The spectral theory of the Neumann-Poincare operator and plasmon resonance

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Neumann-Poincaré operator

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle y - x, \nu_y \rangle}{|x - y|^d} \varphi(y)\, d\sigma(y)$$

The Neumann-Poincaré (NP) operator is a boundary integral operator naturally appearing when solving the Dirichlet (or Neumann) problem using layer potentials.

- Single layer potential:

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\, d\sigma(y)\,, \quad x \in \mathbb{R}^d,$$

where $\Gamma(x)$ be the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|\,, & d = 2\,, \\ -\dfrac{1}{4\pi |x|}\,, & d = 3\,, \end{cases}$$

- Neumann boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

- Look for a solution $u(x) = \mathcal{S}_{\partial\Omega}[\varphi](x)$.
- Jump relation of single layer potential:

$$\frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\varphi]\Big|_{\pm}(x) = \left(\pm \frac{1}{2} I + \mathcal{K}^*_{\partial\Omega} \right)[\varphi](x), \quad x \in \partial\Omega \, ,$$

  where the operator $\mathcal{K}_{\partial\Omega}$ is defined by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle y - x, \nu_y \rangle}{|x - y|^d} \varphi(y) \, d\sigma(y) \, , \quad x \in \partial\Omega,$$

  and $\mathcal{K}^*_{\partial\Omega}$ is its L^2-adjoint.
- $\mathcal{K}^*_{\partial\Omega}$ (or $\mathcal{K}_{\partial\Omega}$) is called the Neumann-Poincaré (NP) operator associated with the domain Ω.
- It carries geometric information of $\partial\Omega$.

- Neumann (1887/1888), Poincaré's variational problem (1896).
- If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$, $\mathcal{K}^*_{\partial\Omega}$ is a compact operator on $L^2(\partial\Omega)$ (and on $H^{-1/2}(\partial\Omega)$) and the Fredholm theory developed (1903).
- If $\partial\Omega$ is Lipschitz, then $\mathcal{K}^*_{\partial\Omega}$ is a singular integral operator (20th Century. Marcinkiewicz, Zygmund, Calderon, Coifman, Meyer, Kenig, ...).
    - L^2-boundedness (Coifman-McIntosh-Meyer, '82)
    - Invertibility of $-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}$ (Verchota, '84)
    - High precision computation (Roklin, Greengard, Helsing)
- Spectral theory
    - Carleman (1916)
    - 1950's, Alfors, Schiffer, ...

Symmetrization of the NP operator

- Plemelj's symmetrization principle (Khavinson-Putinar-Shapiro '07)

$$\mathcal{S}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega} = \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}.$$

- New inner product on $\mathcal{H}^* := H_0^{-1/2}(\partial\Omega)$:

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi] \rangle = -\int_{\partial\Omega}\int_{\partial\Omega} \Gamma(x-y)\varphi(x)\psi(y)d\sigma(x)d\sigma(y).$$

($\mathcal{H}^*$-norm is equivalent to $H^{-1/2}$-norm (K-Kim-Lee-Shin, preprint).

- $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on $\mathcal{H}^*$

$$\langle \varphi, \mathcal{K}^*_{\partial\Omega}[\psi] \rangle_{\mathcal{H}^*} = -\langle \varphi, \mathcal{S}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega}[\psi] \rangle = -\langle \varphi, \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}[\psi] \rangle = \langle \mathcal{K}^*_{\partial\Omega}[\varphi], \psi \rangle_{\mathcal{H}^*}$$

Good news

20th century's fundamental mathematics can be utilized: Among them are

- Functional analysis
- Spectral theory self-adjoint operator and pseudo-differential operator
- Theory of singular integral operator

Spectral property of self-adjoint operator

- Spectral resolution:

$$\mathcal{K}^*_{\partial\Omega} = \int_{-1/2}^{1/2} t \, d\mathcal{E}(t).$$

- If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, then $\mathcal{K}^*_{\partial\Omega}$ is compact on $\mathcal{H}^*$. So, $\mathcal{K}^*_{\partial\Omega}$ has eigenvalues accumulating to 0, and

$$\mathcal{K}^*_{\partial\Omega} = \sum_{j=1}^{\infty} \lambda_j \varphi_j \otimes \varphi_j.$$

- Spectrum: absolutely continuous, singularly continuous, pure point spectrum

$$\sigma = \sigma_{\mathrm{ac}} \cup \sigma_{\mathrm{sc}} \cup \sigma_{\mathrm{pp}}.$$

Why spectrum?

- Plasmon resonance on meta-materials
- Cloaking by anomalous localized resonance
- Stress concentration and field enhancement

Plasmon resonance

- Dielectric constant distribution

$$\epsilon = \begin{cases} \epsilon_c + i\delta & \text{in } \Omega, \\ \epsilon_m & \text{in } \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

$\epsilon_c < 0$ (negative dielectric constant), δ: dissipation.

- Dielectric equation in quasi-static limit:

$$\begin{cases} \nabla \cdot \epsilon \nabla u = a \cdot \nabla \delta_z & \text{in } \mathbb{R}^d, \\ u(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty, \end{cases}$$

where a is a constant vector and δ_z is the Dirac mass at $z \in \mathbb{R}^d \setminus \overline{\Omega}$.

Plasmon resonance on meta-materials

- Representation of solution

$$u_\delta(x) = a \cdot \nabla_x \Gamma(x-z) + \mathcal{S}_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^d,$$

where the potential $\varphi_\delta \in \mathcal{H}_0^*(\partial\Omega)$ is the solution to

$$(\lambda I - \mathcal{K}_{\partial\Omega}^*)[\varphi_\delta] = \partial_\nu F_z \quad \text{on } \partial\Omega.$$

with

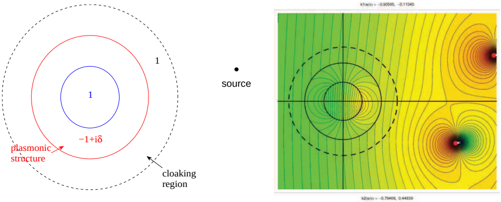
$$\lambda := \frac{\epsilon_c + \epsilon_m + i\delta}{2(\epsilon_c - \epsilon_m) + 2i\delta}.$$

- $\lambda \to \dfrac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)}$ as $\delta \to 0$.
- ϵ_c is negative if and only if $\dfrac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} \in (-1/2, 1/2)$. The NP spectrum lies in $(-1/2, 1/2)$.

Plasmon resonance

- Plasmon resonance in the quasi-static limit occurs at the NP spectrum.
- Perturbation formula for resonance frequencies on nano particles: Ammari- Millien-Ruiz-Zhang '15, Ammari-Ruiz-Yu-Zhang, '16

Cloaking by anomalous localized resonance (CALR)



- If $|z| > r^* := \sqrt{r_e^3/r_i}$, then $E(\delta) = \delta \|\nabla u_\delta\|^2_{L^2(\Omega)} \to 0$ as $\delta \to 0$.
- If $|z| < r^*$, then $E(\delta) \to \infty$ as $\delta \to 0$, and the source is cloaked (Milton-Nicorovici '06)

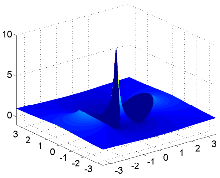
* If $|z| > r^*$, then $u_\delta \to$ the solution without the structure (the structure is cloaked).

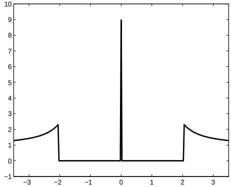
Cloaking by anomalous localized resonance (CALR)

- CALR is a resonance occurring at the limit point of NP eigenvalues (Ammari-Ciraolo-K-Lee-Milton '13)

* Variational approach: Kohn-Lu-Schweizer-Weinstein '13

Stress or field concentration





- The field concentrates in between two inclusions with extreme conductivity (0 or ∞)
- It is because more and more eigenvalues of the corresponding NP operator are getting closer to $1/2$ and $-1/2$. (Bonnetier-Triki '13, Lim-Yu '14)

NP Spectrum on smooth domains

If a boundary is smooth, then the NP operator is compact operator and has eigenvalues of finite multiplicities converging to 0

- If Ω is a disk, then 0 is the only eigenvalue of $\mathcal{K}^*_{\partial\Omega}$. Moreover, disks are the only domain on which the NP operator is of finite rank (Shapiro)
- Concentric disks (Ammari-Ciraolo-K-Lee-Milton '13)
  - Eigenvalues are

$$\pm\frac{1}{2}\left(\frac{r_i}{r_e}\right)^n, \quad n = 1, 2, \ldots$$

    and eigenfunctions are

$$\begin{bmatrix} 0 \\ e^{\pm in\theta} \end{bmatrix}, \quad \begin{bmatrix} e^{\pm in\theta} \\ 0 \end{bmatrix}$$

  - CALR occurs at 0 (accumulation point of eigenvalues).
  - By adding another circle, eigenvalues are perturbed to be non-zero and converges to 0.

- If Ω is an ellipse of the long axis a and short axis b, eigenvalues are

$$\pm\frac{1}{2}\left(\frac{a-b}{a+b}\right)^n, \quad n = 1, 2, \dots,$$

and the corresponding eigenfunctions are elliptic harmonics. (The NP eigenvalues converge to 0 exponentially fast.)

Theorem (Miyanishi-Suzuki, '16)

If Ω is a bounded domain in $\mathbb{R}^2$ with C^k boundary, then for any $\alpha > -k + 3/2$

$$\lambda_n = o(n^\alpha) \quad \text{as } n \to \infty.$$

- If $\partial\Omega$ is smooth, then $\lambda_n = o(n^{-k})$ for any positive integer k.
- What about domains with real analytic boundaries? 3D?

Theorem (Ando-K-Miyanishi, '16)

Let Ω be a bounded planar domain with the analytic boundary $\partial\Omega$ and $\epsilon_{\partial\Omega}$ be the modified maximal Grauert radius of $\partial\Omega$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the eigenvalues of the NP operator $\mathcal{K}_{\partial\Omega}^*$ on $H_0^{-1/2}(\partial\Omega)$ enumerated in descending order. For any $\epsilon < \epsilon_{\partial\Omega}$ there is a constant C such that

$$|\lambda_{2n-1}| = |\lambda_{2n}| \leq Ce^{-n\epsilon} \tag{1}$$

for any n.

- The decay rate is optimal.
- modified maximal Grauert radius of $\partial\Omega$ is determined by the maximal set to which the defining function is analytically extended.

Ball in 3D:

- The NP eigenvalues are

$$\frac{1}{2(2n+1)}, \quad n = 1, 2, \ldots,$$

  and the corresponding eigenfunctions are the spherical harmonics of order n (the multiplicities are $2n+1$).

- CALR does not occur due to slow convergence of eigenvalues: Ammari-Ciraolo-K-Lee-Milton '14, Ando-K '15
- Miyanishi-Suzuki, '16: On smooth domains in 3D,

$$\lambda_n = o(n^{-p}), \quad p < \frac{1}{2}.$$

- The optimal rate seems to be $n^{-1/2}$

NP Spectrum on domains with corners

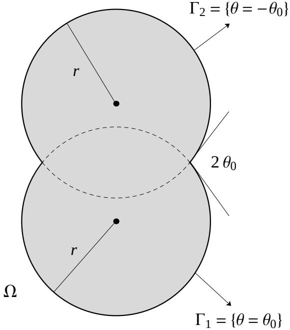
If a domain has a corner, then the NP operator is a singular integral operator and is not compact.

- Bounds on the essential spectrum of the NP operator on planar curvilinear polygonal domains are obtained (Perfekt-Putinar 14).

$$b_{ess} = \max_j \left(1 - \frac{\theta_j}{\pi} \right)$$

Essential spectrum (invariant under unitary transformations): continuous spectrum + limit points of eigenvalues.

Intersecting disks (K-Lim-Yu '15)



* Carleman's thesis (1916)
* Lei et al, ACS Nano (2011)

- Complete spectral resolution of the NP operator is derived.
- Spectrum:

$$\sigma_{ac}(\mathcal{K}^*_{\partial\Omega}) = [-\frac{1}{2} + \frac{\theta_0}{\pi}, \frac{1}{2} - \frac{\theta_0}{\pi}], \quad \sigma_{sc}(\mathcal{K}^*_{\partial\Omega}) = \emptyset, \quad \sigma_{pp}(\mathcal{K}^*_{\partial\Omega}) = \emptyset.$$

- The bound $\frac{1}{2} - \frac{\theta_0}{\pi}$ is exactly the one found by PP.
- At the continuous spectrum, resonance is stronger (or equal to) than $\delta^{-1/2}$ and weaker than δ^{-1}
- If the derivative of the spectral measure is continuous and nonzero, resonance is at the rate of $\delta^{-1/2}$.
- This is the first example of domains with corners where complete spectrum is known

Classification of spectrum by resonance

Helsing-K-Lim '16

- Aim: to show existence (or non-existence) of continuous spectrum & pure point spectrum on domains with corners.
- Idea: Different resonance occurs at continuous spectrum & pure point spectrum.
- A special technique is required for high precision computation on domains with corners.
- Resonance rate:

$$\alpha(t) = -\lim_{\delta \to 0} \frac{\log \|\varphi_{t,\delta}\|_*^2}{\log \delta}$$

where $\varphi_{t,\delta}$ is the solution to

$$\big((t + i\delta)I - \mathcal{K}^*_{\partial\Omega}\big)[\varphi_{t,\delta}] = f$$

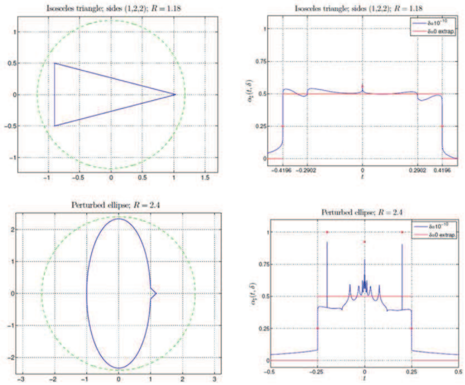
Classification of spectrum by resonance

Theorem (Helsing-K-Lim)

(i) *If* $\alpha(t) > 0$, *then* $t \in \sigma(\mathcal{K}^*_{\partial\Omega})$.

(ii) *If* $\dfrac{1}{2} \le \alpha(t) < 1$, *then* $t \in \sigma_{\mathrm{c}}(\mathcal{K}^*_{\partial\Omega})$.

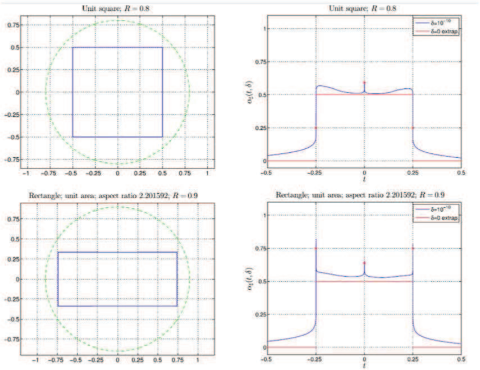
(iii) *If* $\alpha(t) = 1$ *and* t *is isolated, then* $t \in \sigma_{\mathrm{pp}}(\mathcal{K}^*_{\partial\Omega})$.



Triangle: only absolutely continuous spectrum

Perturbed ellipse: absolutely continuous + singularly continuous spectrum

All three kinds of spectra may show up!

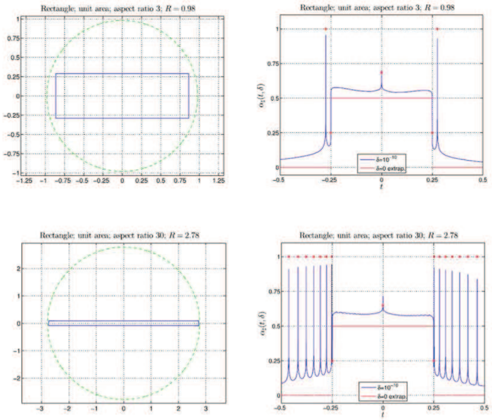


r: aspect ratio

r=1

r=r*~2.201592

If r<r*, then no eigenvalue, only continuous spectrum



r=3

r=30

If r>r*, then more and more eigenvalues appear as r increases.

- Rectangles:
  - What is the threshold $r_* = 2.201592$ of the aspect ratio?
  - What is the relation between aspect ratio and the number of eigenvalues?

Theorem (Helsing-K-Lim)

As the aspect ratio tends to ∞, the spectral bound tends to $1/2$. So eigenvalue exists.

- Essential spectrum seems to be $[-b_{ess}, b_{ess}]$.

Continuous spectrum and discrete spectrum: Converge? Some weak sense?

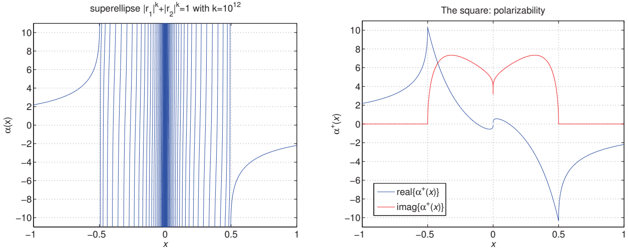


Figure: Polarizablity tensor (J. Helsing)

Theorem (Perfekt-Putinar, '16)

$$\sigma_{ess} = [-b_{ess}, b_{ess}].$$

* Bonnetier-Zhang ('17) found a new proof based on Weyl sequences of generalized eigenfunctions.

Continuous spectrum in 3D (Helsing-Putinar '17)

NP Spectrum on a surface of revolution:

continuous spectrum + eigenvalues

Elastic NP operators

The elastic NP operator is not compact even on smooth domains.

- Kelvin matrix of fundamental solutions to the Lamé operator: $\mathbf{\Gamma} = (\Gamma_{ij})_{i,j=1}^{d}$,

$$\Gamma_{ij}(\mathbf{x}) = \begin{cases} -\dfrac{\alpha_1}{4\pi}\dfrac{\delta_{ij}}{|\mathbf{x}|} - \dfrac{\alpha_2}{4\pi}\dfrac{x_i x_j}{|\mathbf{x}|^3}, & \text{if } d = 3, \\ \dfrac{\alpha_1}{2\pi}\delta_{ij}\ln|\mathbf{x}| - \dfrac{\alpha_2}{2\pi}\dfrac{x_i x_j}{|\mathbf{x}|^2}, & \text{if } d = 2, \end{cases}$$

where

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad \alpha_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right).$$

- The elastic NP operator:

$$\mathbf{K}[\mathbf{f}](\mathbf{x}) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_{\mathbf{y}}} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}) \quad \text{a.e. } \mathbf{x} \in \partial\Omega,$$

where

$$\partial_\nu \mathbf{u} := \lambda(\nabla\cdot\mathbf{u})\mathbf{n} + 2\mu(\widehat{\nabla}\mathbf{u})\mathbf{n} \quad \text{on } \partial\Omega.$$

Elastic NP operator

Polynomial compactness: Let

$$k_* := \frac{\mu}{2\mu + \lambda}.$$

- In 2D, $\mathbf{K}^2 - k_*^2 \mathbf{I}$ is compact (Ando-Ji-K-Kim-Yu, '15).
- In 3D, $\mathbf{K}^3 - k_*^2 \mathbf{K}$ is compact (Ando-K-Miyanishi, '17)

Spectral structure:

- $\mathbf{K}$ on 2D smooth domains has eigenvalues converging to $\pm k_0$.
- $\mathbf{K}$ on 3D has eigenvalues converging to 0, $\pm k_*$.

* Analysis of CALR type resonance: AJKKY '15, Li-Liu '16, Ando-K-Kim-Yu '17.
* Elastic NP eigenvalues on balls: Deng-Li-Liu '17.

Thank you!