Mathematics of super-resolution in resonant media

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Our campus: HKUST
Outline

Background of super-resolution

Super-resolution by using Helmholtz resonators

Super-resolution by using plasmonic particles

Super-resolution by using high refractive index media

Super-resolution in bubbly media
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Background of super-resolution

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Super-resolution in bubbly media
Resolution in microscopy

1. A point source produces diffract pattern on the imaging plane;
2. Diffract pattern = bright spot in the center (airy disc) + rings of much weaker brightness;
3. Resolution: minimum distance between two sources that can be resolved by a microscopy;
4. Resolution \( \approx \) the diameter of the airy disc.
Resolution limit: Abbe’s diffraction limit

1. **Airy disc** is formed by focusing light from the far field onto the imaging plane;

2. **Ernst Abbe (1873):** there is a fundamental limit on the size of the focused spot. The limit is given by $\frac{\lambda}{2}$ where $\lambda$ is the wavelength of the wave emitted from the source;

3. Resolution limit $= \text{Abbe's diffraction limit} = \frac{\lambda}{2}$. 
Resolution limit in inverse source problems

1. Field produced by point sources = **propagating modes** + **evanescent modes**;
   - Propagating modes can propagate into the far field with information on the scale greater than $\lambda$;
   - Evanescent modes are confined near the sources with information on the scale smaller than $\lambda$;

2. Resolution is **limited by $\lambda$**.

3. **Question**: $\lambda$ is well-defined in homogeneous space. What if we are in an inhomogeneous space?
Inverse source problem in inhomogeneous media

1. $G(\cdot, \cdot, k)$: the Green's function for the media with refractive index $n(x)$ at frequency $k$;

2. $u(x) := \mathcal{K}_D[f](x) := \int_D G(x, y, k)f(y) \, dy$;

3. **Inverse source problem**: reconstruct $f$ from the far field data $u$. 

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[Diagram showing unknown sources and known media with $n(x) = 1$]
Methods of solving the inverse source problem

1. Time reversal method (phase conjugate method),

\[ l(x) = K^*_D K_D[f](x). \]

2. \( L^2 \)-norm Minimization method,

\[ \min \| f \|_{L^2(D)} \text{ subject to } K_D[f] = u. \]

3. \( L^1 \)-norm Minimization method,

\[ \min \| f \|_{L^1(D)} \text{ subject to } K_D[f] = u. \]

Remark

The time-reversal method is a direct imaging method without using any a priori information about the sources.
Resolution limit in the inhomogeneous space

1. Time reversal corresponds to the **focusing** in the microscopy.

2. The **imaging functional** is

\[
I(x) = K_D^* K_D[f](x) \approx -\frac{1}{k} \int_D \Im G(x, y, k)f(y) \, dy;
\]

3. The **point spread function** is given by \( \Im G(x, x, k) \).

4. The resolution of TRM is determined by \( \Im G(x, x, k) \); This is also the **resolution limit** for the inverse source problem without any a priori information.
Super-resolution techniques

1. Near-field scanning optical microscopy, E. Synge in 1928, and J. Keefe in 1956;


3. Structured illumination,
   - Nonlinear structured illumination microscopy (Gustafsson, 2005, P.N.A.S);
Super-resolution techniques: meta-materials


2. **Resonant media (consists of sub-wavelength resonators)**: Fink, et al., 2007, Nature; 2011, P.R.L.
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Helmholtz resonators

1. **Helmholtz resonator**: acoustic cavity, large volume with a small opening;

2. **Main property**: scattered field is enhanced when the frequency of incident wave is close to its resonances.
Helmholtz resonator

1. **Helmholtz (1850s)** used the “Helmholtz resonators” to tune musical instruments;

2. **Rayleigh (1916)** calculated the resonance frequency (in the quasi-stationary regime) for spherical resonators;

3. **Miles (1971)** studied the resonances numerically;

4. **Gadyl’shin (1997)** obtained the asymptotic for all resonances by the matched asymptotic method (complicated for multiple coupled resonators; does not give asymptotic for the Green’s function).
The resonances for a single resonator

1. $k$: the resonance; a complex number;
2. $G_\epsilon$: the resonant mode;
3. $\Omega_\epsilon$: the half space + the space occupied by the resonator;
4. $\Lambda_\epsilon$: the opening of the resonator; a surface patch (such as a disc); of size $\epsilon$ which allows for asymptotic analysis.

$$\Delta G_\epsilon(x) + k^2 G_\epsilon(x) = 0$$
Resonances in the quasi-static regime

Theorem

(Ammari and Zhang, Comm. Math. Physic, 2015) There exist exactly two resonances for a single Helmholtz resonator in the quasi-static regime:

\begin{align*}
k_{0,\epsilon,1} &= \sqrt{\frac{1}{|D|}} \epsilon c_\Lambda - \frac{1}{2} \alpha_0 \sqrt{\frac{c_\Lambda}{|D|}} c_\Lambda \epsilon^{\frac{3}{2}} - \frac{1}{2} \alpha_1 \frac{c^2_\Lambda}{|D|} \epsilon^2 + O(\epsilon^{\frac{5}{2}}), \\
k_{0,\epsilon,2} &= -\sqrt{\frac{1}{|D|}} \epsilon c_\Lambda + \frac{1}{2} \alpha_0 \sqrt{\frac{c_\Lambda}{|D|}} c_\Lambda \epsilon^{\frac{3}{2}} - \frac{1}{2} \alpha_1 \frac{c_\Lambda^2}{|D|} \epsilon^2 + O(\epsilon^{\frac{5}{2}}).
\end{align*}
The super-focusing experiment

1. **Main reference:** F. Lemoult, M. Fink and G. Lerosey, P.R.L(2011).
2. Coke cans act as *sub-wavelength resonators*;
3. Super-focusing $\Rightarrow$ super-resolution.
A simplified model of the experiment

1. $\Omega_\epsilon$: the upper half space + spaces occupied by the cavities;
2. Signals are recorded and re-emitted over a half sphere in the far-field;

Key: asymptotic of the Green’s function $G_\epsilon(x, y, k)$ for the Helmholtz equation in $\Omega_\epsilon$ near the quasi-static resonant frequency.
The equations and the focused fields

The physics process can be modeled by the following time domain scattering problem

\[
\begin{align*}
  u_{tt}(x, t) - \Delta u(x, t) &= \delta(x - x_0)f(t), \quad (x, t) \in \Omega_\epsilon \times (0, \infty), \\
  u(x, t) &= 0, \quad x \in \Omega_\epsilon, t < 0, \\
  \frac{\partial u}{\partial \nu}(x, t) &= 0, \quad (x, t) \in \partial \Omega_\epsilon \times (0, \infty).
\end{align*}
\]

The time-reversal focused field has the following form

\[
I(x, x_0) = -\frac{2}{\pi} \int_{0}^{\infty} \Im G_\epsilon(x, x_0, k)\Im(\hat{f}(k)) \, dk + \text{remaining term}.
\]
Main result on super-focusing

Theorem

(Ammari and Zhang, Comm. Math. Physic, 2015) Super-focusing can be achieved if the signal \( f = F(\epsilon^{1/2} t) \) is quasi-static. Moreover, the focused field has the following asymptotic,

\[
I(x) = I(x, x_0) = \int_0^{2\tau_1\epsilon^{1/2}} \frac{\sin \frac{k|x - x_0|}{2\pi|x - x_0|}}{\epsilon} \Im(\hat{F}(\epsilon^{-1/2} k)) \, dk
\]

\[
+ \left( c_\Lambda \right)^{3/2} \frac{\epsilon^{3/2}}{\sqrt{|D|}} \Im(\hat{F}(\tau_1)) \sum_{j=1}^{M} \frac{1}{4\pi|x - z(j)| \cdot |x_0 - z(j)|} + o(\epsilon^{3/2})
\]
Super-resolution by using Helmholtz resonators

Plot of the focused field due to resonators

Super-focusing can indeed be achieved: focused spot has size of order 1; the normal focused spot has size of order \( \frac{1}{\sqrt{\epsilon}} \).

1. \( z^j \)’s: unit grid points on \([0, 6] \times [0, 6] \times 0\);
2. \( x_0 = (2, 2, 0.2) \), \( x \in [0, 6] \times [0, 6] \times 0.2 \), \( z = I(x, x_0) \).
1. **Super-focusing** at be achieved at multiple places;
2. **Super-resolution** (two spots $\Rightarrow$ two airy discs of the image of two sources).

\[ x_0 = (2, 2, 0.2), \quad x'_0 = (2, 4, 0.2), \quad x \in [0, 6] \times [0, 6] \times 0.2, \]
\[ z = I(x, x_0) + I(x, x'_0). \]
Outline

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Super-resolution by using Helmholtz resonators

**Super-resolution by using plasmonic particles**

Super-resolution by using high refractive index media

Super-resolution in bubbly media
Plasmonic particles

1. **Metallic** particles of size ranging from several nm’s to hundreds of nm’s;
2. The electron density may be strongly coupled to EM fields with certain frequencies in the visible and near-infrared regime; This results strong scattering and enhancement of local field;
3. This phenomenon is called **surface plasmon resonance**;
4. Plasmonic particles are **subwavelength resonators** for EM fields.
Super-resolution by using a cluster of plasmonic particles

Super-resolution by using plasmonic particles

Theorem
(with Ammari, Millien, Ruiz, 2017, ARMA) In the presence of $L$ weakly coupled plasmonic particles, the Green's function has the following asymptotic

$$
\Im \Gamma(x, x_0, k_m) \approx \Im G(x, x_0, k_m) + \\
\sum_{j \in J} \sum_{l=1}^{L} \Re \left( H_{j,p}(x_0) \tilde{X}_{j,l,p} X_{j,l,q} S_{j,q}(x, 0) \right) \cdot \Im \left( \frac{1}{\lambda - \lambda_j + \left( \frac{1}{\epsilon_c} - \frac{1}{\epsilon_m} \right)^{-1} \tau_{j,l}} \right).
$$
Super-resolution by using high refractive index media

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Super-resolution by high refractive index media

1. **Goal:** resolve densely packed point sources ($d \ll \lambda$). Impossible in the free space;
2. **Method:** surround the sources by high refractive index media;
3. In **high refractive index** media, waves are compressed and have much smaller wavelength.
Super-resolution by using high refractive index media

The Green’s function in high refractive index media

The Green’s function $G$ in high refractive index media (after re-scaling) satisfies the equation

$$\Delta_x G(x, x_0) + G(x, x_0) + \tau \chi_D(x) G(x, x_0) = \delta(x - x_0) \quad \text{in } \mathbb{R}^d,$$

where $\tau \gg 1$ is the contrast.
Super-resolution by using high refractive index media

Sub-wavelength resonant modes

Definition

\( u \in H^2_{\text{loc}}(\mathbb{R}^d) \) is called a resonant mode if there exists \( \lambda \in \mathbb{C} \) such that

\[
\begin{align*}
(\Delta + 1)u(x) &= \frac{1}{\lambda} u(x) \quad \text{in} \ D, \\
(\Delta + 1)u &= 0 \quad \text{in} \ \mathbb{R}^d \setminus D,
\end{align*}
\]

\( u \) satisfies the Sommerfeld radiation condition.

\( \lambda \) is called the corresponding resonance. Sub-wavelength resonant modes are those corresponds to small \( \lambda \).

Remark: resonant modes can propagate into the far-field.
Theorem

(Ammari and Zhang, Pro. Royal. Soc. A, 2015) The following resonant expansion holds for the Green’s function

\[ G(x, x_0) = G_0(x, x_0) + \sum_{\gamma' \leq \gamma} \sum_{\gamma} \beta_{\gamma, \gamma'} u_\gamma(x) \overline{u_{\gamma'}(x_0)}, \]

where each \( \beta_{\gamma, \gamma'} \) is a linear combination of terms in the form \( \frac{1}{T - \lambda_\gamma} \).
Resonances in high contrast media and the associated Green’s function

1. The domain is a disk with radius two;
2. The wave number outside the disk is set to be one.
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Super-resolution in bubbly media
Bubbles

1. $\rho_b$: density of the air in the bubble; $\rho$: density of water;
2. $k_b$: wavenumber in the bubble; $k$: wavenumber in the water.
3. $\delta = \frac{\rho_b}{\rho}$, $\tau = \frac{k_b}{k} = \frac{v_b}{v} = \sqrt{\frac{\rho_b \kappa}{\rho \kappa_b}}$.

We assume that $\delta \ll 1$ and $\tau = O(1)$
Minnaert resonance

1. **Quasi-static** resonance;
2. First observed by M. Minnaert in 1993;
3. First analytic formula is derived by Minnaert based on mechanic oscillator model, assuming that the bubble is spherical.
Scattering by a single bubble

The scattering by a single bubble $D$ can be modeled by the following equations:

\[
\begin{aligned}
\nabla \cdot \frac{1}{\rho} \nabla u + \frac{\omega^2}{\kappa} u &= 0 \quad \text{in } \mathbb{R}^3 \setminus D, \\
\nabla \cdot \frac{1}{\rho_b} \nabla u + \frac{\omega^2}{\kappa_b} u &= 0 \quad \text{in } D, \\
u_+ - u_- &= 0 \quad \text{on } \partial D, \\
\left. \frac{1}{\rho} \frac{\partial u}{\partial \nu} \right|_+ - \left. \frac{1}{\rho_b} \frac{\partial u}{\partial \nu} \right|_- &= 0 \quad \text{on } \partial D, \\
u^s := u - u^i \text{ satisfies the Sommerfeld radiation condition}
\end{aligned}
\]
Minnaert resonance frequency

Theorem
(with Ammari, et.al, 2016) In the quasi-static regime, there exists two resonances for a single bubble:

\[
\begin{align*}
\omega_{0,0}(\delta) &= \sqrt{\frac{\text{Cap}(D)}{\tau^2 v^2 \text{Vol}(D)}} \delta^{\frac{1}{2}} - i \frac{\text{Cap}(D)^2}{8 \pi \tau^2 v \text{Vol}(D)} \delta + O(\delta^{\frac{3}{2}}), \\
\omega_{0,1}(\delta) &= -\sqrt{\frac{\text{Cap}(D)}{\tau^2 v^2 \text{Vol}(D)}} \delta^{\frac{1}{2}} - i \frac{\text{Cap}(D)^2}{8 \pi \tau^2 v \text{Vol}(D)} \delta + O(\delta^{\frac{3}{2}}).
\end{align*}
\]

where \( \text{Vol}(D) \) is the volume of \( D \) and \( \text{Cap}(D) \) is the capacity of \( D \). The resonance \( \omega_{0,0} \) is called the Minnaert resonance.
Monopole approximation for a single bubble

Theorem
(with Ammari, et. al, 2016) In the Minnaert resonant regime, \( \omega = O\left(\frac{\sqrt{\delta}}{s}\right) \), the scattered field in the far field has the following point-wise behaviour

\[
u^s(x) = g(\omega, \delta, D) \left(1 + O(\omega) + O(\delta) + o(1)\right) G(x, y_0, k),
\]

where the scattering coefficient \( g \) is given by:

\[
g(\omega, \delta, D) = \frac{\text{Cap}(D)}{1 - \left(\frac{\omega_M}{\omega}\right)^2 + i\gamma}.
\]
Super focusing in bubbly medium

Scattering in bubbly medium

Let $D^N := \bigcup_{1 \leq j \leq N} D_j^N \subset \Omega$, where $D_j^N = y_j^N + sB$ for $1 \leq j \leq N$. Consider the scattering by $N$-bubbles:

\[
\begin{cases}
\nabla \cdot \left( \frac{1}{\rho} \nabla u^N \right) + \frac{\omega^2}{\kappa} u^N = 0 & \text{in } \mathbb{R}^3 \setminus D^N, \\
\nabla \cdot \left( \frac{1}{\rho_b} \nabla u^N \right) + \frac{\omega^2}{\kappa_b} u^N = 0 & \text{in } D^N, \\
u_+^N - u_-^N = 0 & \text{on } \partial D^N, \\
\frac{1}{\rho} \frac{\partial u^N}{\partial \nu} \bigg|_+ - \frac{1}{\rho_b} \frac{\partial u^N}{\partial \nu} \bigg|_- = 0 & \text{on } \partial D^N, \\
u^N - u^i & \text{satisfies the Sommerfeld radiation condition,}
\end{cases}
\]
Literature on the effective media theory of bubbles

1. L. L. Foldy, 1945, point interaction approximation, statistic average over all bubble configurations;
Normalized bubble density

We assume that there exists $\tilde{V} \in L^\infty(\Omega)$ such that

$$\Theta^N(A) \to \int_A \tilde{V}(x)dx, \text{ as } N \to \infty,$$

for any measurable subset $A \subset \mathbb{R}^3$, where $\Theta^N(A)$ is defined by

$$\Theta^N(A) = \frac{1}{N} \times \{\text{number of points } y_j^N \text{ in } A \subset \mathbb{R}^3\}.$$
Assumptions on bubble distribution

Assumption

The following conditions hold:

\[
\begin{cases}
\min_{i \neq j} |y_i^N - y_j^N| \geq r_N, \\
s \ll r_N,
\end{cases}
\]

where \( r_N = \eta N^{-\frac{1}{3}} \) for some constant \( \eta \).

Assumption

For any \( f \in C^{0, \alpha}(\Omega) \) with \( 0 < \alpha \leq 1 \),

\[
\max_{1 \leq j \leq N} \left| \frac{1}{N} \sum_{i \neq j} G(y_j^N, y_j^N, k) f(y_j^N) - \int_{\Omega} G(y_j^N, y, k) \tilde{V}(y) f(y) dy \right| \lesssim \frac{1}{N^\frac{\alpha}{3}} \| f \|_{C^{0, \alpha}}
\]
Assumptions on frequency and bubble volume fraction

Assumption

(on frequency) \( \omega_M = O(\frac{\sqrt{\delta}}{s}) = O(1) \) and is independent of \( N \). Moreover,

\[
1 - \left( \frac{\omega_M}{\omega} \right)^2 = \beta_0 s^{\epsilon_1}
\]

for some fixed \( 0 < \epsilon_1 < 1 \) and constant \( \beta_0 \).

Assumption

(on volume fraction)

\[
s^{1-\epsilon_1} \cdot N = \Lambda,
\]

where \( \Lambda \) is a constant independent of \( N \). Moreover, we will assume that \( \Lambda \) is large.
Effective medium theory for bubbly medium

Let

\[ V(x) = \frac{Cap(B)}{\beta_0} \cdot \Lambda \cdot \tilde{V}(x). \]

**Theorem**  
*(Ammari, Zhang, in revision)* Let \( \omega < \omega_M \). Then under Assumptions, the micro wave field converges to the solution to the Helmholtz equation

\[ (\triangle + k^2 - V)\psi = 0 \]

together with the radiation condition imposed on \( \psi - u^i \) at infinity, in the sense that for \( x \in Y_{\epsilon_2}^N \), the following estimate holds uniformly:

\[ |u^N(x) - \psi(x)| \lesssim N^{-\min\{\frac{1-\epsilon_0}{6}, \frac{1-\epsilon_2}{3}, \epsilon_2, \frac{\epsilon_0}{3} - \frac{\epsilon_1}{1-\epsilon_1}\}}. \]
Effective medium theory for bubbly medium

**Theorem**

*(Ammari, Zhang, in revision)* Let $\omega > \omega_M$. Then under Assumptions the solution to the scattering problem converges to the solution to the following **diffusive** equation

$$(\Delta + k^2 - V)\psi = 0$$

**together with the radiation condition imposed on** $\psi - u^i$ **at infinity, in the sense that for** $x \in \mathcal{Y}_{\epsilon_2}$, **the following estimate holds uniformly:**

$$|u^N(x) - \psi(x)| \lesssim N^{-\min\left\{\frac{1-\epsilon_0}{6}, \frac{1-\epsilon_2}{3}, \frac{\epsilon_2}{3} - \frac{\epsilon_1}{1-\epsilon_1}\right\}}.$$
Several Remarks at $\omega = \omega_M$

1. At resonant frequency $\omega = \omega_M$, effect media theory does not hold;
2. For subwavelength periodic bubble structures (bubble metamaterial), there is a band gap in the quasi-static regime (appear soon).
Main idea of proof: point interaction approximation

Proposition

Under Assumptions, the following relation between $u_{j, N}^s$ and $u_{j, N}^i$ holds for all $x$ such that $|x - y_{j, N}^N| \gg s$:

$$u_{j, N}^s(x) = G(x, y_0, k) \cdot g \cdot \left( u_{j, N}^i(y_{j, N}^N) + O[N^{\frac{\epsilon_0}{3} - \frac{\epsilon_1}{1-\epsilon_1}} + \frac{s}{|x - y_{j, N}^N|}] \cdot \max_{1 \leq l \leq N} |u_{j, N}^i(y_{j, N}^N)| \right).$$
Conclusion

1. For far field imaging problems in a general media, the resolution limit is defined by the imaginary part of the associated Green’s function.

2. For media with small number of resonators, super-resolution can be explained by propagating sub-wavelength resonant modes;

3. For media with large number of resonators, super-resolution can be explained by effective high refractive index media theory.

4. The effective media is very dispersive near resonant frequency.

Thank You!