A Dirichlet Process Characterization of RBM in a Wedge

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Let $S$ be a wedge in $\mathbb{R}^2$ whose state space in polar coordinates is given by

$$S = \{(r, \theta) : r \geq 0, 0 \leq \theta \leq \xi\}$$

for some $0 < \xi < 2\pi$. 
Consider a standard 2-d Brownian motion \( X \) started from a point \( z \in S \) and reflected back into the interior of the wedge \( S \) whenever it hits the boundary of \( S \).

The method of reflection is as follows.

Let
\[
\partial S_1 = \{(r, \theta) : r > 0, \theta = 0\}
\]
denote the lower boundary of the wedge, and let
\[
\partial S_2 = \{(r, \theta) : r > 0, \theta = \xi\}
\]
denote its upper boundary.
RBM in a Wedge

- The angle of reflection off of the edge $\partial S_1$ is denoted by $-\pi/2 < \theta_1 < \pi/2$, and the angle of reflection off of the edge $\partial S_2 = \{(r, \theta) : r > 0, \theta = \xi\}$ is denoted by $-\pi/2 < \theta_2 < \pi/2$.

- Both of these angles are measured with respect to the inward facing normal off their respective edge, with angles directed toward the origin assumed to be positive.

- We denote by $n_1$ the inward facing unit normal off of the edge $\partial S_1$, and by $v_1$ the unit vector of reflection off of $\partial S_1$. Similar quantities are defined for $\partial S_2$. 
RBM in a Wedge
In many cases, it turns out to be a difficult problem to obtain sample-pathwise such a reflected process $Z$ when starting from a Brownian motion $X$.

However, a process with the probabilistic characteristics of the reflected process $Z$ was rigorously defined by Varadhan and Williams (1985).
Definition

• Let \( C_S \) denote the space of continuous functions with domain \( \mathbb{R}_+ = [0, \infty) \) and range \( S \).

• For each \( t \geq 0 \) and \( \omega \in C_S \), denote by \( Z(t) : C_S \mapsto S \) the coordinate map \( Z(t)(\omega) = Z(t, \omega) = \omega(t) \), and also define the coordinate mapping process \( Z = \{Z(t), t \geq 0\} \).

• Then, for each \( t \geq 0 \), set \( \mathcal{M}_t = \sigma\{Z(s), 0 \leq s \leq t\} \), and set \( \mathcal{M} = \sigma\{Z(s), s \geq 0\} \).

• Next, for each \( n \geq 1 \) and \( F \subset \mathbb{R}^2 \), denote by \( C^n(F) \) the set of \( n \)-times continuously differentiable functions in some domain containing \( F \), and let \( C^n_b \) be the set of functions in \( C^n(F) \) that have bounded partial derivatives up to and including order \( n \) on \( F \).

• Finally, define the differential operators \( D_j = v_j \cdot \nabla \) for \( j = 1, 2 \), and denote by \( \triangle \) the Laplacian operator.
Definition

**Definition 1.** [Varadhan and Williams (1985)] A family of probability measures \( \{ P^z, z \in S \} \) on \((C_S, \mathcal{M})\) is said to solve the submartingale problem if for each \( z \in S \), the following 3 conditions hold,

1. \( P^z(Z(0) = z) = 1 \),

2. For each \( f \in C^{2}_b(S) \), the process

\[
\left\{ f(Z(t)) - \frac{1}{2} \int_{0}^{t} \Delta f(Z(s))ds, \quad t \geq 0 \right\}
\]

is a submartingale on \((C_S, \mathcal{M}, \mathcal{M}_t, P^z)\) whenever \( f \) is constant in a neighborhood of the origin and satisfies \( D_i f \geq 0 \) on \( \partial S_i \) for \( i = 1, 2 \),

3. \( E^z \left[ \int_{0}^{\infty} 1\{ Z(t) = 0 \} dt \right] = 0 \).
Definition

- Reflected Brownian motion arises as the heavy traffic limit of many queueing networks.

- For instance, the multi-dimensional queue length process of open queueing networks with homogeneous customer populations may be approximated by RBM in a wedge. See Reiman (1984) and Harrison and Williams (1987).

- The generalized processor sharing queue may also be approximated by RBM in a wedge. See Ramanan and Reiman (2003).
The Quantity $\alpha$

- Let

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$ 

- Much of the behavior of $Z$ is determined by the value of $\alpha$. 

\[\alpha < 1 \quad \alpha = 1 \quad \alpha > 1\]
The Quantity $\alpha$

- Varadhan and Williams (1985) proved that if $\alpha < 2$, then there exists a unique solution to the submartingale problem.

- They also showed that if $\alpha \geq 2$, then there is a solution satisfying only Conditions 1 and 2 of the submartingale problem. In this case, the process $Z$ is absorbed at the origin upon reaching it.

- The results of Williams (1985) state that $Z$ is a semi-martingale if and only if $\alpha < 1$ or $\alpha \geq 2$.

- The results of Kang and Ramanan (2010) imply that $Z$ is a Dirichlet process if $\alpha = 1$. 
Definition 2. Let \( z \in S \). A continuous process \( Y \) defined on \((C_S, \mathcal{F}, \mathcal{F}_t, P^z)\) is said to be of zero energy if for each \( T > 0 \),

\[
\sum_{t_i \in \pi^n} \|Y(t_i) - Y(t_{i-1})\|^2 \xrightarrow{P} 0 \text{ as } n \to \infty, \tag{1}
\]

for any sequence \( \{\pi^n, n \geq 1\} \) of partitions of \([0, T]\) with \( \|\pi^n\| \to 0 \) as \( n \to \infty \).

- The notion of a zero-energy process can be used to define a Dirichlet process.
Definition 3. Let \( z \in S \). The stochastic process \( Z \) is said to be a Dirichlet process on \( (C_S, \mathcal{F}, \mathcal{F}_t, P^z) \) if we may write

\[
Z = X + Y,
\]

where \( X \) is an \( \mathcal{F}_t \)-adapted local martingale and \( Y \) is a continuous, \( \mathcal{F}_t \)-adapted zero energy process with \( Y(0) = 0 \).

- The class of Dirichlet processes is larger than the class of semi-martingales.

- Moreover, there exists a change-of-variable formula for the class of Dirichlet processes.

- The decomposition \( Z = X + Y \) appearing in Definition 3 may be shown to be unique.
Dirichlet Processes Result

**Theorem 1.** Suppose that $1 < \alpha < 2$. Then, $Z$ has the decomposition

$$Z = X + Y,$$

where $(X, Y)$ is a pair of processes on $(C_S, F, F_t)$ such that for each $z \in S$, $X$ is a standard Brownian motion started from $z$ and $Y$ is a process of zero energy on $(C_S, F, F_t, P^z)$. In particular, for each $z \in S$, the process $Z$ is a Dirichlet process on $(C_S, F, F_t, P^z)$.

- The pair of processes $(X, Y)$ appearing in Theorem 1 do not depend on $z \in S$. That is, there is a single pair of processes $(X, Y)$ on $(C_S, F, F_t)$ such that the statement of the above theorem hold for each $z \in S$. 
Definition 4. Let $f : \mathbb{R}^+ \mapsto \mathbb{R}^d$ and $p > 0$. Then, $f$ is said to be of finite strong $p$-variation if for each $T \geq 0$,

$$V_p(f, T) = \sup \left\{ \sum_{t_i \in \pi} \|f(t_i) - f(t_{i-1})\|^p : \pi \in \pi(T) \right\} < \infty,$$

where the supremum is taken over all partitions $\pi$ of $[0, T]$.

- Strong $p$-variation is a generalization of the concept of bounded variation.

- If $V_p(f, T) < \infty$, then $V_q(f, T) < \infty$ for $q \geq p > 0$.

- Moreover, functions of finite strong $p$-variation are, up to a time-change, locally Hölder continuous with exponent $1/p$. 
\textbf{p-variation Result}

\textbf{Theorem 2.} Suppose that $1 < \alpha < 2$. Then, for each $p > \alpha$ and $z \in S$,

$$P^z(V_p(Y, [0, T]) < +\infty) = 1, T \geq 0.$$

Furthermore, for each $0 < p \leq \alpha$,

$$P^0(V_p(Y, [0, T]) < +\infty) = 0, T \geq 0.$$

- The sample paths of $Y$ become rougher as $\alpha$ increases from 1 to 2.
The Extended Skorokhod Problem

• For our last result, we provide a rigorous statement of the fact that $Z$ is the reflected version of the Brownian motion $X$ on all of $\mathbb{R}_+$.  

• Let $D(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of $\mathbb{R}^d$-valued functions, with domain $\mathbb{R}_+$, that are right-continuous with left limits.

• Also, let $D_S(\mathbb{R}_+, \mathbb{R}^2)$ be the set of $f \in D(\mathbb{R}_+, \mathbb{R}^2)$ such that $f(0) \in S$.

• Next, let $d(\cdot)$ be a set-valued mapping defined on $S$ such that $d(z)$ is a closed convex cone in $\mathbb{R}^2$ for every $z \in S$. In particular, we define

$$d(z) = \begin{cases} 
\{\alpha v_1, \alpha \geq 0\}, & \text{for } z \in \partial S_1, \\
\{\alpha v_2, \alpha \geq 0\}, & \text{for } z \in \partial S_2, \\
V, & \text{for } z = 0, \\
\{0\}, & \text{for } z \in \text{Int}(S). 
\end{cases}$$

(3)
The Extended Skorokhod Problem

**Definition 5.** [Ramanan (2006)] The pair of processes $(\phi, \eta) \in D_S(\mathbb{R}_+, \mathbb{R}^2) \times D(\mathbb{R}_+, \mathbb{R}^2)$ solve the ESP $(S, d(\cdot))$ for $\psi \in D(\mathbb{R}_+, \mathbb{R}^2)$ if $\phi(0) = \psi(0)$, and if for all $t \in \mathbb{R}_+$, the following properties hold,

1. $\phi(t) = \psi(t) + \eta(t)$,
2. $\phi(t) \in S$,
3. For every $s \in [0, t],
   \[ \eta(t) - \eta(s) \in \overline{\text{Conv}[\bigcup_{u \in (s,t]} d(\phi(u))]}, \]
4. $\eta(t) - \eta(t-) \in \overline{\text{Conv}[d(\phi(t))]}$.

- Note that the pushing function $\eta$ is not required to be of bounded variation.
The Extended Skorokhod Problem

**Theorem 3.** Suppose that $1 < \alpha < 2$. Then, for each $z \in S$, the ESP $(S, d(\cdot))$ for the Brownian motion $X$ on $(C_S, \mathcal{F}, \mathcal{F}_t, P^z)$ has a solution $P^z$-a.s. if and only if

$$\overline{\operatorname{Conv}}(V \cup \{\alpha v_1, \alpha \geq 0\} \cup \{\alpha v_2, \alpha \geq 0\}) = \mathbb{R}^2.$$  \hspace{1cm} (4)

In this case, $(Z, Y)$ solves the ESP $(S, d(\cdot))$ for $X$.

- By setting $V = \mathbb{R}^2$, one may find a $V$ such that (4) holds. However, using the fact that $1 < \alpha < 2$, it may be verified that the smaller set $V = \{\alpha v_0, \alpha \geq 0\}$ for any $v_0$ in the interior of $S$ satisfies (4) as well.

- Note also that we do not claim in Theorem 3 that $(Z, Y)$ is the unique solution to the ESP $(S, d(\cdot))$ for $X$. 

Thank You

THANK YOU!