Stochastic processing networks: steady-state 23 diffusion approximations

Jim Dai

School of Operations Research and Information Engineering
Cornell University
and
Institute for Data and Decision Analytics (iDDA)
Chinese University of Hong Kong, Shenzhen

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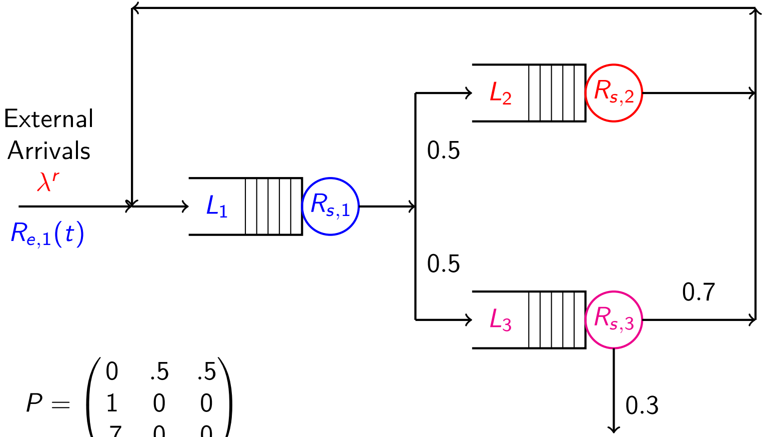
Collaborators

- Masakiyo Miyazawa, Department of Information Sciences, Tokyo University of Science
- Anton Braverman, Kellogg School of Business, Northwestern University
- Chang Cao, Department of Statistical Science, Cornell University
- Xiangyu Zhang, School of ORIE, Cornell University

An example of generalized Jackson networks (GJNs)



$$P = \begin{pmatrix} 0 & .5 & .5 \\ 1 & 0 & 0 \\ .7 & 0 & 0 \end{pmatrix}$$

Indirect method: limit interchange for GJN

Down-Meyn (1994)

$$\begin{array}{ccc} L^{(r)}(t) & \xrightarrow{\;t \to \infty\;} & L^{(r)}(\infty) \\ \Big\downarrow {\scriptstyle r \to 0} & & \Big\downarrow {\scriptstyle r \to 0} \\ L(t) & \xrightarrow{\;t \to \infty\;} & L(\infty) \end{array}$$

Reiman (1984)
L is an SRBM

Gamarnik-Zeevi (2006)

Harrison-Williams (1987)

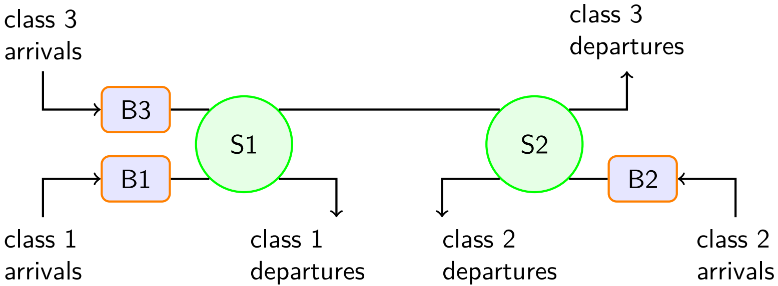
Gamarnik-Zeevi (2006): indirect method

- Budhiraja and Lee (2009): second moments and uniform integrability
- Gurvich (2014), multiclass queueing networks
- Ye-Yao (2015), (head-of-line) bandwidth sharing networks
- There is a growing literature: Tezcan (2008), Zhang-Zwart (2008), Katsuda (2010), Gamarnik-Stolyar (2012), D-Dieker-Gao (2014), and more...

Basic adjoint relationship (BAR): three direct methods

- Stein's method; Gurvich (2014), Braverman-Dai (2017)
  - Strongest results, but difficult for general systems
- Moment generating function (mgf)-BAR approach: Miyazawa (2015), Braverman-Dai-Miyazawa (2017) for GJN
- Drift method (Quadratic-BAR approach): Eryilmaz and Srikant (2012, Maguluri-Srikant (2016), Wang-Maguluri-Srikant-Ying (2017)
  - Surprisingly successful for single-pass systems,

A two-link bandwidth sharing network



- Insensitivity of jobsize distributions in heavy traffic.
- Full version, beyond moments, of the conjecture remains open a problem.

Proportional fairness in heavy traffic: process limit convergence and insensitivity of the limit process

Maria Vlasiou[*], Jiheng Zhang[†], and Bert Zwart[‡]

[*]Eindhoven University of Technology
[†]The Hong Kong University of Science and Technology
[‡]Centrum Wiskunde & Informatica, Amsterdam

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Abstract

Proportional fairness is a popular service allocation mechanism to describe and analyze the performance of data networks at flow level. Recently, several authors have shown that the invariant distribution of networks operating according to proportional fairness admits a product form distribution under critical loading. They focus however on exponential job size distributions, leaving the case of general job size distributions as an open question. Motivated by this, we consider a network operating under proportional fairness where the job

Today's topics

Topics

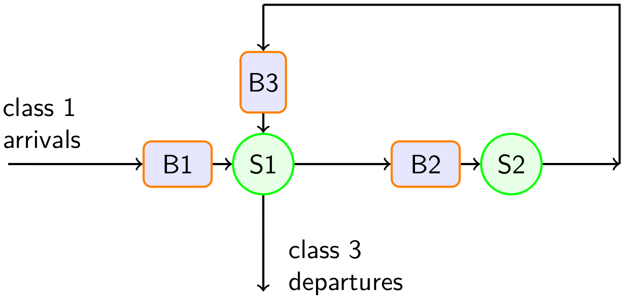
- Mgf approach for multiclass queueing networks under priority policies: Braverman, Dai and Miyazawa.
- Strong state space collapse: Cao, Dai, Miyazawa, Zhang.

Outline

- An illustrative theorem for a re-entrant line
- Mgf approach for $GI/GI/1$ queues
- Proof sketches
- An open problem

Self-contained tools

A 2-station, 3-class reentrant line



- $\Big\{\big(T_e(i), T_{s,1}(i), T_{s,2}(i), T_{s,3}(i)\big), i \geq 1\Big\}$ is an iid sequence with mean $(1, m_1, m_2, m_3)$ and finite second moment.
- The interarrival times are $T_e(i)$, satisfying the heavy traffic condition

$$\lambda = 1 - r, \quad m_1 + m_3 = 1, \quad m_2 = 1, \quad r \downarrow 0.$$

- Load parameters

$$\beta_i = \lambda m_i, \quad i = 1, 2, 3, \quad \rho_1 = \beta_1 + \beta_3 = \lambda < 1, \quad \rho_2 = \beta_2 = \lambda < 1.$$

System dynamics

Assuming (preemptive-resume) LBFS policy, one has the Markovian representation: $X = \{X(t)\}_{t \geq 0}$ is Markov process, where state at time t is

$$X(t) = (L(t), R_e(t), R_s(t)).$$

- $L_i(t) =$ number of customers in buffer i at time t (including the one in service)
- $R_e(t) =$ residual time until next exogenous arrival
- $R_{s,i}(t) =$ residual time until next service completion in buffer i
- The reentrant in the figure has a 7-dimensional representation:

$$\big(L_1(t), R_e(t), R_{s,1}(t)\big), \quad \big(L_2(t), R_{s,2}(t)\big), \quad \big(L_3(t), R_{s,3}(t)\big)$$

A sample result

- When $\lambda = 1 - r < 1$, Markov process X is positive Harris recurrent (Dai 1995): as $t \to \infty$,

$$L^r(t) \Longrightarrow L^r(\infty) = (L_1^r(\infty), L_2^r(\infty), L_3^r(\infty)).$$

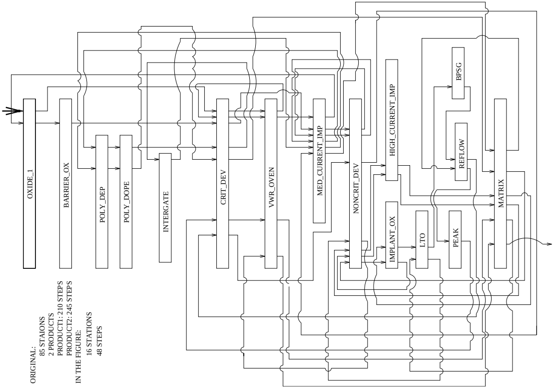
Theorem 1

$$rL^r(\infty) \Longrightarrow \Big(L_1^*(\infty),\ L_2^*(\infty),\ 0\Big) \quad \text{as } r \to 0.$$

- $rL_3^r(\infty) \Rightarrow 0$, state space collapse (SSC).
- $\Big(rL_1^r(\infty), rL_2^r(\infty)\Big) \Longrightarrow \Big(L_1^*(\infty),\ L_2^*(\infty)\Big)$.

State space collapse

The pre-limit is a K-dimensional problem, and the limit is a d-dimensional problem, where K is the number of buffers and d is the number of stations.



Strong SSC

- In proving a version of Theorem 1 for general multiclass queueing networks under priority policies, we need the following strong SSC:

$$r\mathbb{E}_\nu[L_3^r(\infty)] \to 0.$$

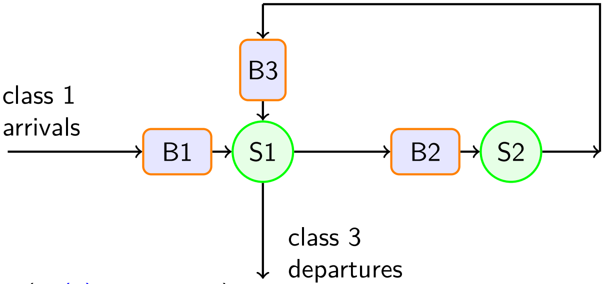
Theorem 2

Assume interarrival and service times have phase-type distributions. Assume that the critically loaded fluid model has SSC. Then

$$\sup_{r\in(0,r_0]} \sum_{i\in\mathcal{H}} \mathbb{E}_\nu[L_i^r(\infty)]^p < \infty \quad \text{for } p \geq 1,$$

which implies the strong SSC.

Fluid model SSC



- Fluid model ($L_1(0)$ can be ∞)

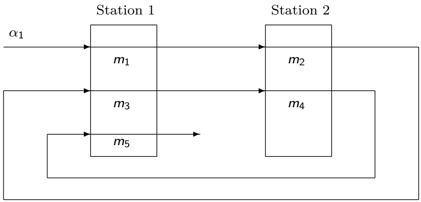
$$L_1(t) = L_1(0) + t - \mu_1 T_1(t),$$
$$L_2(t) = L_2(0) + \mu_1 T_1(t) - \mu_2 T_2(t),$$
$$L_3(t) = L_3(0) + \mu_2 T_2(t) - \mu_3 T_3(t),$$

  where $\mu_i = 1/m_i$ is service rate for class i jobs.
- $\mathcal{H} = \{3\}$. When $L_3(t) > 0$, $\dot{T}_3(t) = 1$, which implies

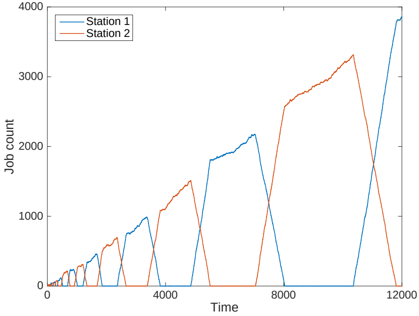
$$\dot{L}_3(t) = \mu_2 \dot{T}_2(t) - \mu_3 \leq \mu_2 - \mu_3 < 0.$$

- $L_3(t) = 0$ for $t \geq L_3(0)/(\mu_3 - \mu_2)$.

LBFS priority policy?



Under the LBFS-FBFS priority policy and $\lambda(m_2 + m_5) > 1$,



Miyazawa's mgf approach

- For a family of nonnegative random vector $L^{(r)}(\infty) \in \mathbb{R}_+^d$,

$$L^{(r)}(\infty) \Rightarrow L(\infty) \quad \text{for some random vector } L(\infty)$$

  if and only if the mgf converges

$$\phi^{(r)}(\theta) = \mathbb{E}[e^{\langle \theta, L^{(r)}(\infty)\rangle}] \to \mathbb{E}[e^{\langle \theta, L(\infty)\rangle}] \quad \text{for all } \theta \le 0.$$

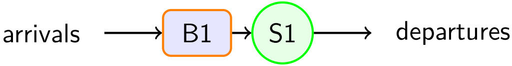
- The family $\{L^{(r)}(\infty)\}$ is tight iff for any sequence $r_n \to 0$,

$$\lim_{n\to\infty} \phi^{(r_n)}(\theta) \to \phi(\theta) \quad \forall \theta \le 0 \text{ implies that}$$

$$\phi(0-) = \phi(0-, \dots, 0-) = 1. \tag{1}$$

- Equation (1) says $\phi(\cdot)$ is left continuous at 0.

$GI/GI/1$ **queue**



- $\{T_e(i), i \geq 1\}$ iid interarrival times; $\lambda = 1/\mathbb{E}[T_e(i)]$;
- $\{T_s(i), i \geq 1\}$ iid service times; $\mu = 1/\mathbb{E}[T_s(i)]$.
- Heavy traffic condition

$$\lambda = \mu - r \quad \text{with } r \downarrow 0.$$

- $X = \{X(t), t \geq 0\}$ is a Markov process, where

$$X(t) = (L(t), R_e(t), R_s(t)),$$

where

  - $L(t)$ is the number of customers in system,
  - $R_e(t)$ is remaining interarrival times,
  - $R_s(t)$ is the remaining service times.

Piecewise deterministic Markov process (PDMP)

- The process $X = (L, R_e, R_s)$ is a piecewise deterministic Markov process (PDMP); Davis (1981)
- A sample path of a PDMP is composed of two parts, deterministic and continuous sections and (random) jumps due to expiration of remaining times.

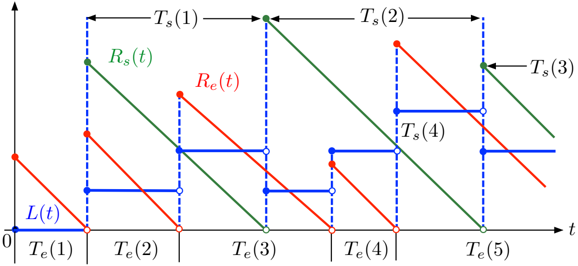


FIGURE: A sample path of remaining times for $GI/G/1$ queue

Change of variables for PDMP

- Consider function $f(x) = f(z, u, v) : \mathbb{Z}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$.
- Define the jump size $\Delta f(X(s)) = f(X(s)) - f(X(s-))$,

$$
\begin{aligned}
f(X(t)) - f(X(0)) &= \int_0^t \frac{d}{ds} f(X(s))\, ds + \sum_{s_i \in (0,t]} \Delta f(X(s_i)) \\
&= -\int_0^t \Big[\frac{\partial}{\partial u} f(X(s)) - 1\{Z(s) > 0\} \frac{\partial}{\partial v} f(X(s))\Big] ds \quad \text{(continuous)} \\
&\quad + \left[\int_0^t \Delta f(X(s)) dN_A(s) + \int_0^t \Delta f(X(s)) dN_D(s)\right] \quad \text{(jumps)}
\end{aligned}
$$

- N_A – arrival process.
- N_D – departure process.

Full BAR in $GI/GI/1$ setting

- Assume X has stationary distribution ν.

Basic Adjoint Relationship (BAR)

$$
\begin{aligned}
0 = & t\mathbb{E}_\nu \Big[-\frac{\partial}{\partial u} f(X(0)) - 1\{Z(0) > 0\} \frac{\partial}{\partial v} f(X(0)\Big] + \text{(continuous)} \\
& + \mathbb{E}_\nu \left[\int_0^t \Delta f(X(s)) dN_A(s) + \int_0^t \Delta f(X(s)) dN_D(s) \right] \quad \text{(jumps)}
\end{aligned}
$$

- Jump terms are intractable; getting rid of arrival-jump term requires

$$
\mathbb{E}_{T_e}\big[f(L+1, T_e, R_s) - f(L, 0, R_s)\big] = 0 \tag{2}
$$

for any given L and R_s.

- Similarly,

$$
\mathbb{E}_{T_s}\big[f(L-1, R_e, T_s) - f(L, R_e, 0)\big] = 0 \tag{3}
$$

for any given $L \geq 1$ and R_e .

Exponential functions (I)

- Fix a $\theta \leq 0$, take

$$f(\theta; z, u, v) = e^{\theta z + a(\theta) u + b\theta v}.$$

- Then

$$\mathbb{E}[f(z+1, T_e, v) - f(z, 0, v)] = 0,$$
$$\mathbb{E}e^{\theta(z+1) + a(\theta) T_e + b(\theta) v} = e^{\theta z + b(\theta) v},$$
$$\mathbb{E}\left[e^{\theta + a(\theta) T_e}\right] = 1.$$

- For each $\theta \leq 0$, find $a = a(\theta)$ such that

$$\mathbb{E}\left[e^{a(\theta) T_e}\right] = e^{-\theta}. \tag{4}$$

Exponential functions (II)

- Similarly, for each $\theta \leq 0$, find $b(\theta)$ such that

$$\mathbb{E}[e^{b(\theta) T_s}] = e^{\theta}. \tag{5}$$

Then

$$\mathbb{E}[f(z-1, u, T_s) - f(z, u, 0)] = 0 \quad \text{for } z \geq 1.$$

Exponential functions (III): Summary

- Define $a(\theta)$ and $b(\theta)$ via (4) and (5). Set

$$f\theta; z, u, v) = e^{\theta z + a(\theta)u + b(\theta)v}.$$

- Then,

$$\frac{\partial}{\partial u} f(\theta; z, u, v) = a(\theta) f(\theta; z, u, v),$$
$$\frac{\partial}{\partial v} f(\theta; z, u, v) = b(\theta) f(\theta; z, u, v).$$

- Thus, BAR becomes

$$\mathbb{E}_\nu \Big[- a(\theta) f(\theta; X(0)) - b(\theta) 1_{\{L(0)>0\}} f(\theta; X(0)) \Big] = 0,$$

or equivalently

$$[a(\theta) + b(\theta)] \mathbb{E}_\nu [f(\theta; X(0))] - b(\theta) \mathbb{E}_\nu \Big[1_{\{L(0)=0\}} f(\theta; X(0)) \Big] = 0. \qquad (6)$$

Prelimit (restricted) BAR

- From (6), for $\theta \le 0$,

$$[a(\theta)+b(\theta)]\mathbb{E}_\nu[f(\theta;X(0))] - b(\theta)\mathbb{P}_\nu(L(0)=0)\mathbb{E}_\nu\Big[f(\theta;X(0))|L(0)=0\Big] = 0.$$

$$[a(\theta)+b(\theta)]\mathbb{E}_\nu[f(\theta;X(0))] - b(\theta)(1-\lambda/\mu)\mathbb{E}_\nu\Big[f(\theta;X(0))|L(0)=0\Big] = 0.$$

- Scaling: changing θ to $r\theta$ for any $\theta \le 0$ to get pre-limit BAR,

$$[a(r\theta)+b(r\theta)]\phi^{(r)}(\theta) - b(r\theta)(1-\lambda/\mu)\phi_0^{(r)}(\theta) = 0, \tag{7}$$

where

$$\phi^r(\theta) \equiv \mathbb{E}[e^{r\theta L^r(0)+a(r\theta)R_e(0)+b(r\theta)R_s(0)}] \approx \mathbb{E}\Big[e^{\theta\left(rL^r(0)\right)}\Big],$$

$$\phi_0^r(\theta) \equiv \mathbb{E}_\nu\Big[f(r\theta;X(0))|L(0)=0\Big] = \mathbb{E}[e^{a(r\theta)T_e(0)+b(r\theta)T_s(0)}] \approx 1.$$

Asymptotic expansion of coefficients in (7)

- As $\theta \to 0$,

$$a(\theta) \approx -\lambda\Big(\theta + c_e^2\theta^2/2\Big), \tag{8}$$

where

$$\lambda = \frac{1}{\mathbb{E}(T_e)}, \quad c_e^2 = \frac{\mathrm{Var}(T_e)}{(\mathbb{E}(T_e))^2}.$$

- Fix $\theta \le 0$. Using (8), as $r \to 0$,

$$\begin{aligned} & a(r\theta) + b(r\theta) \approx -\lambda(r\theta + r^2\theta^2 c_e^2/2) - \mu(-r\theta + r^2\theta^2 c_s^2/2) \\ & = (\mu - \lambda)r\theta - r^2\theta^2(\lambda c_e^2 + \mu c_s^2)/2 \\ & = r^2\theta - r^2\theta^2(\lambda c_e^2 + \mu c_s^2)/2 \\ & \approx r^2\theta - r^2\theta^2\mu(c_e^2 + c_s^2)/2. \quad (\text{ using } \lambda = \mu - r) \end{aligned}$$

- Also

$$\begin{aligned} & - b(r\theta)(1 - \lambda/\mu) \approx (-\mu r\theta + 1/2r^2\theta^2\mu c_s^2)(1 - \lambda/\mu) \\ & \approx (-\mu r\theta)(1 - \lambda/\mu) = -r^2\theta. \end{aligned}$$

Limit BAR

- Recall prelimit BAR (7)

$$[a(r\theta)+b(r\theta)]\phi^{(r)}(\theta)-b(r\theta)(1-\lambda/\mu)\phi_0^{(r)}(\theta)=0.$$

- Assume that as $r\to 0$,

$$\phi^r\to\phi,\quad \phi_0^r\to\phi_0=1.$$

- Dividing (7) by r^2 and taking limit as $r\to 0$, one has the following limit BAR for ϕ and ϕ_0:

$$\Big[\theta^2(\lambda c_e^2+\mu c_s^2)/2-\theta\Big]\phi(\theta)+\theta\phi_0(\theta)=0 \quad \text{for } \theta\le 0. \tag{9}$$

Analysis from the limit BAR (9)

- Thus, for $\theta < 0$,

$$\Big[\theta\mu(c_e^2 + c_s^2)/2 - 1\Big]\phi(\theta) + \phi_0(\theta) = 0,$$

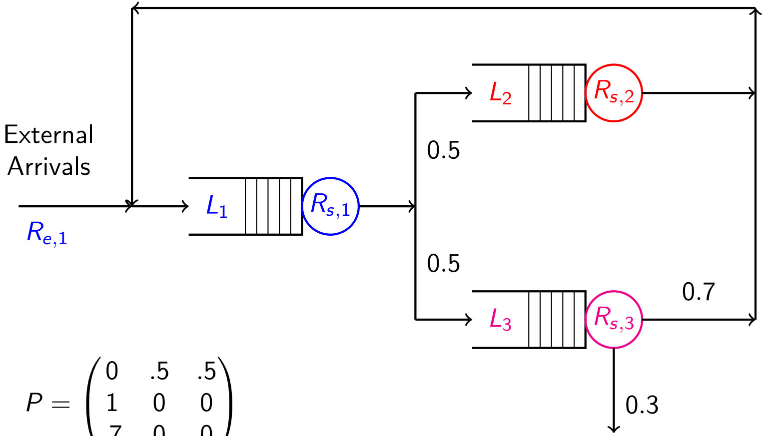
or

$$\phi(\theta) = \frac{\phi_0(\theta)}{\Big[\theta\mu(c_e^2 + c_s^2)/2 - 1\Big]} = \frac{1}{\Big[\theta\mu(c_e^2 + c_s^2)/2 - 1\Big]},$$

- Take $\theta \uparrow 0$, one has $\phi(0-) = \phi_0(0-) = 1$, concluding that $\{rL^r(0)\}$ is tight.
- Furthermore, as $r \to 0$,

$$rL^r(0) \Rightarrow \text{ exponential with mean } \mu(c_e^2 + c_s^2)/2.$$

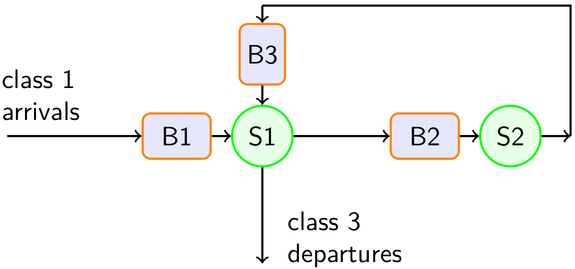
Mgf approach works for generalized Jackson networks



$$P = \begin{pmatrix} 0 & .5 & .5 \\ 1 & 0 & 0 \\ .7 & 0 & 0 \end{pmatrix}$$

Braverman-Dai-Miyazawa (2017)

A sketch for proving Theorem 1



$$f(\theta, X(0)) = \exp\Big(\sum_{i=1}^{3} \theta_i L_i(0) + a(\theta_1) R_e(0)$$
$$+ b_1(\theta_1 - \theta_2) R_{s,1}(0) + b_2(\theta_2 - \theta_3) R_{s,2}(0) + b_3(\theta_3) R_{s,3}(0)\Big).$$

$$\mathbb{E}\left[e^{a(\theta_1) T_e}\right] = e^{-\theta_1}, \quad \mathbb{E}[e^{b_1(\theta_1 - \theta_2) T_{s,1}}] = e^{\theta_1 - \theta_2},$$
$$\mathbb{E}[e^{b_2(\theta_2 - \theta_3) T_{s,2}}] = e^{\theta_2 - \theta_3}, \quad \mathbb{E}[e^{b_3(\theta_3) T_{s,3}}] = e^{\theta_3}.$$

Pre-limit BAR

- Restricted BAR

$$a(\theta_1)\mathbb{E}[f(\theta,X)] + b_1(\theta_1-\theta_2)\mathbb{E}\big[f(\theta,X)1_{\{L_1>0,L_3=0\}}\big] + b_2(\theta_2-\theta_3)\mathbb{E}\big[f(\theta,X)1_{\{L_2>0\}}\big] + b_3(\theta_3)\mathbb{E}\big[f(\theta,X)1_{\{L_3>0\}}\big] = 0,$$

- which is equivalent to

$$\Big[a(\theta_1) + (1-\beta_3)b_1(\theta_1-\theta_2) + b_2(\theta_2-\theta_3) + \beta_3 b_3(\theta_3)\Big]\mathbb{E}[f(\theta,X)] + \Big[b_1(\theta_1-\theta_2) - b_3(\theta_3)\Big]\Big(\mathbb{E}\big[f(\theta,X)1_{\{L_3=0\}}\big] - (1-\beta_3)\mathbb{E}\big[f(\theta,X)\big]\Big) - b_1(\theta_1-\theta_2)\mathbb{E}\big[f(\theta,X)1_{\{L_1=0,L_3=0\}}\big] - b_2(\theta_2-\theta_3)\mathbb{E}\big[f(\theta,X)1_{\{L_2=0\}}\big] = 0. \quad (10)$$

- Replacing θ by $r\theta$, define

$$\phi^r(\theta) = \mathbb{E}[f(r\theta,X)], \quad \phi_1^r(\theta) = \mathbb{E}\big[f(r\theta,X)|L_1=0,L_3=0\big], \quad \phi_2^r(\theta) = \mathbb{E}\big[f(r\theta,X)|L_2=0\big].$$

State space collapse

- Choosing θ_3 to aid the removal of (10).
- Conduct expansion

$$b_1\big(r(\theta_1-\theta_2)\big) \approx \mu_1\Big(r(\theta_1-\theta_2) - r^2(\theta_1-\theta_2)^2 c_{s,1}^2/2\Big),$$
$$b_3(r\theta_3) \approx \mu_3\Big(r\theta_3 - r^2\theta_3^2 c_{s,3}^2/2\Big).$$

- Choose

$$\theta_1 \le 0, \quad \theta_2 \le 0, \quad \theta_3 = \frac{\mu_1}{\mu_3}(\theta_1-\theta_2) \le 0. \tag{11}$$

- There are "enough" points $\theta = (\theta_1, \theta_2, \theta_3)$ satisfying (11).

Limit BAR

- Assume that, for any $\theta_1 \leq \theta_2 \leq 0$, and $\theta_3 = (\mu_1/\mu_3)(\theta_1 - \theta_2)$,

$$\Big(\phi^r(\theta), \phi_1^r(\theta), \phi_2^r(\theta)\Big) \to \Big(\phi(\theta_1, \theta_2, \theta_3), \phi_1(\theta_2), \phi_2(\theta_1, \theta_3)\Big).$$

- The limit satisfies

$$\gamma(\theta_1, \theta_2, \theta_3)\phi(\theta_1, \theta_2, \theta_3) + (\theta_2 - \theta_3)\phi_2(\theta_1, \theta_3) + \mu_3\theta_3\phi_1(\theta_2) = 0$$

where

$$\gamma(\theta_1, \theta_2, \theta_3) = \mu_1(\theta_1 - \theta_2) + (\theta_2 - \theta_3) + \text{ quadratic term of } \theta.$$

Tightness

Assume for any $\theta_1 \le \theta_2 \le 0$, and $\theta_3 = (\mu_1/\mu_3)(\theta_1 - \theta_2)$. Ignoring the quadratic term,

$$(\theta_2 - \theta_3)\Big(\phi_2(\theta_1, \theta_3) - \phi(\theta_1, \theta_2, \theta_3)\Big) + \mu_3\theta_3\Big(\phi_1(\theta_2) - \phi(\theta_1, \theta_2, \theta_3)\Big) = 0.$$

- Setting $\theta_2 = u \uparrow 0$, $\theta_3 = -u^2$, and

  $$\theta_1 = u - \frac{\mu_3}{\mu_1}u^2 < 0,$$

  one has

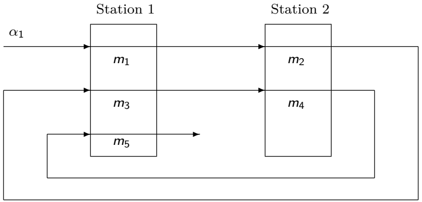
  $$\phi(0-, 0-, 0-) = \phi_2(0-, 0-).$$

- Setting $\theta_2 = 0$ and using $\phi_1(0) = 1$, one has

  $$\phi(0-, 0, 0-) - \phi_2(0-, 0-) + \mu_3(1 - \phi(0-, 0, 0-)) = 0,$$

  which implies that $\phi(0-, 0-, 0-) = \phi_2(0-, 0-) = \phi(0-, 0, 0-) = 1$.

BAR for a 5-class reentrant line



$$
\begin{aligned}
& a(\theta_1)\mathbb{E}\Big[f(\theta; X(0)\Big] \\
& + b_1(\theta_1 - \theta_2)\mathbb{E}\Big[f(\theta; X(0)1_{\{L_1>0, L_3=0, L_5=0\}}\Big] \\
& + b_2(\theta_2 - \theta_3)\mathbb{E}\Big[f(\theta; X(0)1_{\{L_2>0, L_4=0\}}\Big] \\
& + b_3(\theta_3 - \theta_4)\mathbb{E}\Big[f(\theta; X(0)1_{\{L_3>0, L_5=0\}}\Big] \\
& + b_4(\theta_4 - \theta_5)\mathbb{E}\Big[f(\theta; X(0)1_{\{L_4>0\}}\Big] \\
& + b_5(\theta_5)\mathbb{E}\Big[f(\theta; X(0)1_{\{L_5>0\}}\Big] = 0.
\end{aligned}
$$

Choice of θ_3, θ_4 and θ_5 to aid SSC

$$\mu_1(\theta_1 - \theta_2) = \mu_3(\theta_3 - \theta_4) = \mu_5\theta_5, \quad \mu_2(\theta_2 - \theta_3) = \mu_4(\theta_4 - \theta_5). \tag{12}$$

- "Not enough points $\theta < 0$" satisfying (12).
- Truncation on L_3, L_4, L_5 is needed to allow positive values of θ_3, θ_4, θ_5. For example, even if $\theta_3 > 0$,

$$e^{r\theta_3 L_3^r(0)} \le e^{\theta_3} \text{ if } L^3(0) < 1/r.$$

Sufficient to have

$$\sup_{(r\in(0,r_0])} \mathbb{E}[L_3^r(0)] < \infty.$$

Open problem: not using Theorem 2

- In proving a version of Theorem 1 for general multiclass queueing networks under priority policies, we need the following strong SSC:

$$r\mathbb{E}_\nu[L_3^r(\infty)] \to 0.$$

When all distributions are exponential, the mean drift

$$\mathbb{E}[L_3^r(n+1)]^2 | L_3^r(n) = x] - x^2 \le C_1 - C_2 x,$$

where $C_1, C_2 > 0$ do not depend on r. Thus,

$$\mathbb{E}[L_3^r(\infty)] \le \frac{C_1}{C_2}.$$

- SSC

$$rL_3^r(\infty) \Rightarrow 0.$$

can be proved using the BAR for this case. General?