An introduction to cluster superalgebras

Ashish K. Srivastava

Saint Louis University, St. Louis
joint work with Li Li, James Mixco and Biswajit Ransingh

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Motivation behind superalgebras

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- Rotational symmetry in the theory of spin.
- Poincare symmetry in the classification of elementary particles.
- Permutation symmetry in dealing with the systems of identical particles.

Supersymmetry

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Superalgebras were introduced to provide an algebraic framework for describing supersymmtery.

Super vector space

A super vector space V is a vector space that is $\mathbb{Z}_2$-graded, that is, it has a decomposition $V = V_0 \oplus V_1$ with $0, 1 \in \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

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The elements of V_0 are called the *even* (or bosonic) elements and the elements of V_1 are called the *odd* (or fermionic) elements. The elements in $V_0 \cup V_1$ are called *homogeneous* and their *degree*, denoted by d, is defined to be 0 or 1 according as they are even or odd.

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The morphisms in the category of super vector spaces are linear maps which preserve the gradings.

Superalgebra

A superalgebra A is an associative algebra with an identity element (which is necessarily an even element) such that the multiplication map $A \otimes A \to A$ is a morphism in the category of super vector spaces.

This is the same as requiring $d(ab) = d(a) + d(b)$ for any two homogeneous elements a and b in A.

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This means in a supercommutative superalgebra, odd elements anticommute with each other, that is, $ab = -ba$ for any two odd elements $a, b \in A$, whereas even elements commute with any other element (even or odd).

Cluster algebras

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Consider the initial seed (X, B) where $X = \{x_1, \ldots, x_m\}$ and $B = [b_{ij}]$ is an $m \times m$ skew-symmetrizable integer matrix (that is, there exists a diagonal integer matrix D such that DB is a skew-symmetric matrix).

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The set $X = \{x_1, \ldots, x_m\}$ is called the initial cluster and each x_i in this set is called an initial cluster variable.

There is a mutation in the direction of each initial cluster variable x_k which is denoted by μ_k.

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The mutation is defined as $\mu_k(X, B) = (X', B')$ with $X' = \{x'_1, \ldots, x'_m\}$ and $B' = [b'_{ij}]$, where

$$x'_j = x_j, \quad j \neq k$$

$$x'_k = \frac{1}{x_k}\left[\left(\prod_{b_{ik}>0} x_i^{b_{ik}}\right) + \left(\prod_{b_{ik}<0} x_i^{-b_{ik}}\right)\right] \tag{1}$$

and
$b'_{ij} = -b_{ij}$ if $i = k$ or $j = k$, otherwise

$$b'_{ij} = b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}.$$

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Note that in the special case when B is a skew-symmetric integer matrix, B can be encoded by a connected quiver Q which has no loops and no 2-cycles and then mutation of B can be suitably rephrased in terms of mutation of Q as follows;

We write $\mu_k(Q) = Q'$, where Q' is obtained from Q by keeping the same vertices and changing arrows by the following rule:

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(i) If there is a path $x_i \to x_k \to x_j$, add an arrow $x_i \to x_j$ for each distinct path.

(ii) Reverse all arrows incident at x_k.

(iii) Delete any 2-cycle produced in the process.

Scattering amplitudes

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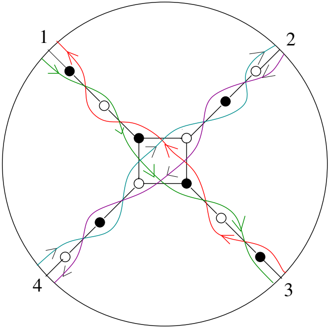
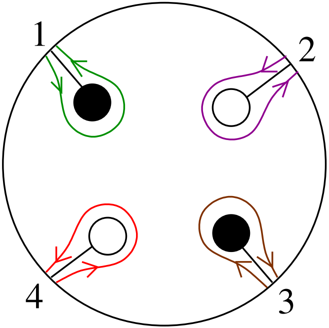
But to visually interpret them in easier manner, one may label particles involved by $\{1, 2, \ldots, n\}$ and interaction of particles involved could be associated with a permutation of $\{1, 2, \ldots, n\}$. One of the ways to represent scattering amplitudes is on-shell diagram.

Scattering amplitudes

Consider the following two four-particle scattering amplitudes. Let us label the faces of the diagram as indicated below and assume that rays bend to the right of black vertices and to the left of white vertices.





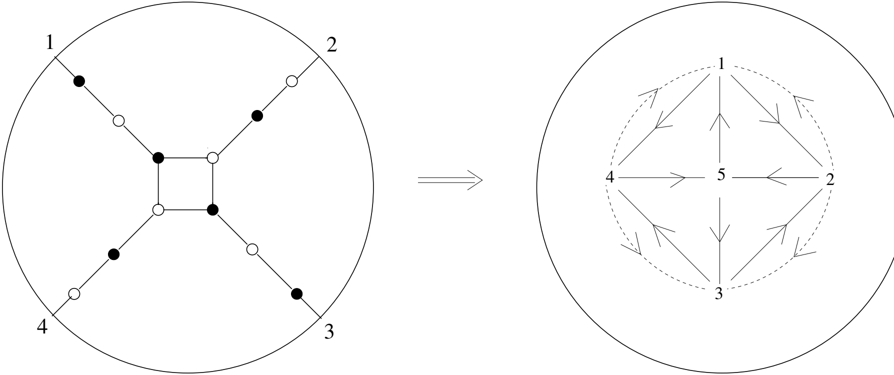
On-shell diagrams

The above on-shell diagrams denote the following two permutations of scattering of four particles, respectively

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Quiver attached to a bipartite graph

Let $\mathcal{N}$ denote the supersymmetry supercharge. Note that $\mathcal{N} = 0$ represents only bosons and $\mathcal{N} > 0$ represents bosons with fermions. The latter case is also known as supersymmetry. The number of fermions and type of Lagrangian determine the value of $\mathcal{N}$. Dual of a bipartite graph in physics of $\mathcal{N} = 4$ supersymmetry is called a quiver. We associate a quiver with a planar bipartite graph by taking a vertex for each face and for each edge in the bipartite graph, we draw an arrow in this quiver in such a way that it sees the white arrow in left as depicted below.



The complicated on-shell diagrams are related with complicated quiver gauge theories and it makes sense to be able to deal with situations when we have loops and oriented 2-cycles in quiver to model more natural physical systems.

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Cluster Superalgebra

Recently, Ovsienko has made an attempt to define cluster superalgebras. He considers extension of a quiver by adding odd vertices and makes it an oriented hypergraph. Note that in an oriented hypergraph, an arrow can connect any number of vertices.

The main limitations of his approach are:

1. absence of arrows between odd vertces.
2. lack of exchange relations for the odd variables.

Consider an initial seed $(X|Y, Q)$ with $X = \{x_1, \ldots, x_m\}$ is the set of variable that commute with each other and $Y = \{y_1, \ldots, y_n\}$ is the set of Grassman (or odd) variables that anticommute with each other.

These variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ are called initial supercluster variables.

Let Q be a qiver with $m+n$ vertices which are labelled as $x_1, \ldots, x_m, y_1, \ldots, y_n$. We will call the vertices labelled with $x_1, \ldots, x_m$ as even vertices and the vertices labelled with $y_1, \ldots, y_n$ as odd vertices.

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Just as in the case of classical cluster algebras, we do not allow 2-cycles between two even variables or two odd variables. However, we allow 2-cycles between an even vertex and an odd vertex. Just as in the case of classical cluster algebras, we do not allow loops on even vertices however, we do allow loops on odd vertices. We will call such a quiver a *supercluster quiver.*

Even and odd mutations

We first define Fomin-Zelevinsky type mutation for our setting.

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There may be two types of even (odd) variables: exchangeable even (odd) variables and frozen even (odd) variables. We define two types of mutations: an even mutation and an odd mutation. We will denote the even mutation in the direction of an exchangeable vertex x_k as μ_k and odd mutation in the direction of an exchangeable vertex y_k as η_k.

Even mutation

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$$\mu_k(x_1, \cdots, x_m, y_1, \cdots, y_n, Q) = (\mu_k(x_1), \cdots, \mu_k(x_m), \mu_k(y_1), \cdots, \mu_k(y_n), \tag{2}$$

where

$$\begin{aligned} \mu_k(y_i) &= y_i \quad \text{for each } i \\ \mu_k(x_i) &= x_i, \quad \text{for each } i \neq k \\ \mu_k(x_k) &= \frac{1}{x_k}\left[(-1)^u \left(\prod_{x_i \to x_k} x_i\right) + (-1)^v \left(\prod_{x_k \to x_j} x_j\right) + \left(\sum_{y_i \to x_k \to y_j, y_i \to y_j} y_i y \right.\right. \end{aligned} \tag{3}$$

Even mutation

where u is the total number of loops on all odd vertices which have arrows between them and even vertices x_i with arrow $x_i \to x_k$, v is the total number of loops on all odd vertices which have arrows between them and even vertices x_j with arrow $x_k \to x_j$. In the expression $\left(\sum_{y_i \to x_k \to y_j, y_i \to y_j} y_i y_j \right)$ above, both conditions $y_i \to x_k \to y_j$ and $y_i \to y_j$ must be satisfied and we consider the multiplicity in the sense that if there are a arrows $y_i \to x_k$, b arrows $x_k \to y_j$ and c arrows $y_i \to y_j$, then it will give contribute $abcy_i y_j$ in the sum.

Even mutation

The new quiver $\mu_k(Q)$ is obtained from Q by modifying vertices in view of the above mentioned exchange rules and changing arrows as follows:

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(i) If there is a path $x_i \to x_k \to x_j$, add an arrow $x_i \to x_j$ for each distinct path.

(ii) Reverse all arrows connecting x_k to another even vertex.

(iii) Delete any 2-cycle produced between two even variables in the process.

Odd mutation

We define the odd mutation in the direction of an exchangeable odd vertex y_i as

$$\eta_i(x_1, \cdots, x_m, y_1, \cdots, y_n, Q) = (\eta_i(x_1), \cdots, \eta_i(x_m), \eta_i(y_1), \cdots, \eta_i(y_n), \eta_i(\tag{4}$$

where

$$\begin{aligned}
\eta_i(y_j) &= y_j, \quad j \neq i \\
\eta_i(y_i) &= \delta(y_i)y_i + \left(\prod_{x_k \leftrightarrows y_i} \frac{1}{x_k}\right)\left[\left(\sum_{y_i \to y_j} y_j\right)\left(\prod_{y_i \to x_l \to y_j} x_l\right) + \left(\sum_{y_j \to y_i} y_j\right)\left(\right.\right. \\
\eta_i(x_k) &= x_k \quad \text{for each } k
\end{aligned} \tag{5}$$

Here $\delta(y_i) = 1$ if there is no arrow between y_i and another odd vertex and $\delta(y_i) = 0$ otherwise. In the expressions $\left(\sum_{y_i \to y_j} y_j\right)$ and

Odd mutation

The new quiver $\eta_i(Q)$ is obtained from Q by modifying vertices in view of the above mentioned exchange rules and changing arrows as follows:

(i) If there is a path $y_k \to y_i \to y_j$ for $k \neq i \neq j$, then add an arrow $y_k \to y_j$ for each distinct path.

(ii) Reverse all arrows incident on y_i.

(iii) Delete any 2-cycle produced between two odd variables in the process.

It is not difficult to see that the even mutation is involutive, that is, $\mu_k^2 = 1$. So, if there are m' number of exchangeable even variables, then the exchange pattern for even vertices is an m'-regular tree. Also, we have $\eta_k^3(y) = \eta_k(y)$ for each $y \in Y$ and each element in $\eta_k^2(Y)$ can be generated by X, Y and $\eta_k(Y)$.

Cluster Superalgebra

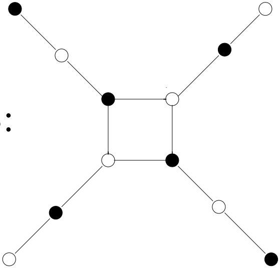
Let $\mathcal{X}_{even}$ be the set of all supercluster even variables that can be obtained by applying a sequence of even mutations to the initial seed $(X|Y,Q)$ and $\mathcal{X}_{odd}$ be the set of all supercluster odd variables that can be obtained by applying a sequence of odd mutations to the initial seed $(X|Y,Q)$. Then the cluster superalgebra $\mathcal{C}_{\mathbb{K}}(X|Y,Q)$ over a field $\mathbb{K}$ (of characteristic different from 2) is defined to be the supercommutative $\mathbb{K}$-superalgebra generated by $\mathcal{X}_{even} \cup \mathcal{X}_{odd}$.

If there are no Grassman variables, then even mutation is exactly the same as Fomin-Zelevinsky mutation for classical cluster algebras.

Combinatorial geometric model of even and odd mutations

Consider a bipartite planar graph B given as follows:

B:



We associate a quiver $Q = Q(B)$ with the bipartite graph B in the following manner. Take a vertex for each face and for each edge in the graph B, we draw an arrow in this quiver in such a way that it sees the white arrow in left. So we get the following quiver:

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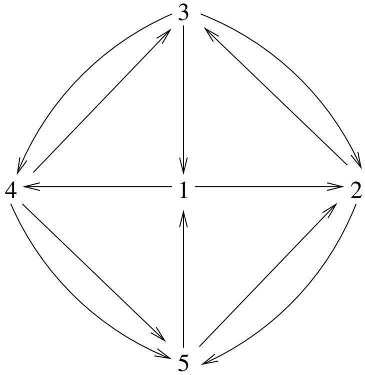
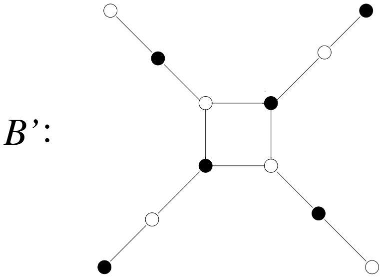
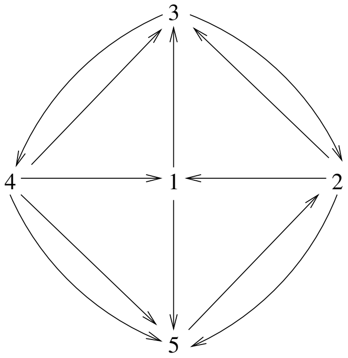


If we do the "flip" move on B (which swaps white and black vertices), we get the new bipartite graph B' as



Clearly the quiver $Q' = Q(B')$ associated to this bipartite graph is

$Q' = Q(B')$:



Note that if we take vertices $1, 3, 5$ as even vertices and $2, 4$ as odd, then under our definition of even and odd mutations we have $\eta_4 \eta_2 \mu_1(Q) = Q'$.

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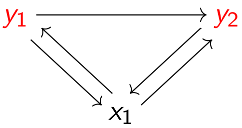
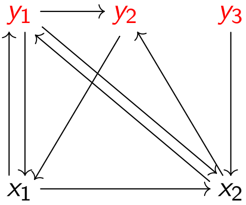
This shows that the "flip" move of planar bipartite graphs provides a geometric combinatorial model for a sequence of even and odd mutations.



(6)

We have:

$$\mu_1(x_1) = \frac{1}{x_1}(2 + y_1 y_2),\ \ \eta_1(y_1) = y_2,\ \ \eta_2(y_2) = y_1 \tag{7}$$

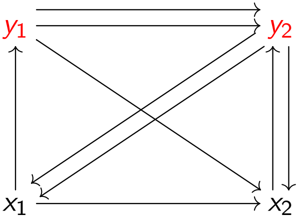


(8)

We have:

$$\mu_1(x_1) = \frac{1}{x_1}(1+x_2), \;\; \mu_2(x_2) = \frac{1}{x_2}(1+x_1+y_1y_2), \;\; \mu_1(x_2) = x_2, \;\; \mu_2(x_1) = x \tag{9}$$

$$\eta_1(y_1) = \frac{y_2}{x_1}, \quad \eta_2(y_2) = y_1x_2, \quad \eta_3(y_3) = y_3 \tag{10}$$



(11)

$$\mu_1(x_1) = \frac{1 + x_2}{x_1},\ \ \mu_2(x_2) = \frac{1}{x_2}(1 + x_1 + 2y_1y_2) \tag{12}$$

$$\eta_1(y_1) = 2y_2x_2,\ \ \eta_2(y_2) = 2y_1 \tag{13}$$

A supermatrix over a superalgebra $\mathcal{A}$ is a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the matrices A, D have even entries and they are of sizes $m \times m$ and $n \times n$, respectively. The matrices B, C have odd entries and are of sizes $m \times n$, $n \times m$, respectively.

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The general linear supergroup $GL(m|n)$ over a superlagera $\mathcal{A}$ is the group of all invertible supermatrices M of size $(m+n) \times (m+n)$, i.e. the supermatrices M with the superdetminant $sdet(M) = det(A - BD^{-1}C)det(D^{-1})$ invertible in $\mathcal{A}$.

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$$M^{st} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}$$

Let us write

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

$$K_{2k+1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -I_k \\ 0 & -I_k & 0 \end{pmatrix}$$

$$K_{2k} = \begin{pmatrix} 0 & -I_k \\ -I_k & 0 \end{pmatrix}$$

and

$$J_{m,n} = diag(J_m, K_n).$$

The symplectic-orthogonal superalgebra $SpO(2m|n)$ over a superring $R = R_0 \oplus R_1$ is defined as the superalgebra consisting of $(2m+n) \times (2m+n)$ supermatrices M with entries in R such that $sdet(M)$ is invertible in R and $M^{st} J_{m,n} M = J_{m,n}$.

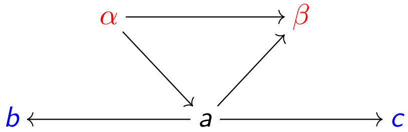
The symplectic-orthogonal superalgebra $SpO(2|1)$ over any superring $R = R_0 \oplus R_1$ admits a cluster superalgebra structure.

Let $R = R_0 \oplus R_1$ be a superring. The symplectic-orthogonal superalgebra $SpO(2|1)$ over a superring $R = R_0 \oplus R_1$ consists of supermatrices $\begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$ such that

$$ad = 1 + bc + \alpha\beta, e = 1 + \alpha\beta, \gamma = a\beta - b\alpha, \delta = c\beta - d\alpha. \quad (14)$$

The elements $a, b, c, d, e \in R_0$ and $\alpha, \beta, \gamma, \delta \in R_1$. Note that the elements $a, b, c, d, \alpha, \beta$ generate the symplectic-orthogonal superalgebra $SpO(2|1)$.

Choose $a, b, c, \in R_0$ and $\alpha, \beta \in R_1$. Consider the initial seed $(X|Y)$ with $X = \{a, b, c\}$ where b, c are frozen and $Y = \{\alpha, \beta\}$ and consider the following quiver Q:

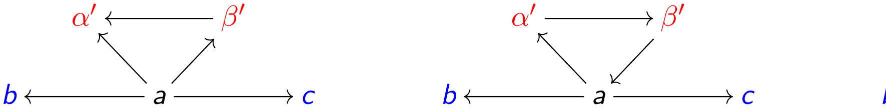


We have

$$\mu_a(a) = \frac{1}{a}[bc + 1 + \alpha\beta] \tag{15}$$

Set $\mu_a(a) = d$. Then $ad = 1 + bc + \alpha\beta$. This gives us the first relation of the Equation 14. Note that $\mu_a^2(a) = a$, so iterating even mutations does not produce more new exchangeable even variables other than a and d, thus $\mathcal{X}_{even} = \{a, b, c, d\}$.

Next, we show that every odd variable can be generated by $\{a, \alpha, \beta\}$. Indeed, iterated odd mutations may produce three more distinct quivers:



For all the four quivers, we must have $\eta_{\alpha'}(\alpha') = \beta'$ or $a\beta'$, and $\eta_{\beta'}(\beta') = \alpha'$ or $a\alpha'$; thus the new odd cluster variables are generated by a, α', β'. Therefore all odd cluster variables are generated by a, α, β by induction. In fact, a simple computation shows

$$\mathcal{X}_{odd} = \{a^i\alpha \mid i \geq 0\} \cup \{a^i\beta \mid i \geq 0\}.$$

This shows that the symplectic-orthogonal superalgebra $SpO(2|1)$ over any superring $R = R_0 \oplus R_1$ admits a cluster superalgebra structure.

We do not know whether the more general result that the symplectic-orthogonal superalgebra $SpO(2m|n)$ over any superring $R = R_0 \oplus R_1$ admits a cluster superalgebra structure is true or not.
For $m = 1$ and $n = 2$, we have
The symplectic-orthogonal superalgebra $SpO(2|2)$ over any superring $R = R_0 \oplus R_1$ is quotient of a subalgebra of a cluster superalgebra.

The symplectic-orthogonal superalgebra $SpO(2|2)$ over a superring $R = R_0 \oplus R_1$ consists of supermatrices

$$M = \begin{pmatrix} a & b & \gamma_1 & \gamma_2 \\ c & d & \delta_1 & \delta_2 \\ \alpha_1 & \beta_1 & e_1 & e_2 \\ \alpha_2 & \beta_2 & e_3 & e_4 \end{pmatrix}$$

such that $sdet(M)$ is invertible in R and

$$M^{st} J_{1,2} M = J_{1,2}.$$

The above condition gives us the following set of equations

$$
\begin{aligned}
ad &= 1 + bc - \alpha_1\beta_2 - \alpha_2\beta_1 \\
e_1e_4 + e_2e_3 &= 1 - \gamma_1\delta_2 - \gamma_2\delta_1 \\
e_1e_3 &= -\gamma_1\delta_1 \\
e_2e_4 &= -\gamma_2\delta_2 \\
-c\gamma_1 + a\delta_1 &= e_1\alpha_2 + e_3\alpha_1 \\
-c\gamma_2 + a\delta_2 &= e_2\alpha_2 + e_4\alpha_1 \\
-d\gamma_1 + b\delta_1 &= e_1\beta_2 + e_3\beta_1 \\
-d\gamma_2 + b\delta_2 &= e_2\beta_2 + e_4\beta_1
\end{aligned} \tag{16}
$$

Choose $a, b, c, e_1, e_2, e_3 \in R_0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in R_1$.

Consider the initial seed $(X|Y)$ with $X = \{a, b, c, e_1, e_2, e_3\}$ where b, c, e_2, e_3 are frozen and $Y = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2\}$ and consider the following quiver:

Mutating along the directions of vertices a and e_1, we get

$$\mu_a(a) = \frac{1}{a}[bc + 1 + \beta_2\alpha_1 + \beta_1\alpha_2] \tag{17}$$

$$\mu_{e_1}(e_1) = \frac{1}{e_1}[1 - e_2 e_3 + \delta_2\gamma_1 + \delta_1\gamma_2] \tag{18}$$

Set $\mu_a(a) = d$ and $\mu_{e_1}(e_1) = e_4$.
This shows that the superalgebra $SpO(2|2)$ is a quotient of a subalgebra of the cluster superalgebra $C(X|Y, Q)$.

A superfrieze, or a supersymmetric frieze pattern over a $\mathbb{Z}_2$-graded ring $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$ has been defined as the following array

$\dots \quad 0 \quad 0$

$\dots \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$1 \quad 1 \quad 1$

$\varphi_{0,0} \quad \varphi_{\frac{1}{2},\frac{1}{2}} \quad \varphi_{1,1} \quad \varphi_{\frac{3}{2},\frac{3}{2}} \quad \varphi_{2,2}$

$f_{0,0} \quad f_{1,1}$

$\varphi_{-\frac{1}{2},\frac{1}{2}} \quad \varphi_{0,1} \quad \varphi_{\frac{1}{2},\frac{3}{2}} \quad \varphi_{1,2} \quad \varphi_{\frac{3}{2},\frac{5}{2}}$

$f_{-1,0} \quad f_{0,1} \quad f_{1,2}$

$\dots \quad \dots \quad \dots \quad \dots \quad \dots$

$f_{2-m,1} \quad f_{0,m-1}$

$\dots \quad \varphi_{\frac{3}{2}-m,\frac{3}{2}} \quad \varphi_{2-m,2} \quad \dots \quad \varphi_{0,m} \quad \varphi_{\frac{1}{2},m-}$

$1 \quad 1 \quad 1$

where $f_{i,j} \in \mathcal{R}_0$ and $\varphi_{i,j} \in \mathcal{R}_1$, and where every *elementary diamond*:

$$\begin{array}{ccccc} & & B & & \\ & \Xi & & \Psi & \\ A & & & & D \\ & \Phi & & \Sigma & \\ & & C & & \end{array}$$

satisfies the following conditions:

$$\begin{aligned} AD - BC &= 1 + \Sigma\Xi, \\ B\Phi - A\Psi &= \Xi, \\ B\Sigma - D\Xi &= \Psi, \end{aligned} \tag{19}$$

that we call the frieze rule.

The number of even rows between the rows of 1's is called the width of the superfrieze.

The last two equations are equivalent to

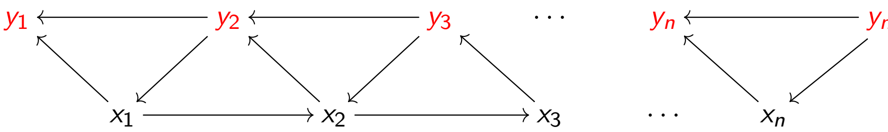
$$A\Sigma - C\Xi = \Phi, \qquad D\Phi - C\Psi = \Sigma.$$

Note also that these equations also imply $\Xi\Sigma = \Phi\Psi$, so that the first equation can also be written as follows: $AD - BC = 1 - \Phi\Psi$.

One can associate an elementary diamond with every element of $\mathrm{SpO}(2|1)$ using the following formula:

$$\left(\begin{array}{cc|c} a & b & \gamma \\ c & d & \delta \\ \hline \alpha & \beta & e \end{array}\right) \longleftrightarrow \begin{array}{ccccc} & & -a & & \\ & \color{red}{\gamma} & & \color{red}{\alpha} & \\ b & & & & -c \\ & \color{red}{-\beta} & & \color{red}{\delta} & \\ & & d & & \end{array}$$

Consider the following extension of the standard Dynkin quiver of type A_n with a set of n even vertices $X = \{x_1, \dots, x_n\}$ and a set of $n+1$ odd vertices $Y = \{y_1, \dots, y_{n+1}\}$:



(20)

Let $\widetilde{Q}$ denote the supercluster quiver and consider $\mathcal{C}_{\mathbb{K}}(X|Y, \widetilde{Q})$ with $\mathbb{K}$ a field of characteristic different from 2. We have the following:
The supercommutative superalgebra generated by all the entries of a superfrieze of width n is a subalgebra of the cluster superalgebra $\mathcal{C}_{\mathbb{K}}(X|Y, \widetilde{Q})$.

Choose the following entries of the superfrieze on parallel diagonals:

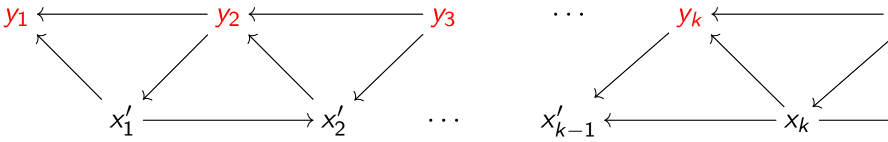
$$
\begin{array}{cccccccccccccc}
1 & & & & 1 & & & & & & & & & \\
 & {\color{red}*} & & {\color{red}y_1} & & {\color{red}*} & & {\color{red}y_1'} & & & & & & \\
 & & x_1 & & & & x_1' & & & & & & & \\
 & & & {\color{red}*} & & {\color{red}y_2} & & {\color{red}*} & & {\color{red}y_2'} & & & & \\
 & & & & x_2 & & {\color{red}\ddots} & & x_2' & & {\color{red}\ddots} & & & \\
 & & & & & \ddots & & {\color{red}y_n} & & \ddots & & {\color{red}y_n'} & & \\
 & & & & & & x_n & & & & x_n' & & & \\
 & & & & & & & {\color{red}*} & & {\color{red}y_{n+1}} & & {\color{red}*} & & {\color{red}y_{n+1}'} \\
 & & & & & & & & 1 & & & & 1 &
\end{array}
$$

All entries of the superfrieze are determined by $x_1, \ldots, x_n, y_1, \ldots, y_{n+1}$ and hence these can be taken as initial coordinates. Note that we are done if we can show that the superfrieze entries $x'_1, \ldots, x'_n, y'_1, \ldots, y'_{n+1} \in \mathcal{C}_{\mathbb{K}}(X|Y, \widetilde{Q})$.

We have that

$$x_k x'_k = 1 + x_{k+1} x'_{k-1} + y_{k+1} y_k \tag{21}$$

Now perform even mutations on $\widetilde{Q}$ at vertices x_1, then x_2, until x_n. After the first $k-1$ mutations we obtain the following quiver:



Mutating at vertex x_k we obtain the following from our rule for even mutation:

$$x_k x'_k = 1 + x_{k+1} x'_{k-1} + y_{k+1} y_k. \tag{22}$$

This shows that the entries $x'_1, \ldots, x'_n$ from the superfrieze are the same as the supercluster variables $x'_1, \ldots, x'_n$ obtained by iterated even mutations at consecutive even vertices in $\widetilde{Q}$.
We now consider the odd entries of the superfrieze $y'_1, \ldots, y'_{n+1}$.
We have that

$$y'_1 = y_2 - x'_1 y_1,$$

so clearly $y'_1 \in \mathcal{C}_{\mathbb{K}}(X|Y, \widetilde{Q})$. We have the following for odd entries of the superfrieze for all k:

$$y'_k = y'_{k-1} - y_1 x'_k.$$

As y'_k is a linear combination of y'_{k-1}, y_1, and x'_k for all k, it follows that $y'_k \in \mathcal{C}_{\mathbb{K}}(X|Y, \widetilde{Q})$ for all k since it has already been established that $y'_1 \in \mathcal{C}_{\mathbb{K}}(X|Y, \widetilde{Q})$.
By a similar argument, this holds for all parallel diagonals and it can be established that all entries of the superfrieze are contained in $\mathcal{C}_{\mathbb{K}}(X|Y, \widetilde{Q})$.

Failure of Laurent Phenomenon

As a major contrast to the classical cluster algebra setting, the Laurent phenomenon fails to hold in the case of cluster superalgebras as we will see in the example below. Consider the initial seed $(X|Y,Q)$ with $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ and quiver Q given as:

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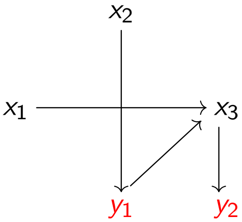

(23)

We have

$$\mu_1\mu_3\mu_1\{x_1, x_2, x_3, y_1, y_2, Q\} = \{x_1'', x_2, x_3', y_1, y_2, \mu_1\mu_3\mu_1(Q)\}$$

where

$$x_1'' = \frac{1 + x_3 + x_1(1 - y_1y_2) + x_1x_3}{x_3(1 + x_3)}$$

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where

$$x_1'' = \frac{1 + x_3 + x_1(1 - y_1 y_2) + x_1 x_3}{x_3(1 + x_3)}$$

Note that x_1'' obtained above is not a Laurent polynomial in initial cluster variables x_1, x_2, x_3, y_1, y_2 and this shows that the Laurent phenomenon fails to hold in the case of cluster superalgebras.

A supercluster quiver Q' is said to be *mutation equivalent* to another supercluster quiver Q if there exists mutations $\sigma_1, \ldots, \sigma_r$ with each σ_i being either an even mutation or an odd mutation such that $\sigma_r \circ \cdots \circ \sigma_1(Q) = Q'$.

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A cluster superalgebra $\mathcal{C}_K(X|Y, Q)$ is said to be of

1. *finite type* if the number of supercluster variables is finite.
2. *finite mutation type* if the number of supercluster quivers Q' that are mutation equivalent to Q is finite.

We propose following problems for further development of the notion of cluster superalgebras.

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Infinite supercluster variables

It is well-known that the cluster algebra $\mathcal{A}(X, Q)$ is of finite type if the uderlying graph of the quiver Q is mutation equivalent to a simply laced Dynkin diagram.

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It is natural to ask then if a quiver whose underslying graph is a simply laced Dynkin diagram is extended by some odd vertices, do we still end up getting only finitely many supercluster variables in the corresponding cluster superalgebra?

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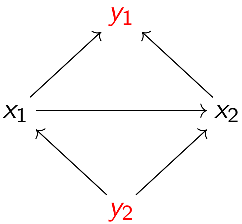
It is natural to ask then if a quiver whose underslying graph is a simply laced Dynkin diagram is extended by some odd vertices, do we still end up getting only finitely many supercluster variables in the corresponding cluster superalgebra?
We answer this question in the negative in following example.

Consider an initial seed $(X|Y, Q)$ where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and quiver Q (whose even vertices form the Dynkin diagram of type A_2) is as given below



(24)

Let $x_3 = \mu_1(x_1) = x_1^{-1}(1 + x_2 + y_1 y_2)$.

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Then apply μ_2, let $x_4 = \mu_2(x_2) = x_2^{-1}(1 + x_3 + y_1 y_2)$, etc. So $x_n = x_{n-2}^{-1}(1 + x_{n-1} + y_1 y_2)$.

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Then apply μ_2, let $x_4 = \mu_2(x_2) = x_2^{-1}(1 + x_3 + y_1 y_2)$, etc. So $x_n = x_{n-2}^{-1}(1 + x_{n-1} + y_1 y_2)$.

We claim that the cluster superalgebra $\mathcal{C}_K(X|Y, Q)$ is generated by infinitely many super cluster variables.

Define a function

$$
\begin{aligned}
f: \quad \mathcal{C}_K(X|Y,Q) &\longrightarrow \mathbb{Q}[w]/w^2 \\
x_1 &\longrightarrow 1 \\
x_2 &\longrightarrow 1 \\
y_1 y_2 &\longrightarrow w
\end{aligned}
\tag{25}
$$

It turns out that $\{f(x_i)\}$ is an infinite set.

$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$
1	1	$2+w$	$3+2w$	$2+\frac{1}{2}w$
$f(x_6)$	$f(x_7)$	$f(x_8)$	$f(x_9)$	$f(x_{10})$
$1-\frac{1}{6}w$	$1+\frac{1}{6}w$	$2+\frac{3}{2}w$	$3+2w$	2
$f(x_{11})$	$f(x_{12})$	$f(x_{13})$	$f(x_{14})$	$f(x_{15})$
$1-\frac{1}{3}w$	$1+\frac{1}{3}w$	$2+2w$	$3+2w$	$2-\frac{1}{2}w$

In general,

$$\begin{array}{ccccc} f(x_{5k+1}) & f(x_{5k+2}) & f(x_{5k+3}) & f(x_{5k+4}) & f(x_{5k+5}) \\ 1-\frac{k}{6}w & 1+\frac{k}{6}w & 2+\frac{k+2}{2}w & 3+2w & 2-\frac{k-1}{2}w \end{array} \quad (27)$$

for $k = 1, 2, 3, \cdots ,$.

In general,

$f(x_{5k+1})$	$f(x_{5k+2})$	$f(x_{5k+3})$	$f(x_{5k+4})$	$f(x_{5k+5})$
$1-\frac{k}{6}w$	$1+\frac{k}{6}w$	$2+\frac{k+2}{2}w$	$3+2w$	$2-\frac{k-1}{2}w$

(27)

for $k = 1, 2, 3, \cdots,$.

This shows that there are infinitely many different values of $\{f(x_i)\}$.

In general,

$$
\begin{array}{ccccc}
f(x_{5k+1}) & f(x_{5k+2}) & f(x_{5k+3}) & f(x_{5k+4}) & f(x_{5k+5}) \\
 & & & & \\
1-\frac{k}{6}w & 1+\frac{k}{6}w & 2+\frac{k+2}{2}w & 3+2w & 2-\frac{k-1}{2}w
\end{array}
\qquad (27)
$$

for $k = 1, 2, 3, \cdots ,$.

This shows that there are infinitely many different values of $\{f(x_i)\}$.

This establishes the claim that the cluster superalgebra $\mathcal{C}_K(X|Y,Q)$ is generated by infinitely many super cluster variables.