Inverse Source Problems for Wave Propagation

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Joint Work with

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Outline

- Motivation and model problems
- Inverse random source
- Increasing stability
- Ongoing and future work

Source scattering problems are concerned with the relationship between radiating sources and wave fields.

- Direct problem: To determine the wave field from the given source and the differential equation governing the wave motion.

- Inverse problem: To determine the radiating source which produces the measured wave field.

- The Helmholtz equation - acoustic wave

$$\Delta u + \kappa^2 u = f \quad \text{in } \mathbb{R}^d.$$

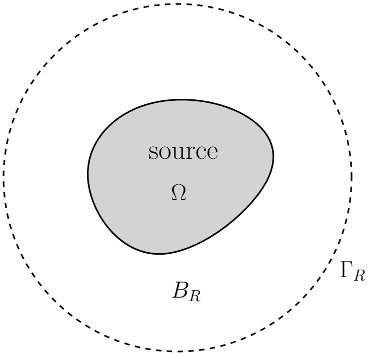
- The Navier equation - elastic wave

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^d.$$

- The Maxwell equations - electromagnetic wave

$$\nabla \times \boldsymbol{E} - \mathrm{i}\omega\mu \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} + \mathrm{i}\omega\varepsilon \boldsymbol{E} = \boldsymbol{J} \quad \text{in } \mathbb{R}^3.$$

Problem geometry



Applications

- Antenna synthesis
- Tomography (PAT)
- Medical imaging (MEG, EEG, ENG)
- Fluorescence microscopy
- Neuroscience (brain imaging)

- Deterministic problems

  Devaney and Sherman ('82), Marengo and Devaney ('99), Ammari, Bao, and Fleming ('02), Fokas, Kurylev, and Marinakis ('04), Hauer, Kühn, Potthast ('05), Albanese and Monk ('06), Devaney, Marengo, and Li ('07), Eller and Valdivia ('09), Bao, Lin, and Triki ('10), Badia and Nara ('11), Tittelfitz ('15), Bao, Lu, Rundell, and Xu ('15), Zhang and Guo ('15), Bao, L., Lin, and Triki ('15), Cheng, Isakov, and Lu ('16)

- Stochastic problems

  Devaney ('79), L. ('11), Bao and Xu ('13), Bao, Chow, L., and Zhou ('14), Bao, Chen, and L. ('16)

- Homogeneous media
- Inhomogeneous media

Uncertainties are widely introduced to the mathematical models for three major reasons:

- Randomness directly appears in the studied systems
- Incomplete knowledge of the systems may be modeled by uncertainties
- Stochastic techniques are introduced to couple the interference between different scales

- Homogenization theory: J. Keller, J. Lions, G. Papanicolaou
- Bayesian statistics: D. Donoho, A. Stuart, E. Somersalo
- Wiener chaos expansion: R. Cameron, W. Martin, G. Karniadakis

The Helmholtz equation

$$\Delta u(x,\kappa) + \kappa^2 u(x,\kappa) = f(x), \quad x \in \mathbb{R}^2.$$

The source function

$$f(x) = g(x) + \sigma(x)\dot{W}_x.$$

The Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}(\partial_r u - \mathrm{i}\kappa u) = 0, \quad r = |x|.$$

The direct problem: Given g and σ, to determine the random wave field u.

The inverse problem: To recover g and σ^2 from $u|_{\Gamma_R}$ at $\kappa_j, j = 1, \ldots, m$.

W_x is the 2-parameter Brownian motion on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu)$.

White noise

$$\dot{W}_x := \frac{\partial^2 W_x}{\partial x_1 \partial x_2}.$$

Stochastic integral

$$\int_{\mathbb{R}^2} \phi(x) \mathrm{d}W_x = \int_{\mathbb{R}^2} W_x \frac{\partial^2 \phi(x)}{\partial x_1 \partial x_2} \mathrm{d}x.$$

Proposition

$$\mathbf{E} \int_{\mathbb{R}^2} \phi(x) \mathrm{d}W_x = 0, \quad \mathbf{E} \left| \int_{\mathbb{R}^2} \phi(x) \mathrm{d}W_x \right|^2 = \int_{\mathbb{R}^2} |\phi(x)|^2 \mathrm{d}x.$$

Consider the deterministic scattering problem

$$\begin{cases} \Delta u + \kappa^2 u = g, & x \in \mathbb{R}^2, \\ \partial_r u - \mathrm{i}\kappa u = o(r^{-1/2}), & r \to \infty. \end{cases}$$

Given $g \in L^2(\Omega)$, it has a unique solution

$$u(x) = \int_\Omega G(x, y) g(y) \mathrm{d}y,$$

where the Green function

$$G(x, y) = -\frac{\mathrm{i}}{4} H_0^{(1)}(\kappa |x - y|).$$

Consider the stochastic scattering problem

$$\begin{cases} \Delta u + \kappa^2 u = g + \sigma \dot{W}_x, & x \in \mathbb{R}^2, \\ \partial_r u - \mathrm{i}\kappa u = o(r^{-1/2}), & r \to \infty. \end{cases}$$

Theorem (Bao-Chen-L)

There exists a unique continuous stochastic process (mild solution) u, which satisfies

$$u(x) = \int_\Omega G(x,y) g(y) \mathrm{d}y + \int_\Omega G(x,y)\sigma(y) \mathrm{d}W_y.$$

- $g \in L^2(\Omega)$.
- σ is chosen such that the stochastic integral

$$\int_\Omega G(x, y, \kappa)\sigma(y)\mathrm{d}W_y$$

satisfies

$$\mathbf{E}\Big|\int_\Omega G(x, y, \kappa)\sigma(y)\mathrm{d}W_y\Big|^2 = \int_\Omega |G(x, y, \kappa)|^2\sigma^2(y)\mathrm{d}y < \infty.$$

Hence $\sigma \in L^p(\Omega), p > 2$ and $\sigma \in C^{0,\eta}(\Omega), 0 < \eta < 1$.

- Step 1: continuous modification of the Gaussian random field

$$v(x) = \int_\Omega G(x,y)\sigma(y)\mathrm{d}W_y.$$

- Step 2: construct an approximation sequence

$$\dot{W}_x^n = \sum_{j=1}^{n} |K_j|^{-\frac{1}{2}} \xi_j \chi_{K_j}(x), \quad \xi_j = |K_j|^{-\frac{1}{2}} \int_{K_j} \mathrm{d}W_x, \quad \xi_j \sim \mathcal{N}(0,1).$$

- Step 3: consider an approximated solution

$$u^n(x) = \int_\Omega G(x,y)g(y)\mathrm{d}y + \int_\Omega G(x,y)\sigma(y)\mathrm{d}W_x^n.$$

Recall the mild solution

$$u(x,\kappa_j) = \int_\Omega G(x,y,\kappa_j)g(y)\mathrm{d}y + \int_\Omega G(x,y,\kappa_j)\sigma(y)\mathrm{d}W_y.$$

Taking the expectation yields

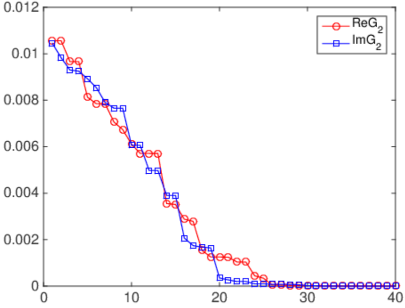
$$\mathbf{E}u(x,\kappa_j) = \int_\Omega G(x,y,\kappa_j)g(y)\mathrm{d}y.$$

Consider real and imaginary parts

$$\mathrm{Re}G(x,y,\kappa_j) = \frac{1}{4}Y_0(\kappa_j|x-y|), \quad \mathrm{Im}G(x,y,\kappa_j) = -\frac{1}{4}J_0(\kappa_j|x-y|).$$

Real-valued Fredholm integral equations

$$\mathbf{E}\mathrm{Re}u(x,\kappa_j) = \frac{1}{4}\int_\Omega Y_0(\kappa_j|x-y|)g(y)\mathrm{d}y,$$
$$\mathbf{E}\mathrm{Im}u(x,\kappa_j) = -\frac{1}{4}\int_\Omega J_0(\kappa_j|x-y|)g(y)\mathrm{d}y.$$

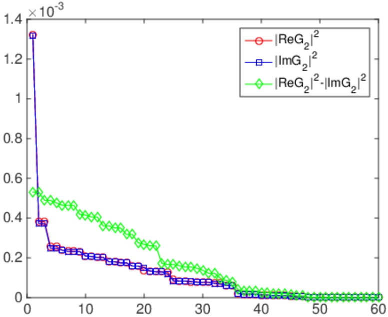


Taking the variance of the mild solution yields

$$\mathbf{V}\mathrm{Re}u(x,\kappa_j) = \frac{1}{16}\int_\Omega Y_0^2(\kappa_j|x-y|)\sigma^2(y)\mathrm{d}y,$$

$$\mathbf{V}\mathrm{Im}u(x,\kappa_j) = \frac{1}{16}\int_\Omega J_0^2(\kappa_j|x-y|)\sigma^2(y)\mathrm{d}y.$$

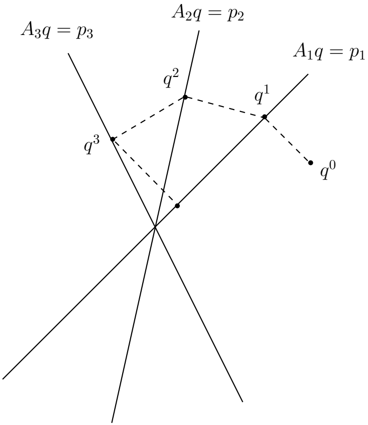
$$\mathbf{V}\mathrm{Re}u(x,\kappa_j)-\mathbf{V}\mathrm{Im}u(x,\kappa_j) = \frac{1}{16}\int_\Omega \left(Y_0^2(\kappa_j|x-y|) - J_0^2(\kappa_j|x-y|)\right)\sigma^2(y)\mathrm{d}y$$



Numerical method - Kaczmarz algorithm

Consider the linear system of equations

$$A_j q = p_j, \quad j = 1, \ldots, m.$$



Regularized Kaczmarz algorithm

Let $q^0 = 0$, do $k = 0, 1, \ldots$

$$\begin{cases} q_0 = q^k, \\ q_j = q_{j-1} + A_j^{\mathrm{T}}(\mu I + A_j A_j^{\mathrm{T}})^{-1}(p_j - A_j q_{j-1}), \quad j = 1, \ldots, m, \\ q^{k+1} = q_m, \end{cases}$$

where $\mu > 0$ is a regularization parameter.

Consider the approximated scattering problem

$$\begin{cases} \Delta u + \kappa^2 u = g + \sigma \dot{W}_x^n, & x \in \mathbb{R}^2, \\ \partial_r u - \mathrm{i}\kappa u = o(r^{-1/2}), & r \to \infty, \end{cases}$$

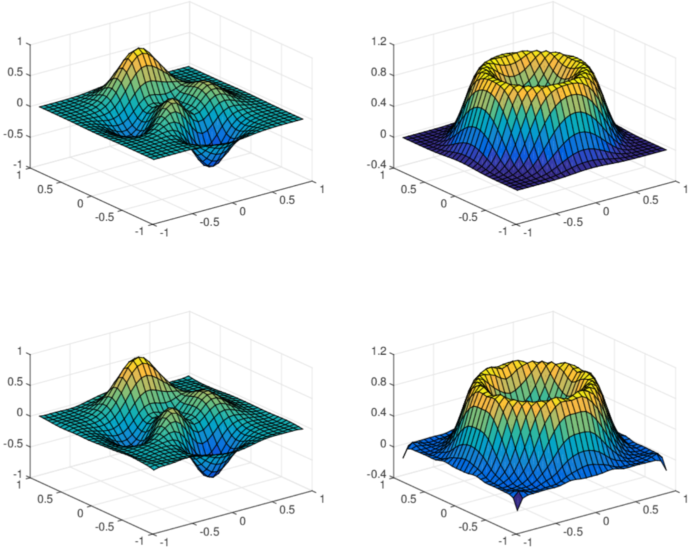
where

$$\dot{W}_x^n = \sum_{j=1}^{n} |K_j|^{-\frac{1}{2}} \xi_j \chi_{K_j}(x).$$

Direct solver: FEM with PML, Monte Carlo.

Parameters: $\kappa_j = (j - 0.5)\pi, j = 1, \ldots, 5$, $\mu = 1.0 \times 10^{-7}$, outloop $k = 5$.

Numerical result



Consider the stochastic scattering problem

$$\begin{cases} \Delta u + \kappa^2(1+q)u = g + \sigma \dot{W}_x, & x \in \mathbb{R}^2, \\ \partial_r u - \mathrm{i}\kappa u = o(r^{-1/2}), & r \to \infty. \end{cases}$$

Theorem (Bao-Chen-L)

The stochastic scattering problem admits a unique continuous stochastic process u, which satisfies

$$u(x) = -\kappa^2 \int_\Omega G(x,y)q(y)u(y)\mathrm{d}y + \int_\Omega G(x,y)g(y)\mathrm{d}y + \int_\Omega G(x,y)\sigma(y)\mathrm{d}W_y.$$

Consider the inhomogeneous stochastic Helmholtz equation

$$\Delta u(x,\kappa) + \kappa^2(1+q(x))u(x,\kappa) = g(x) + \sigma(x)\dot{W}_x \quad \text{in } B_R.$$

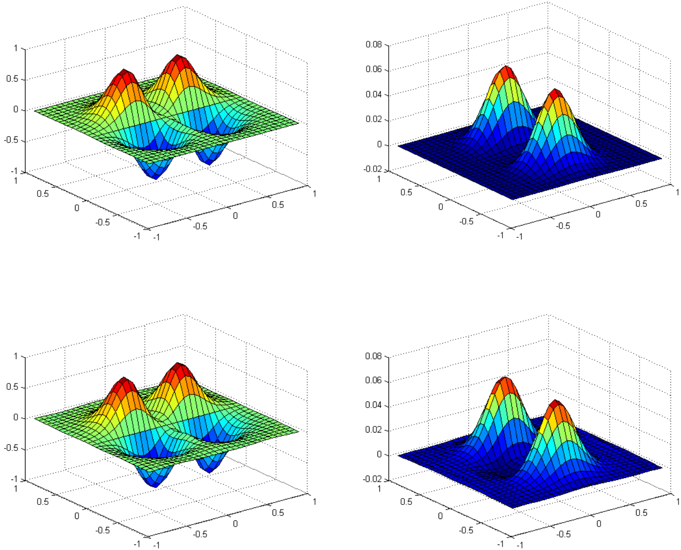
Let v be the eigenfunction for the following problem:

$$\begin{cases} \Delta v(x,\kappa) + \kappa^2(1+q(x))v(x,\kappa) = 0 & \text{in } B_R, \\ v(x,\kappa) = 0 & \text{on } \Gamma_R. \end{cases}$$

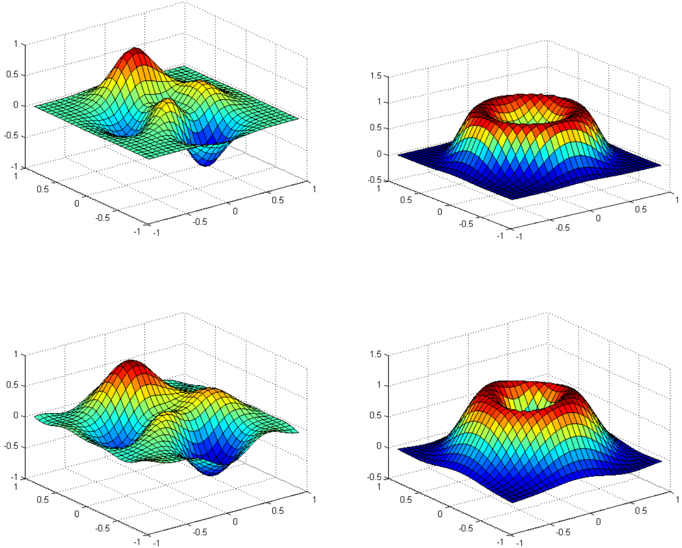
We have from the integation by parts that

$$-\int_{\Gamma_R} \partial_\nu v(x,\kappa)u(x,\kappa)\mathrm{d}\gamma = \int_{B_R} g(x)v(x,\kappa)\mathrm{d}x + \int_{B_R} \sigma(x)v(x,\kappa)\mathrm{d}W_x.$$

Numerical result - homogeneous medium



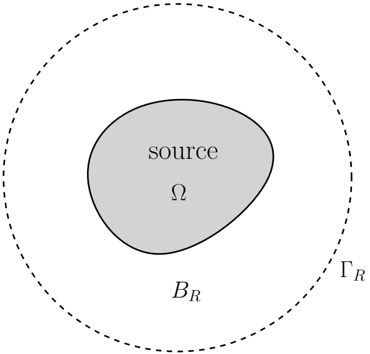
Numerical result - inhomogeneous medium



Increasing stability

- Continuous frequency data
- Discrete frequency data

Problem geometry



The Helmholtz equation

$$\Delta u(x,\kappa) + \kappa^2 u(x,\kappa) = f(x), \quad x \in \mathbb{R}^2.$$

The Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}(\partial_r u - \mathrm{i}\kappa u) = 0, \quad r = |x|.$$

Given $f \in L^2(\Omega)$, it has a unique solution:

$$u(x,\kappa) = \int_\Omega G(x,y;\kappa) f(y)\mathrm{d}y,$$

where

$$G(x,y;\kappa) = -\frac{\mathrm{i}}{4} H_0^{(1)}(\kappa|x-y|).$$

Given function u on Γ_R, it has the Fourier series expansion:

$$u(R,\theta) = \sum_{n\in\mathbb{Z}} \hat{u}_n(R)e^{\mathrm{i}n\theta}, \quad \hat{u}_n(R) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)e^{-\mathrm{i}n\theta}\mathrm{d}\theta.$$

Introduce the DtN operator $T : H^{\frac{1}{2}}(\Gamma_R) \to H^{-\frac{1}{2}}(\Gamma_R)$

$$(Tu)(R,\theta) = \kappa \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)'}(\kappa R)}{H_n^{(1)}(\kappa R)} \hat{u}_n(R)e^{\mathrm{i}n\theta}.$$

Transparent boundary condition:

$$\partial_\nu u = Tu \quad \text{on } \Gamma_R.$$

Consider reduced problem

$$\begin{cases} \Delta u + \kappa^2 u = f & \text{in } B_R, \\ \partial_\nu u = Tu & \text{on } \Gamma_R. \end{cases}$$

Define boundary data

$$\|u(\cdot,\kappa)\|_{\Gamma_R}^2 = \int_{\Gamma_R} \left(|Tu(x,\kappa)|^2 + \kappa^2 |u(x,\kappa)|^2 \right) \mathrm{d}\gamma(x).$$

ISP 1. Let f be a complex function with a compact support $\Omega \subset B_R$. The ISP is to determine f from the data $u(x,\kappa), x \in \Gamma_R, \kappa \in (0,K)$, where $K > 1$ is a positive constant.

Define

$$\mathbb{F}_M = \{f \in H^m(\Omega) : \|f\|_{H^m(B_R)} \leq M, \ \mathrm{supp} f = \Omega \subset B_R\},$$

where $m > 2$ is an integer and $M > 1$ is a constant.

Theorem (L-Yuan)

Let $f \in \mathbb{F}_M$ and u be the solution of the scattering problem corresponding to f. Then

$$\|f\|^2_{L^2(\Omega)} \lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}}|\ln \epsilon|^{\frac{1}{4}}}{(6m-15)^3}\right)^{2m-5}},$$

where

$$\epsilon = \left(\int_0^K \kappa \|u(\cdot,\kappa)\|^2_{\Gamma_R} \mathrm{d}\kappa\right)^{\frac{1}{2}}.$$

- Step 1: energy estimate

$$\|f\|^2_{L^2(\Omega)} \lesssim \int_0^\infty \kappa \|u(\cdot,\kappa)\|^2_{\Gamma_R} \mathrm{d}\kappa.$$

- Step 2: low frequency estimate

$$I_1(s) = \int_0^s \kappa^3 \int_{\Gamma_R} \left| \int_\Omega H_0^{(1)}(\kappa|x-y|) f(y) \mathrm{d}y \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa,$$

$$I_2(s) = \int_0^s \kappa \int_{\Gamma_R} \left| \int_\Omega \partial_{\nu_x} H_0^{(1)}(\kappa|x-y|) f(y) \mathrm{d}y \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa.$$

- Step 3: high frequency tail estimate

$$\int_s^{+\infty} \kappa \|u(\cdot,\kappa)\|^2_{\Gamma_R} \mathrm{d}\kappa \lesssim s^{-(2m-5)} \|f\|^2_{H^m(\Omega)}.$$

- Step 4: link between low and high frequencies.

Consider reduced problem

$$\begin{cases} \Delta u + \kappa^2 u = f & \text{in } B_R, \\ \partial_\nu u = Tu & \text{on } \Gamma_R. \end{cases}$$

Define boundary data at discrete frequency

$$\|u(\cdot, \kappa_n)\|_{\Gamma_R}^2 = \int_{\Gamma_R} \left(|Tu(x, \kappa_n)|^2 + \kappa_n^2 |u(x, \kappa_n)|^2 \right) \mathrm{d}\gamma(x),$$

where

$$\kappa_n = n \left(\frac{\pi}{R} \right), \quad n = |\boldsymbol{n}|, \quad \boldsymbol{n} \in \mathbb{Z}^2 \setminus \{0\}.$$

ISP 2. Let f be a complex function with a compact support $\Omega \subset B_R$. The ISP is to determine f from the data $u(x, \kappa), x \in \Gamma_R, \kappa \in (0, \frac{\pi}{R}] \cup \cup_{n=1}^{N} \{\kappa_n\}$, where $1 < N \in \mathbb{N}$.

Define

$$\tilde{\mathbb{F}}_M = \{f \in \mathbb{F}_M : \int_\Omega f(x)\mathrm{d}x = 0\}.$$

Theorem (L-Yuan)

Let $f \in \tilde{\mathbb{F}}_M$ and u be the solution of the scattering problem corresponding to f. Then

$$\|f\|^2_{L^2(\Omega)} \lesssim \epsilon_1^2 + \frac{M^2}{\left(\frac{N^{\frac{5}{8}}|\ln \epsilon_2|^{\frac{1}{9}}}{(6m-15)^3}\right)^{2m-5}},$$

where

$$\epsilon_1 = \left(\sum_{n\leq N} \|u(\cdot,\kappa_n)\|^2_{\Gamma_R}\right)^{\frac{1}{2}}, \quad \epsilon_2 = \sup_{\kappa\in(0,\frac{\pi}{R}]} \|u(\cdot,\kappa)\|_{\Gamma_R}.$$

- Step 1: energy estimate

$$|\hat{f}_{\boldsymbol{n}}|^2 \lesssim \|u(\cdot, \kappa_n)\|^2_{\Gamma_R}, \quad \boldsymbol{n} \in \mathbb{Z}^2 \setminus \{0\}.$$

- Step 2: low frequency estimate

$$I(s) = \left| \int_{B_R} f(x) e^{-\mathrm{i} s x \cdot \boldsymbol{d}} \mathrm{d}x \right|^2, \quad s \in (0, \frac{\pi}{R}].$$

- Step 3: high frequency tail estimate

$$\sum_{n=N_0}^{\infty} |\hat{f}_{\boldsymbol{n}}|^2 \leq N_0^{-(2m-5)} \|f\|^2_{H^m(B_R)}.$$

- Step 4: link between low and high frequencies.

References

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- (with G. Bao and C. Chen) Inverse random source scattering for elastic waves, preprint.
- (with G. Bao and C. Chen) Inverse random source scattering for the Helmholtz equation in inhomogeneous media, preprint.
- (with G. Yuan) Stability on the inverse random source scattering problem for the one-dimensional Helmholtz equation, J. Math. Anal. Appl., to appear.
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Ongoing and future work

- Partial data
- Inhomogeneous media
- Inverse random medium problem
- Time-domain inverse problems

Thank You !