Pulse reflection in random waveguide with turning point

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Analysis of waveguides

Randomly perturbed waveguides with reflecting boundaries.

• In ideal waveguides (with straight walls and filled with homogeneous media) the wave equation is separable and wave field can be decomposed in 1-D waves (modes) that propagate along the waveguide axis or decay (evanescent).

Mode coupling induced by random perturbations:

Waveguides with random media Kohler, Papanicolaou; Dozier, Tappert; Marcuse 1970's. Time reversal & imaging Garnier, Papanicolaou 2007; Gomez 2011; E&M Alonso, B. 2015. Random boundaries Alonso, B., Garnier 2011; B., Garnier 2014.

• Waveguides with slow variations: Ahluwalia, Keller, Matkowsky; Anyanwu, Keller 1970's and Ting, Miksis 1980's.

• We study waveguides with slowly varying cross-section and bends, that have randomly perturbed boundary.

Goal of study

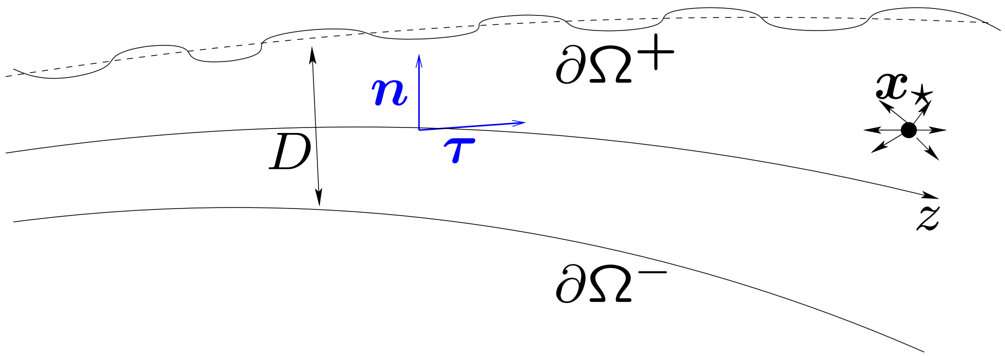
• **Motivating question:** Can we determine the waveguide geometry from data collected by sensors located at one or two ends of the region of interest?

Sensors emit probing pulses and measure the resulting wave field. We need to understand the evolution of these pulses in the waveguide. In particular, can we measure travel time?

• The most interesting aspect of this problem is pulse reflection due to narrowing of the waveguide $\rightsquigarrow$ turning waves.

We quantify the effect of random fluctuations of the boundary on the turning waves so we can understand when the reflected pulse holds together (and thus we can measure travel time) or it falls apart (and thus we cannot measure travel time).

Problem formulation in 2-D acoustic waveguide



Curvilinear coordinates

$$\boldsymbol{x}(r,z) = \boldsymbol{x}_{\|}(z) + r\boldsymbol{n}\left(\frac{z}{L}\right)$$

$$\partial_z \boldsymbol{x}_{\|}(z) = \boldsymbol{\tau}\left(\frac{z}{L}\right)$$

- Wave equation: $\left(\Delta - \frac{1}{c^2}\partial_t^2\right) p(t,r,z) = f(t)\delta(\boldsymbol{x} - \boldsymbol{x}_\star).$

- Sound soft boundaries $\rightsquigarrow$ pressure $p(t, r^{\pm}(z), z) = 0$

$$r^+(z) = \frac{D(z/L)}{2}\left[1 + 1_{(-\infty,0)}(z)\sigma\nu\left(\frac{z}{\ell}\right)\right], \quad r^-(z) = -\frac{D(z/L)}{2}.$$

- D is increasing and smooth, ν is mean zero, stationary, mixing random process, smooth and bounded a.s.

Bounded, differentiable curvature $\kappa(z/L)$ of axis, L = scale of propagation, ℓ = correlation length of fluctuations of strength σ.

Scaling regime

- To analyze the pressure field we use stochastic asymptotic analysis in the asymptotic parameter $\varepsilon = \ell/L \ll 1$.

Amplitude of fluctuations is $\sigma = \sqrt{\varepsilon}\,\sigma_\varepsilon$. Mode coupling theory is for $\sigma_\varepsilon \sim 1$ but we will see an effect even at $\sigma_\varepsilon \ll 1$.

- The pulse is $f(t) = \cos(\omega_o t) F(Bt)$ with Fourier transform

$$\widehat{f}(\omega) = \frac{1}{2B}\left[\widehat{F}\left(\frac{\omega - \omega_o}{B}\right) + \widehat{F}\left(\frac{\omega + \omega_o}{B}\right)\right], \qquad \text{supp}\,\widehat{F} = O(1).$$

- Wavelengths $\lambda = \frac{2\pi c}{\omega} \sim \ell$ and duration of $f(t)$ is $\frac{1}{B} \ll \frac{L}{c} \sim \frac{1}{\varepsilon \omega_o}$.

- Number of propagating modes is $N(z/L, \omega) = \left\lfloor \frac{2D(z/L)}{\lambda} \right\rfloor$.

Take $B \ll \omega_o$ (e.g. $B/\omega_o \sim \sqrt{\varepsilon}$) and ω_o so that

$$N(z/L, \omega) = \begin{cases} 1, & z_T(\omega) < z \leq 0 \\ 0, & z < z_T(\omega). \end{cases}$$

Random change of variables

• To satisfy the boundary conditions at all ε change variables

$$r = \rho + \frac{(2\rho + D(z/L))}{4}\sigma\nu\left(\frac{z}{\ell}\right), \quad |\rho| \leq \frac{D(z/L)}{2}, \quad \sigma = \sqrt{\varepsilon}\,\sigma_\varepsilon,$$

where in our scaling

$$\frac{z}{L} = O(1), \quad \frac{z}{\ell} = O\left(\frac{1}{\varepsilon}\right), \quad \rho \sim D \sim \ell.$$

• The homogeneous Dirichlet (sound soft) boundary conditions are at $\rho = \pm D/2$, independent of ε.

• This change of variables maps the random fluctuations to the wave operator, which will have an asymptotic expansion in ε.

The perturbed wave equation

• We obtain, frequency by frequency, for $k(\omega) = \omega/c$,

$$\left[\partial_z^2 + \frac{k^2(\omega) + \partial_\rho^2}{\varepsilon^2} + \frac{\sigma_\varepsilon \mathcal{L}_1}{\varepsilon^{3/2}} + \frac{\mathcal{L}_2}{\varepsilon} + \dots\right] \widehat{p}^\varepsilon(\omega, \rho, z) = \frac{\widehat{f}^\varepsilon(\omega)}{\varepsilon} \delta(\rho - \rho_\star)\delta(z)$$

$\mathcal{L}_1$ is differential operator with coefficients proportional to ν.

$\mathcal{L}_2 = \mathcal{L}_2^{(o)} + \sigma_\varepsilon^2 \mathcal{L}_2^{(1)}$, where $\mathcal{L}_2^{(o)}$ is differential operator with deterministic coefficients depending on curvature, and $L_2^{(1)}$ is differential operator with coefficients proportional to ν^2.

• At boundary $\widehat{p}^\varepsilon\left(\omega, \frac{\pm D(z)}{2}, z\right) = 0$ and radiation conditions.

• To see effect of the random medium, we take care of the leading deterministic part by decomposing wave in modes.

Waveguide modes

• Operator $k^2(\omega) + \partial_\rho^2$ with Dirichlet conditions at $\pm D/2$ has:

eigenfunctions $y_j(\rho, z) = \sqrt{\frac{2}{D(z)}} \sin\left[\frac{(2\rho + D(z))\pi j}{2D(z)}\right]$,

eigenvalues $k^2(\omega) - \left(\frac{\pi j}{D(z)}\right)^2$, for $j = 1, 2, \ldots$

• Mode decomposition is expansion in basis of the eigenfunctions

$$\widehat{p}^{\varepsilon}(\omega, \rho, z) = \widehat{u}^{\varepsilon}(\omega, z) y_1(\rho, z) + \sum_{j>2} \widehat{u}_j^{\varepsilon}(\omega, z) y_j(\rho, z).$$

• Modes $\widehat{u}_j^{\varepsilon}(\omega, z) e^{-i\omega t}$ are waves with wavenumbers given by square root of eigenvalues. Only one mode propagates.

• Evanescent waves have negligible effect on propagating one in our scaling. We focus attention on $\widehat{u}^{\varepsilon}(\omega, z)$.

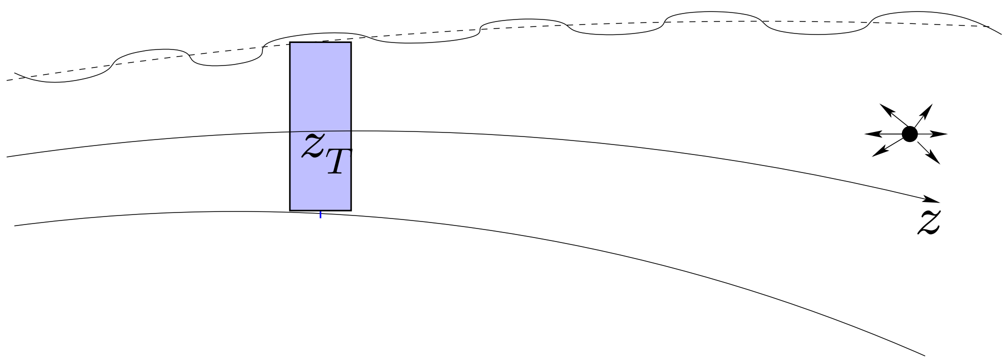
Propagating mode equation and turning point

$$\left[\partial_z^2 + \frac{1}{\varepsilon^2}\left(k^2(\omega) - \frac{\pi^2}{D^2(z)}\right) + \frac{\pi^2 \sigma_\varepsilon}{\varepsilon^{3/2} D^2(z)} \nu\left(\frac{z}{\varepsilon}\right) + \ldots\right] \widehat{u}^\varepsilon(\omega, z) = 0$$

with source conditions at $z = 0$ and radiation conditions.

• Turning point $z_T(\omega)$ where $k(\omega) = \pi / D(z_T(\omega))$.

For $z > z_T(\omega)$ propagating wave and for $z < z_T(\omega)$ evanescent.



• Strong scattering near $z_T \rightsquigarrow$ matched asymptotics is difficult so we give a global treatment.

Global turning wave decomposition

- Decomposition uses flow $\mathcal{M}^\varepsilon(\omega, z)$ that cancels leading term

$$\begin{pmatrix} \widehat{u}^\varepsilon(\omega, z) \\ -i\varepsilon\partial_z \widehat{u}^\varepsilon(\omega, z) \end{pmatrix} = \begin{pmatrix} \mathcal{M}^\varepsilon(\omega, z) & -\overline{\mathcal{M}^\varepsilon(\omega, z)} \\ -i\varepsilon\partial_z \mathcal{M}^\varepsilon(\omega, z) & i\varepsilon\partial_z \overline{\mathcal{M}^\varepsilon(\omega, z)} \end{pmatrix} \begin{pmatrix} a^\varepsilon(\omega, z) \\ b^\varepsilon(\omega, z) \end{pmatrix}$$

- Assuming a simple turning point i.e., $D'(z_T) > 0$,

$$\mathcal{M}^\varepsilon(\omega, z) = \frac{\sqrt{\pi}|\eta_\omega^\varepsilon(z)|^{\frac{1}{4}}}{\left|k^2(\omega) - \frac{\pi^2}{D^2(z)}\right|^{\frac{1}{4}}} e^{\frac{i\pi}{4} - \frac{i\phi_\omega(0)}{\varepsilon}} \left[A_i\left(-\eta_\omega^\varepsilon(z)\right) - iB_i\left(-\eta_\omega^\varepsilon(z)\right)\right].$$

where A_i and B_i are Airy functions with arguments defined by

$$\phi_\omega(z) = \int_{z_T(\omega)}^{z} ds \left[k^2(\omega) - \frac{\pi^2}{D^2(s)}\right]^{\frac{1}{2}}, \quad \eta_\omega^\varepsilon(z) = \mathrm{sign}(z - z_T)\left[\frac{3|\phi_\omega(z)|}{2\varepsilon}\right]^{\frac{2}{3}}.$$

Properties of propagator

- $\boldsymbol{M}^\varepsilon$ is invertible, with $\det \boldsymbol{M}^\varepsilon(\omega, z) = 2$, for all z.

- It ensures energy conservation $\partial_z \left[|a^\varepsilon(\omega, z)|^2 - |b^\varepsilon(\omega, z)|^2 \right] = 0$.

- For $z > z_T(\omega) + O(1)$ it gives, with $\beta(\omega, z) = \sqrt{k^2(\omega) - \pi^2/D^2(z)}$,

$$\widehat{u}^\varepsilon(\omega, z) \approx \frac{a^\varepsilon(\omega, z)}{\beta^{1/2}(\omega, z)} e^{\frac{i}{\varepsilon}\int_0^z dz' \beta(\omega, z')} - \frac{\widehat{b}^\varepsilon(\omega, z)}{\beta^{1/2}(\omega, z)} e^{-\frac{i}{\varepsilon}\int_0^z dz' \beta(\omega, z')}$$

- $\boldsymbol{M}^\varepsilon$ grows at $z < z_T(\omega)$. To counter growth set radiation cond:

$$a^\varepsilon(\omega, z_b) = i e^{2i\frac{\phi_\omega(0)}{\varepsilon}} b^\varepsilon(\omega, z_b) \quad \text{for some} \quad z_b < z_T(\omega).$$

The reflection coefficient

- Radiation condition and conservation of energy give

$$R^\varepsilon(\omega,z) = \frac{a^\varepsilon(\omega,z)}{b^\varepsilon(\omega,z)} = ie^{2i\frac{\phi_\omega(0)}{\varepsilon}+i\psi^\varepsilon_\omega(z)}, \quad \psi^\varepsilon_\omega(z) = \text{random phase}.$$

- The phase satisfies nonlinear stochastic equation

$$\partial_z \psi^\varepsilon_\omega(z) = \frac{2\pi^3 |\eta^\varepsilon_\omega(z)|^{\frac{1}{2}} \left[A_i^2(-\eta^\varepsilon_\omega(z)) + B_i^2(-\eta^\varepsilon_\omega(z))\right]}{D^2(z)\left|k^2(\omega) - \frac{\pi^2}{D^2(z)}\right|^{\frac{1}{2}}} \frac{\sigma_\varepsilon}{\sqrt{\varepsilon}}\nu\left(\frac{z}{\varepsilon}\right)$$
$$\times \cos^2\left\{\frac{\psi^\varepsilon_\omega(z)}{2} - \arg\left[A_i(-\eta^\varepsilon_\omega(z)) + iB_i(-\eta^\varepsilon_\omega(z))\right]\right\}$$

for $z > z_b$ with initial condition $\psi^\varepsilon_\omega(z_b) = 0$.

- Source gives $b^\varepsilon(\omega,0)$. Reflected wave: $a^\varepsilon(\omega,0) = R^\varepsilon(\omega,0)b^\varepsilon(\omega,0)$.

The reflection coefficient

• When $z_T - z \gg O(\varepsilon^{2/3})$, arguments of Airy functions are

$$-\eta_\omega^\varepsilon(z) = \left[\frac{3}{2\varepsilon}\int_z^{z_T} ds \sqrt{\frac{\pi^2}{D^2(s)} - k^2(\omega)}\right]^{\frac{2}{3}} \gg 1.$$

• Using asymptotic expansions of A_i and B_i we obtain

$$\partial_z \psi_\omega^\varepsilon(z) = O\left(\exp\left[-\frac{4}{3}|\eta_\omega^\varepsilon(z)|^{3/2}\right]\right) \quad \rightsquigarrow$$

starting from z_b, $\psi_\omega^\varepsilon(z) \approx 0$ until z approaches z_T from the left.

• In the remaining domain the problem is similar to a diffusion (central limit type) limit, except for coefficient that blows up like $|z - z_T|^{-1/2}$ at the turning point.

The results*

Theorem 1: $\psi_\omega^\varepsilon(0)$ is asymptotically Gaussian distributed in the limit $\varepsilon \to 0$, with mean zero and variance

$$v_\omega^2 = \frac{k^4(\omega)}{\gamma_\omega}\widehat{\mathcal{R}}(0) \lim_{\varepsilon\to 0} \sigma_\varepsilon^2 \ln\left(\frac{\phi_\omega(0)}{\varepsilon}\right), \qquad \gamma_\omega = -\partial_z \frac{\pi^2}{D^2(z)}\Big|_{z=z_T(\omega)},$$

where $\widehat{\mathcal{R}} =$ power spectral density of process ν and we recall

$$\phi_\omega(0) = \int_{z_T(\omega)}^0 dz\, \beta(\omega, z), \qquad \beta(\omega, z) = \sqrt{k^2(\omega) - \pi^2/D^2(z)}.$$

Theorem 2: Let $\psi_j^\varepsilon = \psi_{\omega_o + \sqrt{\varepsilon} w_j}^\varepsilon(0)$ since bandwidth $\sim \sqrt{\varepsilon}\omega_o$. $\mathbf{\Psi}^\varepsilon = \left(\psi_1^\varepsilon, \ldots, \psi_m^\varepsilon\right)$ converges in distribution as $\varepsilon \to 0$ to Gaussian vector with mean zero and covariance matrix

$$\mathbf{C} = \frac{v_{\omega_o}^2}{3}(\mathbf{I}_m + 2\mathbf{J}_m)$$

where $\mathbf{I}_m =$ identity, $\mathbf{J}_m =$ matrix with all entries equal to one.

*Use diffusion limit theorem due to J. H. Kim 1996.

The reflected wave

- At $z = 0$, the sensor receives the reflected wave

$$p^{\varepsilon}_{ref}(t, \rho, 0-) \approx \frac{y_1(\rho, 0)}{\sqrt{\beta(\omega_o, 0)}} \int \frac{d\omega}{2\pi} e^{-i\omega t} b^{\varepsilon}(\omega, 0)\, ie^{\frac{2i}{\varepsilon}\phi_\omega(0) + i\psi^{\varepsilon}_{\omega}(0)}$$

with

$$b^{\varepsilon}(\omega, 0) = \frac{iy_1(\rho_\star, 0)}{4\sqrt{\varepsilon\beta(\omega_o, 0)}\, B}\Big[\widehat{F}\Big(\frac{\omega - \omega_o}{\sqrt{\varepsilon}B}\Big) + \overline{\widehat{F}\Big(\frac{-\omega - \omega_o}{\sqrt{\varepsilon}B}\Big)}\Big]$$

- Let $\omega = \omega_o + \sqrt{\varepsilon} w$ and expand

$$\frac{2}{\varepsilon}\phi_{\omega_o + \sqrt{\varepsilon}w}(0) = \frac{2}{\varepsilon}\phi_{\omega_o}(0) + \frac{w}{\sqrt{\varepsilon}} T_{\omega_o} + w^2 \theta_{\omega_o} + \dots$$

- Observe p^{ε}_{ref} near travel time $T_{\omega_o} = \frac{2}{c}\int_{z_T(\omega_o)}^{0} dz\, \frac{k(\omega_o)}{\beta(\omega_o, z)}$ in time window $t^{\varepsilon} = T_{\omega_o} + \sqrt{\varepsilon} t$.

Theorem 3 The reflected wave at t^ε is, up to multiplicative constant independent of ε,

$$p^\varepsilon_{ref}(t^\varepsilon, \rho, 0-) \sim y_1(\rho, 0)\mathrm{Re}\Big[e^{\frac{i[2\phi_{\omega_o}(0)-\omega_o t^\varepsilon]}{\varepsilon}} \mathcal{F}^\varepsilon_{ref}(t)\Big].$$

It oscillates at frequency ω_o/ε as emitted pulse, with envelope

$$\mathcal{F}^\varepsilon_{\mathrm{ref}}(t) = \int \frac{dw}{2\pi B}\widehat{F}\Big(\frac{w}{B}\Big)e^{iw^2\theta_{\omega_o}+i\psi^\varepsilon_{\omega_o+\sqrt{\varepsilon}w}(0)-iwt}.$$

As $\varepsilon \to 0$, $\mathcal{F}^\varepsilon_{\mathrm{ref}}(t)$ converges in distribution, in space of continuous functions on compact sets in $\mathbb{R}$, to

$$\mathcal{F}_{\mathrm{ref}}(t) = e^{i\varphi - \frac{\upsilon^2_{\omega_o}}{6}} \int \frac{dw}{2\pi B}\widehat{F}\Big(\frac{w}{B}\Big)e^{iw^2\theta_{\omega_o}-iwt}, \quad \varphi \sim \mathcal{N}\left(0, 2\upsilon^2_{\omega_o}/3\right).$$

Sketch of proof

- $\forall\, t_1, \dots, t_m$ calculate finite-order moments

$$\mathbb{E}\left[\prod_{j=1}^{m} \mathcal{F}^{\varepsilon}_{\text{ref}}(t_j)\right] = \int \frac{dw_1}{2\pi B}\widehat{F}\left(\frac{w_1}{B}\right) \cdots \int \frac{dw_m}{2\pi B}\widehat{F}\left(\frac{w_m}{B}\right) e^{i\sum_{j=1}^{m}\left(w_j^2\theta_{\omega_o} - w_j t_j\right)}$$
$$\times\, \mathbb{E}\left[e^{i\sum_{j=1}^{m}\psi_j^{\varepsilon}}\right]$$

- By Theorem 2 $\quad \lim_{\varepsilon\to 0} \mathbb{E}\left[e^{i\sum_{j=1}^{m}\psi_j^{\varepsilon}}\right] = e^{-\frac{m(2m+1)\upsilon_{\omega_o}^2}{6}} = \mathbb{E}\left[\prod_{j=1}^{m} e^{-\frac{\upsilon_{\omega_o}^2}{6} + i\varphi}\right]$

- We obtain $\quad \lim_{\varepsilon\to 0} \mathbb{E}\left[\prod_{j=1}^{m} \mathcal{F}^{\varepsilon}_{\text{ref}}(t_j)\right] = \mathbb{E}\left[\prod_{j=1}^{m} \mathcal{F}_{\text{ref}}(t_j)\right]$

- Moreover $\sup_{|t'-t|\leq \Delta_t} \left|\mathcal{F}^{\varepsilon}_{\text{ref}}(t') - \mathcal{F}^{\varepsilon}_{\text{ref}}(t)\right| \leq C\Delta_t \rightsquigarrow$ tightness of laws

of $\{\mathcal{F}^{\varepsilon}_{\text{ref}}(t)\}$ for t in compact set in $\mathbb{R}$.

Conclusions

• Pulse stabilization result in waveguide displays strong interaction of the turning wave with the random boundary.

• Travel time and dispersion deformation of reflected wave $\rightsquigarrow$ tomographic info about $D(z)$. For inversion we need many propagating modes. In our scaling this makes no difference as mode coupling is negligible.

• If we did not have the turning point, the waves would see no effect at $L = \ell/\varepsilon$, for random fluctuations of amplitude $\sigma = \sqrt{\varepsilon}\sigma_\varepsilon$, with $\sigma_\varepsilon \sim |1/\ln(\varepsilon)|^{1/2}$.

• At $\sigma \sim \sqrt{\varepsilon}$ the modes are coupled by the random fluctuations.

We analyzed this case (with Garnier and Wood) for time harmonic waves. Again, the turning wave picks up a random phase in vicinity of z_T, but there is no pulse stabilization in this case.

• Non-monotone $D(z) \rightsquigarrow$ trapped modes. This is difficult.