Homogenization of a Transmission Problem

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Periodic Scatterer

\[ u = u_\varepsilon \]

\[ u^i \]

\[ u^s \]

\[ \mathbb{R}^d \setminus \overline{D} \]
Scattering by periodic inhomogeneous media
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Most existing homogenization results pertain to periodic with no boundary or bounded domain
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Bounded domain usually Dirichlet boundary conditions
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Scatterers in free space obey transmission boundary conditions
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Scatterers in free space obey transmission boundary conditions

This matters; boundary layers play a large role
Model of a periodic scatterer

Scattering of time harmonic incident field $u^i$ by periodic inhomogeneity $D$. Scattered field given by $u^s$, total field $u = u^s + u^i$.

\[
\nabla \cdot a(x/\epsilon) \nabla u + k^2 n(x/\epsilon) u = 0 \quad \text{in} \quad D \quad (1)
\]
\[
\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \quad (2)
\]
\[
(u^s + u^i) = u \quad \text{on} \quad \partial D \quad (3)
\]
\[
\nabla (u^s + u^i) \cdot \nu = a(x/\epsilon) \nabla u \cdot \nu \quad \text{on} \quad \partial D \quad (4)
\]
\[
\lim_{r \to \infty} r^{\frac{d-1}{2}} \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0 \quad (5)
\]
Model of a periodic scatterer

- Coefficients $a(y)$, $n(y)$ periodic in $y = x/\epsilon$, $y \in Y = [0, 1]^d$
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- The period size \( \epsilon > 0 \) assumed to be very small in comparison to size of \( D \) and wavelength.
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- Assume $a(y)$ is positive definite symmetric matrix valued function
- Assume $n(x/\epsilon), a(x/\epsilon) \in C^\infty(D)$ positive
So this is the transmission problem for $u_\varepsilon := u$ in $D$ and $u_\varepsilon := u^s$ in $\mathbb{R}^d \setminus \overline{D}$

\[
\nabla \cdot a(x/\varepsilon) \nabla u_\varepsilon + k^2 n(x/\varepsilon) u_\varepsilon = 0 \quad \text{in} \quad D
\]
\[
\Delta u_\varepsilon + k^2 u_\varepsilon = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}
\]
\[
u^+ - u^-_\varepsilon = f \quad \text{on} \quad \partial D
\]
\[
(a(x/\varepsilon) \nabla u_\varepsilon \cdot \nu)^+ - (a(x/\varepsilon) \nabla u_\varepsilon \cdot \nu)^- = g \quad \text{on} \quad \partial D
\]

where $u_\varepsilon$ satisfies the Sommerfeld radiation condition at infinity, $f := -u^i$ and $g := -\nu \cdot \nabla u^i$ on $\partial D$. 
Model of a periodic scatterer

\[ u^i \rightarrow D \rightarrow u^s \]

\[ u = u_\epsilon \]

\[ \mathbb{R}^d \backslash \overline{D} \]
Bounded domains

- Well known homogenization theory
- Bensoussan, Lions, Papanicolaou ’78
- Tartar, Sanchez-Palencia
Homogenized problem (expected)

\begin{align*}
\nabla \cdot A \nabla u_0 + k^2 \bar{n} u_0 &= 0 \quad \text{in} \quad D \\
\Delta u_0 + k^2 u_0 &= 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\
u_0^+ - u_0^- &= f \quad \text{on} \quad \partial D \\
\left(\nabla u_0 \cdot \nu\right)^+ - \left(A \nabla u_0 \cdot \nu\right)^- &= g \quad \text{on} \quad \partial D 
\end{align*}

(7)

where $u_0$ satisfies the Sommerfeld radiation condition at infinity.
\( \bar{n} \) denotes the unit cell average of \( n \), i.e.

\[
\bar{n} = \int_Y n(y) \, dy,
\]

and \( A \) is the constant-valued homogenized matrix

\[
A_{ij} = \int_Y \left( a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) \, dy,
\]

(8)

Here \( \chi^j(y) \) are the so-called cell functions which represent the \( Y \)-periodic solutions to

\[
\frac{\partial}{\partial y_i} \left( a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) = 0.
\]

(9)

The additive constant for \( \chi^j \) is always chosen so that

\[
\int_Y \chi^j \, dy = 0.
\]
Standard technique which regards the solution as that depending on a “slow” variable $x$, and a “fast” variable $y = x/\epsilon$
How to get this

- Standard technique which regards the solution as that depending on a “slow” variable $x$, and a “fast” variable $y = x/\epsilon$
- Write the equation for $u_\epsilon$ inside of $D$ as a first-order system

\[
a(x/\epsilon) \nabla u_\epsilon - v_\epsilon = 0
\]
\[
\nabla \cdot v_\epsilon + k^2 n(x/\epsilon) u_\epsilon = 0
\] (10)
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- Write the equation for $u_\epsilon$ inside of $D$ as a first-order system

$$a(x/\epsilon) \nabla u_\epsilon - v_\epsilon = 0$$
$$\nabla \cdot v_\epsilon + k^2 n(x/\epsilon) u_\epsilon = 0$$  \hspace{1cm} (10)

- Write the ansatz

$$u_\epsilon = u_0(x, x/\epsilon) + \epsilon u^{(1)}(x, x/\epsilon) + \epsilon^2 u^{(2)}(x, x/\epsilon) + \ldots$$
$$v_\epsilon = v_0(x, x/\epsilon) + \epsilon v^{(1)}(x, x/\epsilon) + \epsilon^2 v^{(2)}(x, x/\epsilon) + \ldots$$  \hspace{1cm} (11)
For the bulk expansion in the exterior it suffices to use

\[ u_\epsilon = u_0(x), \quad v_\epsilon = v_0(x). \]
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Plug into equations using

\[ \nabla = \nabla_x + \frac{1}{\epsilon} \nabla_y, \]

and equate like powers of \( \epsilon \).
This yields

\[ a(y) \nabla_y u_0 = 0 \]  \hspace{1cm} (12)
\[ \nabla_y \cdot v_0 = 0 \]  \hspace{1cm} (13)
\[ a(y) \nabla_y u^{(1)} + a(y) \nabla_x u_0 - v_0 = 0 \]  \hspace{1cm} (14)
\[ \nabla_y \cdot v^{(1)} + \nabla_x \cdot v_0 + k^2 n(y) u_0 = 0 \]  \hspace{1cm} (15)
Take the $y$ divergence of (14) and apply (13) to yield the cell problem (9) for $\chi^j$:

$$\nabla_y \cdot a(y) \nabla_y (\chi^j - y_j) = 0$$

to get

$$u^{(1)} = -\chi^j(y) \frac{\partial u_0}{\partial x_j}. \tag{16}$$

We then have an expression for $v_0$ from (14)

$$(v_0(x, y))_i = \left( a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j}{\partial y_k}(y) \right) \frac{\partial u_0}{\partial x_j}. \tag{17}$$
Homogenized problem

Taking the cell average in (15) yields the homogenized PDE in the interior of \( D \) (7).

\[
\nabla \cdot A \nabla u_0 + k^2 \nu u_0 = 0 \quad (18)
\]

Furthermore

\[
a(x/\epsilon) \nabla u_\epsilon = v_\epsilon \approx v_0
\]

where

\[
v_0 = \int_Y v_0 \, dy = A \nabla u_0.
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$$a(x/\epsilon) \nabla u_{\epsilon} = v_\epsilon \approx v_0$$

where

$$\bar{v}_0 = \int_Y v_0 \, dy = A \nabla u_0.$$
We go further to find

\[ u^{(2)} = \chi^{ij}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j} + k^2 \beta(y) u_0 \]  

(19)

\[ v^{(1)} = -a\chi^j \nabla_x \frac{\partial u_0}{\partial x_j} + a \nabla_y \chi^{ij} \frac{\partial u_0}{\partial x_i \partial x_j} + k^2 a \nabla_y \beta u_0 \]  

(20)

where \( \chi^{ij} \) are defined by

\[ \nabla_y \cdot (a \nabla_y \chi^{ij}) = b_{ij}(y) - \bar{b}_{ij} \]  

(21)

with

\[ b_{ij}(y) = -a_{ij} + a_{ik} \frac{\partial \chi^j}{\partial y_k} + \frac{\partial}{\partial y_k}(a_{ki} \chi^j), \]  

(22)
and where we also have a cell function from the lower order term

$$\nabla_y \cdot (a \nabla_y \beta(y)) = \bar{n} - n(y),$$  \hspace{1cm} (23)
Corrections (first order)

- All of these bulk corrections are zero in the exterior of $D$. 

- Correct Dirichlet transmission conditions are satisfied by $u_0$, but this is disturbed by $u^{(1)}$.
- Due to the presence of $a(x/\epsilon)$, Neumann transmission conditions are not satisfied by $u_0$. This is further disturbed by the presence of $v^{(1)}$.
- To compensate need a boundary corrector.
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This is further disturbed by the presence of $\nu^{(1)}$.

To compensate, need a boundary corrector.
First order boundary correction

The first order boundary corrector function is

\[
\nabla \cdot a(x/\epsilon) \nabla \theta_\epsilon + k^2 n(x/\epsilon) \theta_\epsilon = 0 \quad \text{in} \quad D
\]
\[
\Delta \theta_\epsilon + k^2 \theta_\epsilon = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}
\]
\[
\theta_\epsilon^+ - \theta_\epsilon^- = u^{(1)} \quad \text{on} \quad \partial D
\]
\[
(\nabla \theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon) \nabla \theta_\epsilon \cdot \nu)^- = \left( \frac{v_0 - \overline{v}_0}{\epsilon} + v^{(1)} \right) \cdot \nu \quad \text{on} \quad \partial D(24)
\]

with Sommerfeld radiation conditions at infinity.
Error estimates

**Theorem**

Let $u_\epsilon$ be the solution to (6), $u_0$ the solution to (7), and the bulk and boundary corrections $u^{(1)}$ and $\theta_\epsilon$ given by (16) and (24) respectively. Then for any ball $B_R$ of radius $R > 0$ which contains $D$,

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon \theta_\epsilon)\|_{L^2(B_R)} \leq C_R \epsilon^2 \|u_0\|_{H^4(D)}$$

and

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon \theta_\epsilon)\|_{H^1(D)} + \|u_\epsilon - (u_0 + \epsilon \theta_\epsilon)\|_{H^1(B_R \setminus D)} \leq C_R \epsilon \|u_0\|_{H^4(D)}$$

where the constant $C_R$ is independent of $\epsilon$ and $u_0$. 

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Note that the above result shows (as was done for Dirichlet and Neumann problems $n = 0$ Santosa, Vogelius ’93 Vogelius, M. ’97) that the only bulk effect at first order is $u^{(1)} = -\chi^j(y) \frac{\partial u_0}{\partial x_j}$. 
Note that the above result shows (as was done for Dirichlet and Neumann problems $n = 0$, Santosa, Vogelius '93 Vogelius, M. '97) that the only bulk effect at first order is $u^{(1)} = -\chi^j(y) \frac{\partial u_0}{\partial x_j}$.

We can show this with the formal asymptotics directly- assume $u^{(1)} = -\chi^j(y) \frac{\partial u_0}{\partial x_j} + \hat{u}(x)$
Remarks-bulk mean field

One finds that \( \hat{u} \) solves

\[
\nabla \cdot A \nabla \hat{u} + k^2 \bar{n} \hat{u} =
\]

\[
- \left( - a_{ki} \chi^j + a_{kl} \frac{\partial \chi^{ij}}{\partial y_l} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} - k^2 \left( a_{ki} \frac{\partial \beta}{\partial y_i} - n \chi^k \right) \frac{\partial u_0}{\partial x_k}.
\]

The source term can be shown to sum to zero by integration by parts.

So there is no mean field at order \( \epsilon \).
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They used Bloch decomposition to derive dispersion equations for periodic media in time domain

They showed dispersion will appear for large times, numerically demonstrate effects at $t \sim 1/\epsilon^2$
Remarks-bulk mean field

Here we find that if we set

$$u^{(2)} = \chi \ddot{y}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j} + k^2 \beta(y) u_0 + \hat{u}^{(2)}(x)$$  \hspace{1cm} (25)
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(25)

We get

$$A : \nabla \nabla \hat{u}^{(2)} + k^2 \hat{n} \hat{u}^{(2)} = - \left( A : \nabla \nabla \nabla \nabla u_0 + k^2 \mathcal{N} : \nabla \nabla u_0 \right),$$

(26)

where \(A\) and \(\mathcal{N}\) are respectively fourth- and second-order constant tensors and

$$A : \nabla \nabla \nabla \nabla u_0 = A_{ijkl} \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l},$$

exhibiting dispersive effects.
Remarks-Boundary effects

- The boundary corrector $\theta_\epsilon$ is harder to compute than the original $u_\epsilon$. 

This makes it useless as a numerical corrector. However, the presence of $\theta_\epsilon$ or something close to it is necessary to obtain a higher order approximation. We must understand its behavior to go past first order.
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- Presented in BLP
- Behavior first analyzed in Santosa Vogelius ’93 and further Vogelius, M ’97. Found limit as $\epsilon \to 0$ not necessarily unique.
- Tails analyzed in Allaire, Amar ’99
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- If boundary is flat at a rational angle with respect to the periodic structure, precise value of $\epsilon$ dictates what portion of the medium the boundary "sees".

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- Periodic half strip for rational normals Vogelius, M ’97
- Half space for irrational normals Gérard-Varet, Masmoudi ’12
Recent results for transmission problem in halfspace- Claeys, Fliss, Vinoles ’2016.

Use matched asymptotics to develop expansion which captures boundary behavior

Computable expansion gives high order approximation.
The transmission boundary corrector

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with Sommerfeld radiation conditions at infinity.
An example of a boundary layer limit

- Here we try to find its limit for a square domain $D = (0, 1) \times (0, 1)$. (or more precisely, its effective boundary values on the flat side of a smooth domain)
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- Consider the boundary $\Gamma = \{(1, x_2)\}_{0 \leq x_2 \leq 1}$
- The lower order part of the Neumann transmission data $v^{(1)}$ will contribute to the limit at this order by going to its boundary weak limit $\overline{v^{(1)}_{\partial \Omega}}$. 

Data is sum of oscillating functions for $j = 1, 2$ and consider each separately

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- Use some techniques from above works but requires new analysis due to transmission condition
- Consider the boundary \( \Gamma = \{(1, x_2)\} \) \(0 \leq x_2 \leq 1\)
- The lower order part of the Neumann transmission data \( \nu^{(1)} \) will contribute to the limit at this order by going to its boundary weak limit \( \nu^{(1)} \partial \Omega \).
- Data is sum of oscillating functions for \( j = 1, 2 \) and consider each separately
An example of a boundary layer limit

\[ \nabla \cdot a(x/\epsilon)\nabla \theta_\epsilon + k^2 n(x/\epsilon)\theta_\epsilon = 0 \quad \text{in} \quad D \]

\[ \Delta \theta_\epsilon + k^2 \theta_\epsilon = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{D} \]

\[ \theta^+_\epsilon - \theta^-_\epsilon = \chi^1(x/\epsilon)\frac{\partial u_0}{\partial x_1} \quad \text{on} \quad \Gamma \]

\[ \theta^+_\epsilon - \theta^-_\epsilon = 0 \quad \text{on} \quad \partial D \setminus \Gamma \]

\[ (\nabla \theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla \theta_\epsilon \cdot \nu)^- = \frac{1}{\epsilon} g_1(x/\epsilon)\frac{\partial u_0}{\partial x_1} + \nu^{(1)} \partial \Omega \quad \text{on} \quad \Gamma \]

\[ (\nabla \theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla \theta_\epsilon \cdot \nu)^- = 0 \quad \text{on} \quad \partial D \setminus \Gamma \quad (27) \]

together with the Sommerfeld radiation condition (5) at infinity, where

\[ g_1(x/\epsilon) = a_{11}(x/\epsilon) - a_{1k}(x/\epsilon)\frac{\partial \chi^1}{\partial y_k}(x/\epsilon) - A_{11}. \quad (28) \]
An example of a boundary layer limit

Notice that in the above problem, the transmission data on the right side of the square depends heavily on the choice of $\epsilon$. 

Suppose that the fractional part of $1/\epsilon$ is constant, i.e.

$$1/\epsilon - \lfloor 1/\epsilon \rfloor = \delta,$$
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- Notice that in the above problem, the transmission data on the right side of the square depends heavily on the choice of $\epsilon$.
- If, for example $\epsilon_k = 1/k$ for $k$ an integer, this boundary layer problem would see only the boundary slice of the periodic functions $\chi^1(y)$ and $g_1(y)$.
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It is for this reason that one can expect different limits of the boundary layer function for different sequences of $\epsilon$ going to zero.
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  $$\frac{1}{\epsilon_k} - \left\lfloor \frac{1}{\epsilon_k} \right\rfloor = \delta,$$
An example of a boundary layer limit

- Set the oscillatory boundary functions to their restrictions:

\[ \chi^1(y_2) = \chi^1(\delta, y_2); \quad g_1(y_2) = g_1(\delta, y_2). \]  

- Scale up boundary cell to strip,

\[ G = \{ -\infty < y_1 < \infty; y_2 \in [0, 1] \} \]

with its two halves

\[ G^+ = \{ y_1 > 0; y_2 \in [0, 1] \} \]

and

\[ G^- = \{ y_1 < 0; y_2 \in [0, 1] \}. \]
An example of a boundary layer limit

Let \( \hat{w}(y_1, y_2) \) solve

\[
\begin{align*}
\nabla_y \cdot a(y_1 + \delta, y_2) \nabla \hat{w} &= 0 \quad \text{in } G^- \\
\Delta_y \hat{w} &= 0 \quad \text{in } G^+ \\
\hat{w}(0, y_2)^+ - \hat{w}(0, y_2)^- &= \chi^1(y_2) \\
\partial_{y_1} \hat{w}(0, y_2)^+ - a_{1i}(\delta, y_2) \partial_{y_i} \hat{w}(0, y_2)^- &= g_1(y_2) \\
\hat{w} \quad [0, 1] - \text{periodic in } y_2
\end{align*}
\]

There exists \( \gamma > 0 \) such that

\[
\begin{align*}
\ e^{\gamma y_1} \nabla \hat{w} &\in L^2(G^+) \\
\text{and} \quad e^{-\gamma y_1} \nabla \hat{w} &\in L^2(G^-).
\end{align*}
\]
An example of a boundary layer limit

- Above problem has unique solution up to additive constant across entire strip $G$. 

Exponential decay yields that $\hat{w}$ approaches a constant as $y_1 \to \pm \infty$.

Set $d^+ = \lim_{y_1 \to \infty} \hat{w}$ and $d^- = \lim_{y_1 \to -\infty} \hat{w}$.

We can now define our limiting boundary value

$$\chi^* = d^+ - d^- \quad (31)$$
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  \[ \chi_1^* = d^+ - d^- \quad (31) \]
An example of a boundary layer limit

Then, we can define $w(y_1, y_2)$ similarly

$$\nabla_y \cdot a(y_1 + \delta, y_2) \nabla w = 0 \quad \text{in} \quad G^- \quad (32)$$

$$\Delta_y w = 0 \quad \text{in} \quad G^+$$

$$w(0, y_2)^+ - w(0, y_2)^- = \chi^1(y_2) - \chi^*_1$$

$$\partial_{y_1} w(0, y_2)^+ - a_{1i}(\delta, y_2) \partial_{y_i} w(0, y_2)^- = g_1(y_2)$$

$w$ $[0, 1]$ – periodic in $y_2$

There exists $\gamma > 0$ such that $e^{\gamma y_1} \nabla w \in L^2(G^+)$

and $e^{-\gamma y_1} \nabla w \in L^2(G^-)$.

Now $w$ itself also decays to zero exponentially as $|y_1| \to \infty$. 

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An example of a boundary layer limit

**Theorem**

Let \( \epsilon_k \) be a sequence approaching zero such that \( \frac{1}{\epsilon_k} - \left\lfloor \frac{1}{\epsilon_k} \right\rfloor = \delta \) for all \( k \). Then if \( \theta_{\epsilon_k} \) solves (27) for \( \epsilon = \epsilon_k \), we have that \( \theta_{\epsilon_k} \to \theta^* \) strongly in \( L^2_{loc}(\mathbb{R}^2) \) where \( \theta^* \) solves

\[
\nabla \cdot A \nabla \theta^* + k^2 \bar{n} \theta^* = 0 \quad \text{in } D
\]
\[
\Delta \theta^* + k^2 \theta^* = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}
\]
\[
(\theta^*)^+ - (\theta^*)^- = \chi_1^* \frac{\partial u_0}{\partial x_1} \quad \text{on } \Gamma
\]
\[
(\theta^*)^+ - (\theta^*)^- = 0 \quad \text{on } \partial D \setminus \Gamma
\]
\[
(\nabla \theta^* \cdot \nu)^+ - (A \nabla \theta^* \cdot \nu)^- = \frac{a_{12}(\delta, y_2)w(0, y_2)^-}{\partial x_1 \partial x_2} \frac{\partial^2 u_0}{\partial x_1} + v^{(1)} \frac{\partial \Omega}{\partial x_1} \quad \text{on } \Gamma
\]
\[
(\nabla \theta^* \cdot \nu)^+ - (A \nabla \theta^* \cdot \nu)^- = 0 \quad \text{on } \partial D \setminus \Gamma
\]
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Boundary layer effects specific to transmission problems appear at all orders. We characterize these effects for a square. Boundary layer limits difficult to characterize explicitly in general.

If $a$ is constant (periodicity only in lower order term $n$), we can say more, work in progress.

For a square, its limit is not zero in general, so its effects are higher order than dispersion!
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