit representation of $\chi'(0-)$

If $B$ is stable, then we have an explicit representation

$$
\chi'(0-) = \lim_{\theta \to 0} \tilde{\chi}(\theta) = \frac{1}{2} \text{tr}[\Lambda \Lambda^* \tilde{P}(0-)] + b^* \tilde{q}(0-) + \frac{1}{2} \tilde{a} (\Sigma \Sigma^*)^{-1} \tilde{a} + g_0
$$

$$
= \frac{1}{2} \text{tr}[\Lambda \Lambda^* \tilde{P}(0-)] + \frac{1}{2} [\tilde{A} B^{-1} b - \tilde{a}] (\Sigma \Sigma^*)^{-1} [\tilde{A} B^{-1} b - \tilde{a}]
$$

$$
- (B^{-1} b)^* g_1 + g_0,
$$

where

$$
\tilde{P}(0-) = \int_0^\infty e^{sB^*} \tilde{A}^* (\Sigma \Sigma^*)^{-1} \tilde{A} e^{sB} ds
$$

and "the law of large numbers" is seen to be ruling at $\chi'(0-)$:

$$
\chi'(0-) = \chi_0 := \sup_h \lim_{T \to \infty} \frac{1}{T} E[\log \frac{V_T(h)}{L_T}]
$$
As $\theta \to -\infty$: If $G = B - \Lambda \Sigma^*(\Sigma \Sigma^*)^{-1} \tilde{A}$ is stable, then we have

$$0 \geq \bar{P} \geq -\int_0^\infty e^{sG} \tilde{A}^*(\Sigma \Sigma^*)^{-1} \tilde{A} e^{sG} ds$$

and $\bar{P}_{-\infty} = \exists \lim_{\theta \to -\infty} \bar{P}(\theta)$ satisfies

$$G^* \bar{P}_{-\infty} + \bar{P}_{-\infty} G + \bar{P}_{-\infty} \Lambda \tilde{N}_{-\infty} \Lambda^* \bar{P}_{-\infty} - \tilde{A}^*(\Sigma \Sigma^*)^{-1} \tilde{A} = 0,$$

where $\tilde{N}_{-\infty} = I - \Sigma^*(\Sigma \Sigma^*)^{-1} \Sigma = \lim_{\theta \to -\infty} N_\theta^{-1}$.

- $(K_1 + \Lambda N_\theta^{-1} \Lambda^* \bar{P})^* \tilde{q} + \bar{P}\left\{\frac{b}{\theta} + \Lambda N_\theta^{-1}(\Sigma^*(\Sigma \Sigma^*)^{-1} \tilde{a} - \Xi)\right\}$
  $$+ \frac{1}{1 - \theta} \tilde{A}^*(\Sigma \Sigma^*)^{-1}(\tilde{a} - \theta \Sigma \Xi) + g_1 = 0.$$

- $\frac{\chi(\theta)}{\theta^2} = \frac{1}{2\theta} \text{tr}[\Lambda \Lambda^* \bar{P}] + \left\{\frac{b}{\theta} + \Lambda N_\theta^{-1}(\Sigma^*(\Sigma \Sigma^*)^{-1} \tilde{a} - \Xi)\right\} \tilde{q} + \frac{1}{2} \tilde{q}^* \Lambda N_\theta^{-1} \Lambda^* \tilde{q}$
  $$+ \frac{1}{2} \left\{\Sigma^*(\Sigma \Sigma^*)^{-1} \tilde{a} - \Xi\right\} N_\theta^{-1} \left\{\Sigma^*(\Sigma \Sigma^*)^{-1} \tilde{a} - \Xi\right\} + \frac{1}{2\theta} \tilde{a}(\Sigma \Sigma^*)^{-1} \tilde{a} + \frac{1}{\theta} g_0$

- $(G + \Lambda \tilde{N}_{-\infty} \Lambda^* \bar{P}_{-\infty})^* \tilde{q}_{-\infty} - \bar{P}_{-\infty} \Lambda \tilde{N}_{-\infty} \Xi + \tilde{A} (\Sigma \Sigma^*)^{-1} \Sigma \Xi + g_1 = 0$
Asymptotic behavior of $\chi'(\theta)$ as $\theta \to -\infty$

**Proposition 4** If $G = B - \Lambda \Sigma^*(\Sigma \Sigma^*)^{-1} \bar{A}$ is stable, then we have

$$\lim_{\theta \to -\infty} \frac{\chi(\theta)}{\theta^2} = \frac{1}{2} \left\{ \Lambda^* \bar{q}_{-\infty} - \Xi \right\}^* \bar{N}_{-\infty} \left\{ \Lambda^* \bar{q}_{-\infty} - \Xi \right\}$$

where

$$\bar{N}_{-\infty} = I - \Sigma^*(\Sigma \Sigma^*)^{-1} \Sigma$$

**Lemma 2** If $G$ is stable and

$$\Lambda^* (G^*)^{-1} \left\{ \bar{A}(\Sigma \Sigma^*)^{-1} \Sigma \Xi + g_1 \right\} + \Xi \notin \mathcal{R}(\Sigma^*),$$

then

$$\Lambda^* \bar{q}_{-\infty} - \Xi \notin \mathcal{R}(\Sigma^*)$$

holds and

$$\left\{ \Lambda^* \bar{q}_{-\infty} - \Xi \right\}^* \bar{N}_{-\infty} \left\{ \Lambda^* \bar{q}_{-\infty} - \Xi \right\} > 0.$$
Remark.

- Previous cases (cf. H-N-S ’10) : If $S^0_T := rT$, $g_1 = 0$, $g_0 = r$, then,

\[
\inf h P(\log \frac{V_T(h)}{S^0_T} \leq \kappa T) \sim e^{-TI_0(\kappa)}, \quad I_0(\kappa) = \sup_{\theta < 0} \{\theta \kappa - \chi_0(\theta)\}
\]

**effective domain** \( (\chi'_0(-\infty), \chi'_0(0-)) = (0, \chi'_0(0-)) \)

\[
\chi'_0(0-) = \frac{1}{2} \text{tr}[\Lambda^* \tilde{P}(0-)] + \frac{1}{2} [AB^{-1}b - \tilde{a}]^* (\Sigma \Sigma^*)^{-1} [AB^{-1}b - \tilde{a}] + g_0,
\]

- Current case : If $g_1 = 0$, $g_0 = r - \gamma - \frac{1}{2} \Xi^* \Xi$, \( \Xi \notin \mathcal{R}(\Sigma^*) \cup \mathcal{R}(\Lambda^*) \), then,

\[
\inf h P(\log \frac{V_T(h)}{L_T} \leq \kappa T) \sim e^{-TI(\kappa)}, \quad I(\kappa) = \sup_{\theta < 0} \{\theta \kappa - \chi(\theta)\}
\]

**effective domain** \( (\chi'(-\infty), \chi'(0-)) = (-\infty, \chi'(0-)) \)

\[
\chi'(0-) = \frac{1}{2} \text{tr}[\Lambda^* \tilde{P}(0-)] + \frac{1}{2} [AB^{-1}b - \tilde{a}]^* (\Sigma \Sigma^*)^{-1} [AB^{-1}b - \tilde{a}] + g_0,
\]
References


Nagai, H. and Sheu, S.J. (2018). Large deviation control arising from downside risk minimization against a benchmark, Preprint


Thank you for your attention !