The spectral theory of the Neumann-Poincare operator and plasmon resonance

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The Neumann-Poincaré (NP) operator is a boundary integral operator naturally appearing when solving the Dirichlet (or Neumann) problem using layer potentials.
• Single layer potential:

\[ S_{\partial \Omega}[\varphi](x) := \int_{\partial \Omega} \Gamma(x - y) \varphi(y) \, d\sigma(y) , \quad x \in \mathbb{R}^d, \]

where \( \Gamma(x) \) be the fundamental solution to the Laplacian, \( i.e. \)

\[ \Gamma(x) = \begin{cases} \frac{1}{2\pi} \ln |x| , & d = 2 , \\ -\frac{1}{4\pi|x|} , & d = 3 , \end{cases} \]

• Neumann boundary value problem:

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = g & \text{on } \partial \Omega.
\end{cases}
\]
• Look for a solution $u(x) = S_{\partial \Omega} [\varphi](x)$.

• Jump relation of single layer potential:

$$\frac{\partial}{\partial \nu} S_{\partial \Omega} [\varphi] \big|_\pm (x) = \left( \pm \frac{1}{2} I + K_{\partial \Omega}^* \right) [\varphi](x), \quad x \in \partial \Omega,$$

where the operator $K_{\partial \Omega}$ is defined by

$$K_{\partial \Omega} [\varphi](x) = \frac{1}{\omega_d} \int_{\partial \Omega} \frac{\langle y-x, \nu_y \rangle}{|x-y|^d} \varphi(y) \, d\sigma(y), \quad x \in \partial \Omega,$$

and $K_{\partial \Omega}^*$ is its $L^2$-adjoint.

• $K_{\partial \Omega}^*$ (or $K_{\partial \Omega}$) is called the Neumann-Poincaré (NP) operator associated with the domain $\Omega$.

• It carries geometric information of $\partial \Omega$. 

The spectral theory of the Neumann-Poincare operator and plasmon resonance

Hyeonbae Kang (Inha University)
• Neumann (1887/1888), Poincaré's variational problem (1896).

• If $\partial \Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$, $K^{*}_{\partial \Omega}$ is a **compact operator** on $L^{2}(\partial \Omega)$ (and on $H^{-1/2}(\partial \Omega)$) and the Fredholm theory developed (1903).

• If $\partial \Omega$ is Lipschitz, then $K^{*}_{\partial \Omega}$ is a **singular integral operator** (20th Century. Marcinkiewicz, Zygmund, Calderon, Coifman, Meyer, Kenig, ... ).
  
  • $L^{2}$-boundedness (Coifman-McIntosh-Meyer, '82)
  • Invertibility of $-\frac{1}{2}I + K^{*}_{\partial \Omega}$ (Verchota, '84)
  • High precision computation (Roklin, Greengard, Helsing)

• **Spectral theory**
  
  • Carleman (1916)
  • 1950’s, Alfors, Schiffer, ...
Symmetrization of the NP operator

- **Plemelj’s symmetrization principle** (Khavinson-Putinar-Shapiro '07)
  \[ S_{\partial\Omega}K^*_{\partial\Omega} = K_{\partial\Omega}S_{\partial\Omega}. \]

- **New inner product** on \( \mathcal{H}^* := H_0^{-1/2}(\partial\Omega) \):
  \[ \langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, S_{\partial\Omega}[\psi] \rangle = -\int_{\partial\Omega} \int_{\partial\Omega} \Gamma(x - y)\varphi(x)\psi(y)d\sigma(x)d\sigma(y). \]

  (\( \mathcal{H}^* \)-norm is equivalent to \( H^{-1/2} \)-norm (K-Kim-Lee-Shin, preprint).

- **\( K^*_{\partial\Omega} \) is self-adjoint** on \( \mathcal{H}^* \)
  \[ \langle \varphi, K^*_{\partial\Omega}[\psi] \rangle_{\mathcal{H}^*} = -\langle \varphi, S_{\partial\Omega}K^*_{\partial\Omega}[\psi] \rangle = -\langle \varphi, K_{\partial\Omega}S_{\partial\Omega}[\psi] \rangle = \langle K^*_{\partial\Omega}[\varphi], \psi \rangle_{\mathcal{H}^*} \]
Good news

20th century’s fundamental mathematics can be utilized: Among them are

- Functional analysis
- Spectral theory self-adjoint operator and pseudo-differential operator
- Theory of singular integral operator
Spectral property of self-adjoint operator

• Spectral resolution:

$$\mathcal{K}_{\partial \Omega}^* = \int_{-1/2}^{1/2} t \, d\mathcal{E}(t).$$

• If $\partial \Omega$ is $\mathcal{C}^{1,\alpha}$, then $\mathcal{K}_{\partial \Omega}^*$ is compact on $\mathcal{H}^*$. So, $\mathcal{K}_{\partial \Omega}^*$ has eigenvalues accumulating to 0, and

$$\mathcal{K}_{\partial \Omega}^* = \sum_{j=1}^{\infty} \lambda_j \varphi_j \otimes \varphi_j.$$

• Spectrum: absolutely continuous, singularly continuous, pure point spectrum

$$\sigma = \sigma_{ac} \cup \sigma_{sc} \cup \sigma_{pp}.$$
Why spectrum?

- Plasmon resonance on meta-materials
- Cloaking by anomalous localized resonance
- Stress concentration and field enhancement
Plasmon resonance

- **Dielectric constant distribution**

\[
\epsilon = \begin{cases} 
\epsilon_c + i\delta & \text{in } \Omega, \\
\epsilon_m & \text{in } \mathbb{R}^d \setminus \overline{\Omega}.
\end{cases}
\]

\(\epsilon_c < 0\) (negative dielectric constant), \(\delta\): dissipation.

- **Dielectric equation in quasi-static limit:**

\[
\begin{cases}
\nabla \cdot \epsilon \nabla u = a \cdot \nabla \delta_z & \text{in } \mathbb{R}^d, \\
u(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty,
\end{cases}
\]

where \(a\) is a constant vector and \(\delta_z\) is the Dirac mass at \(z \in \mathbb{R}^d \setminus \overline{\Omega}\).
Plasmon resonance on meta-materials

- Representation of solution

\[ u_\delta(x) = a \cdot \nabla_x \Gamma(x - z) + S_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^d, \]

where the potential \( \varphi_\delta \in H^*_0(\partial\Omega) \) is the solution to

\[ (\lambda I - \mathcal{K}_{\partial\Omega}^*)[\varphi_\delta] = \partial_\nu F_z \quad \text{on} \ \partial\Omega. \]

with

\[ \lambda := \frac{\epsilon_c + \epsilon_m + i\delta}{2(\epsilon_c - \epsilon_m) + 2i\delta}. \]

- \( \lambda \to \frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} \) as \( \delta \to 0 \).

- \( \epsilon_c \) is negative if and only if \( \frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} \in (-1/2, 1/2) \). The NP spectrum lies in \((-1/2, 1/2)\).
Plasmon resonance

• Plasmon resonance in the quasi-static limit occurs at the NP spectrum.
Cloaking by anomalous localized resonance (CALR)

- If $|z| > r^* := \sqrt{r_e^3/r_i}$, then $E(\delta) = \delta \|\nabla u_\delta\|_{L^2(\Omega)}^2 \to 0$ as $\delta \to 0$.

- If $|z| < r^*$, then $E(\delta) \to \infty$ as $\delta \to 0$, and the source is cloaked (Milton-Nicorovici '06)

* If $|z| > r^*$, then $u_\delta \to$ the solution without the structure (the structure is cloaked).
Cloaking by anomalous localized resonance (CALR)

- CALR is a resonance occurring at the limit point of NP eigenvalues (Ammari-Ciraolo-K-Lee-Milton ’13)

* Variational approach: Kohn-Lu-Schweizer-Weinstein ’13
Stress or field concentration

- The field concentrates in between two inclusions with extreme conductivity (0 or $\infty$)
- It is because more and more eigenvalues of the corresponding NP operator are getting closer to $1/2$ and $-1/2$. (Bonnetier-Triki ’13, Lim-Yu ’14)
NP Spectrum on smooth domains

If a boundary is smooth, then the NP operator is compact operator and has eigenvalues of finite multiplicities converging to 0.
• If $\Omega$ is a disk, then 0 is the only eigenvalue of $K^*_{\partial \Omega}$. Moreover, disks are the only domain on which the NP operator is of finite rank (Shapiro)

• Concentric disks (Ammari-Ciraolo-K-Lee-Milton ’13)
  
  • Eigenvalues are
  \[ \pm \frac{1}{2} \left( \frac{r_i}{r_e} \right)^n, \quad n = 1, 2, \ldots \]
  
  and eigenfunctions are
  \[ \begin{bmatrix} 0 \\ e^{\pm in\theta} \end{bmatrix}, \quad \begin{bmatrix} e^{\pm in\theta} \\ 0 \end{bmatrix} \]

• CALR occurs at 0 (accumulation point of eigenvalues).
• By adding another circle, eigenvalues are perturbed to be non-zero and converges to 0.
• If \( \Omega \) is an ellipse of the long axis \( a \) and short axis \( b \), eigenvalues are

\[
\pm \frac{1}{2} \left( \frac{a - b}{a + b} \right)^n, \quad n = 1, 2, \ldots,
\]

and the corresponding eigenfunctions are elliptic harmonics. (The NP eigenvalues converge to 0 exponentially fast.)
Theorem (Miyanishi-Suzuki, ’16)

If $\Omega$ is a bounded domain in $\mathbb{R}^2$ with $C^k$ boundary, then for any $\alpha > -k + 3/2$

$$\lambda_n = o(n^\alpha) \quad \text{as } n \to \infty.$$ 

- If $\partial \Omega$ is smooth, then $\lambda_n = o(n^{-k})$ for any positive integer $k$.
- What about domains with real analytic boundaries? 3D?
Theorem (Ando-K-Miyanishi, ’16)

Let $\Omega$ be a bounded planar domain with the analytic boundary $\partial \Omega$ and $\epsilon_{\partial \Omega}$ be the modified maximal Grauert radius of $\partial \Omega$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the eigenvalues of the NP operator $K_{\partial \Omega}^*$ on $H_0^{-1/2}(\partial \Omega)$ enumerated in descending order. For any $\epsilon < \epsilon_{\partial \Omega}$ there is a constant $C$ such that

$$|\lambda_{2n-1}| = |\lambda_{2n}| \leq Ce^{-\epsilon n}$$

for any $n$.

- The decay rate is optimal.
- modified maximal Grauert radius of $\partial \Omega$ is determined by the maximal set to which the defining function is analytically extended.
Ball in 3D:

- The NP eigenvalues are
  \[ \frac{1}{2(2n + 1)}, \quad n = 1, 2, \ldots, \]
  and the corresponding eigenfunctions are the spherical harmonics of order \( n \) (the multiplicities are \( 2n + 1 \)).

- CALR does not occur due to slow convergence of eigenvalues: Ammari-Ciraolo-K-Lee-Milton ’14, Ando-K ’15

- Miyanishi-Suzuki, ’16: On smooth domains in 3D,
  \[ \lambda_n = o(n^{-p}), \quad p < \frac{1}{2}. \]

- The optimal rate seems to be \( n^{-1/2} \)
NP Spectrum on domains with corners

If a domain has a corner, then the NP operator is a singular integral operator and is not compact.
• **Bounds on the essential spectrum** of the NP operator on planar curvilinear polygonal domains are obtained (Perfekt-Putinar 14).

\[
b_{\text{ess}} = \max_j \left( 1 - \frac{\theta_j}{\pi} \right)
\]

**Essential spectrum** (invariant under unitary transformations): continuous spectrum + limit points of eigenvalues.
Intersecting disks (K-Lim-Yu '15)

* Carleman’s thesis (1916)
* Lei et al, ACS Nano (2011)
• Complete spectral resolution of the NP operator is derived.

• **Spectrum:**

\[
\sigma_{ac}(K_{\partial \Omega}^*) = \left[ -\frac{1}{2} + \frac{\theta_0}{\pi}, \frac{1}{2} - \frac{\theta_0}{\pi} \right], \quad \sigma_{sc}(K_{\partial \Omega}^*) = \emptyset, \quad \sigma_{pp}(K_{\partial \Omega}^*) = \emptyset.
\]

• The bound \( \frac{1}{2} - \frac{\theta_0}{\pi} \) is exactly the one found by PP.

• At the continuous spectrum, resonance is stronger (or equal to) than \( \delta^{-1/2} \) and weaker than \( \delta^{-1} \).

• If the derivative of the spectral measure is continuous and nonzero, resonance is at the rate of \( \delta^{-1/2} \).

• This is the first example of domains with corners where complete spectrum is known.
Classification of spectrum by resonance

Helsing-K-Lim ’16

• Aim: to show existence (or non-existence) of continuous spectrum & pure point spectrum on domains with corners.

• Idea: Different resonance occurs at continuous spectrum & pure point spectrum.

• A special technique is required for high precision computation on domains with corners.

• Resonance rate:

\[
\alpha(t) = -\lim_{\delta \to 0} \frac{\log \| \varphi_{t,\delta} \|^2}{\log \delta}
\]

where \( \varphi_{t,\delta} \) is the solution to

\[
((t + i\delta)I - K^*_{\partial\Omega})[\varphi_{t,\delta}] = f
\]
Classification of spectrum by resonance

**Theorem (Helsing-K-Lim)**

(i) *If* $\alpha(t) > 0$, *then* $t \in \sigma(K_{\partial \Omega}^*)$.

(ii) *If* $\frac{1}{2} \leq \alpha(t) < 1$, *then* $t \in \sigma_c(K_{\partial \Omega}^*)$.

(iii) *If* $\alpha(t) = 1$ *and* $t$ *is isolated*, *then* $t \in \sigma_{pp}(K_{\partial \Omega}^*)$. 
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Triangle: only absolutely continuous spectrum

Perturbed ellipse: absolutely continuous + singularly continuous spectrum

All three kinds of spectra may show up!
r: aspect ratio

r=1

r=r^* \approx 2.201592

If r<r^*, then no eigenvalue, only continuous spectrum
If $r > r^*$, then more and more eigenvalues appear as $r$ increases.
Rectangles:
- What is the threshold $r_* = 2.201592$ of the aspect ratio?
- What is the relation between aspect ratio and the number of eigenvalues?

Theorem (Helsing-K-Lim)

*As the aspect ratio tends to $\infty$, the spectral bound tends to $1/2$. So eigenvalue exists.*

- Essential spectrum seems to be $[-b_{\text{ess}}, b_{\text{ess}}]$. 
Continuous spectrum and discrete spectrum: Converge? Some weak sense?

Figure: Polarizability tensor (J. Helsing)
### Theorem (Perfekt-Putinar, ’16)

\[
\sigma_{\text{ess}} = [-b_{\text{ess}}, b_{\text{ess}}].
\]

* Bonnetier-Zhang (’17) found a new proof based on Weyl sequences of generalized eigenfunctions.
Continuous spectrum in 3D (Helsing-Putinar ’17)

NP Spectrum on a surface of revolution:

continuous spectrum + eigenvalues
Elastic NP operators

The elastic NP operator is not compact even on smooth domains.
• Kelvin matrix of fundamental solutions to the Lamé operator:

$\Gamma = (\Gamma_{ij})_{i,j=1}^d,$

\[ \Gamma_{ij}(x) = \begin{cases} 
-\frac{\alpha_1}{4\pi} \delta_{ij} \frac{\ln |x|}{|x|^3}, & \text{if } d = 3, \\
\frac{\alpha_1}{2\pi} \delta_{ij} \ln |x| - \frac{\alpha_2}{2\pi} \frac{x_i x_j}{|x|^2}, & \text{if } d = 2,
\end{cases} \]

where

\[ \alpha_1 = \frac{1}{2} \left( \frac{1}{\mu} + \frac{1}{2\mu + \lambda} \right) \quad \text{and} \quad \alpha_2 = \frac{1}{2} \left( \frac{1}{\mu} - \frac{1}{2\mu + \lambda} \right). \]

• The elastic NP operator:

$K[f](x) := \text{p.v.} \int_{\partial \Omega} \partial_{\nu} \Gamma(x - y) f(y) d\sigma(y) \quad \text{a.e. } x \in \partial \Omega,$

where

$\partial_{\nu} u := \lambda (\nabla \cdot u) n + 2\mu (\nabla u) n \quad \text{on } \partial \Omega.$
Elastic NP operator

**Polynomial compactness:** Let

\[ k_* := \frac{\mu}{2\mu + \lambda}. \]

- In 2D, \( K^2 - k_*^2 I \) is compact (Ando-Ji-K-Kim-Yu, ’15).
- In 3D, \( K^3 - k_*^2 K \) is compact (Ando-K-Miyahashi, ’17)

**Spectral structure:**

- \( K \) on 2D smooth domains has eigenvalues converging to \( \pm k_0 \).
- \( K \) on 3D has eigenvalues converging to 0, \( \pm k_* \).

* Elastic NP eigenvalues on balls: Deng-Li-Liu ’17.
Thank you!