Homogenization of a Transmission Problem

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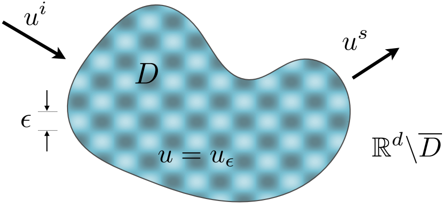
Periodic Scatterer



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- Most existing homogenization results pertain to periodic with no boundary or bounded domain
- Bounded domain usually Dirichlet boundary conditions
- Scatterers in free space obey transmission boundary conditions
- This matters; boundary layers play a large role

Model of a periodic scatterer

Scattering of time harmonic incident field u^i by periodic inhomogeneity D. Scattered field given by u^s, total field $u = u^s + u^i$.

$$
\begin{aligned}
\nabla \cdot a(x/\epsilon)\nabla u + k^2 n(x/\epsilon)u &= 0 \qquad \text{in} \quad D && (1)\\
\Delta u^s + k^2 u^s &= 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} && (2)\\
(u^s + u^i) &= u \qquad \text{on} \quad \partial D && (3)\\
\nabla(u^s + u^i)\cdot \nu &= a(x/\epsilon)\nabla u \cdot \nu \qquad \text{on} \quad \partial D && (4)\\
\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s\right) &= 0 && (5)
\end{aligned}
$$

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- Assume $a(y)$ is positive definite symmetric matrix valued function
- Assume $n(x/\epsilon), a(x/\epsilon) \in C^\infty(D)$ positive

Model of a periodic scatterer

So this the transmission problem for $u_\epsilon := u$ in D and $u_\epsilon := u^s$ in $\mathbb{R}^d \setminus \overline{D}$

$$\begin{aligned} \nabla \cdot a(x/\epsilon)\nabla u_\epsilon + k^2 n(x/\epsilon) u_\epsilon &= 0 \quad \text{in} \quad D \\ \Delta u_\epsilon + k^2 u_\epsilon &= 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\ u_\epsilon^+ - u_\epsilon^- &= f \quad \text{on} \quad \partial D \\ (\nabla u_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla u_\epsilon \cdot \nu)^- &= g \quad \text{on} \quad \partial D \end{aligned} \tag{6}$$

where u_ϵ satisfies the Sommerfeld radiation condition at infinity, $f := -u^i$ and $g := -\nu \cdot \nabla u^i$ on ∂D.

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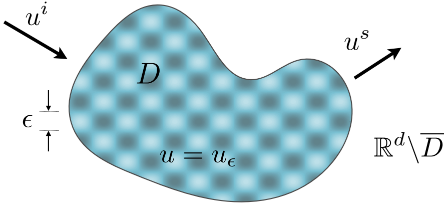

Bounded domains

- Well known homogenization theory
- Bensoussan, Lions, Papanicolaou '78
- Tartar, Sanchez-Palencia

Homogenized problem (expected)

$$
\begin{aligned}
\nabla \cdot A \nabla u_0 + k^2 \overline{n} u_0 &= 0 \quad \text{in} \quad D \\
\Delta u_0 + k^2 u_0 &= 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\
u_0^+ - u_0^- &= f \quad \text{on} \quad \partial D \\
(\nabla u_0 \cdot \nu)^+ - (A \nabla u_0 \cdot \nu)^- &= g \quad \text{on} \quad \partial D
\end{aligned}
\tag{7}
$$

where u_0 satisfies the Sommerfeld radiation condition at infinity.

Homogenized problem (expected)

$\overline{n}$ denotes the unit cell average of n, i.e.

$$\overline{n} = \int_Y n(y)dy,$$

and A is the constant-valued homogenized matrix

$$A_{ij} = \int_Y \left(a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}(y) \right) dy, \tag{8}$$

Here $\chi^j(y)$ are the so-called cell functions which represent the Y-periodic solutions to

$$\frac{\partial}{\partial y_i}\left(a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}(y) \right) = 0. \tag{9}$$

The additive constant for χ^j is always chosen so that

$$\int_Y \chi^j \,\mathrm{d}y = 0.$$

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- Write the equation for u_ϵ inside of D as a first-order system

$$\begin{aligned} a(x/\epsilon)\nabla u_\epsilon - v_\epsilon &= 0 \\ \nabla \cdot v_\epsilon + k^2 n(x/\epsilon) u_\epsilon &= 0 \end{aligned} \qquad (10)$$

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- Write the ansatz

$$\begin{aligned} u_\epsilon &= u_0(x, x/\epsilon) + \epsilon u^{(1)}(x, x/\epsilon) + \epsilon^2 u^{(2)}(x, x/\epsilon) + \dots \\ v_\epsilon &= v_0(x, x/\epsilon) + \epsilon v^{(1)}(x, x/\epsilon) + \epsilon^2 v^{(2)}(x, x/\epsilon) + \dots \end{aligned} \tag{11}$$

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- Plug into equations using

$$\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y,$$

and equate like powers of ϵ.

How to get this

This yields

$$a(y)\nabla_y u_0 = 0 \tag{12}$$

$$\nabla_y \cdot v_0 = 0 \tag{13}$$

$$a(y)\nabla_y u^{(1)} + a(y)\nabla_x u_0 - v_0 = 0 \tag{14}$$

$$\nabla_y \cdot v^{(1)} + \nabla_x \cdot v_0 + k^2 n(y) u_0 = 0 \tag{15}$$

Corrections (first order)

- Take the y divergence of (14) and apply (13) to yield the cell problem (9) for χ^j:

$$\nabla_y \cdot a(y)\nabla_y \left(\chi^j - y_j\right) = 0$$

  to get

$$u^{(1)} = -\chi^j(y)\frac{\partial u_0}{\partial x_j}. \tag{16}$$

- We then have an expression for v_0 from (14)

$$(v_0(x,y))_i = \left(a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}(y)\right)\frac{\partial u_0}{\partial x_j}, \tag{17}$$

Homogenized problem

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- Taking the cell average in (15) yields the homogenized PDE in the interior of D (7).

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- Furthermore

$$a(x/\epsilon) \nabla u_\epsilon = v_\epsilon \approx v_0$$

where

$$\overline{v}_0 = \int_Y v_0 \, \mathrm{d}y = A \nabla u_0.$$

Corrections (first order and second order)

We go further to find

$$u^{(2)} = \chi^{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j} + k^2\beta(y)u_0 \tag{19}$$

$$v^{(1)} = -a\chi^j\nabla_x\frac{\partial u_0}{\partial x_j} + a\nabla_y\chi^{ij}\frac{\partial u_0}{\partial x_i \partial x_j} + k^2 a\nabla_y\beta u_0 \tag{20}$$

where χ^{ij} are defined by

$$\nabla_y \cdot \left(a\nabla_y\chi^{ij}\right) \;=\; b_{ij}(y) - \overline{b}_{ij} \tag{21}$$

with

$$b_{ij}(y) = -a_{ij} + a_{ik}\frac{\partial\chi^j}{\partial y_k} + \frac{\partial}{\partial y_k}(a_{ki}\chi^j), \tag{22}$$

Corrections (first order and second order)

and where we also have a cell function from the lower order term

$$\nabla_y \cdot (a \nabla_y \beta(y)) = \bar{n} - n(y), \qquad (23)$$

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- Due to the presence of $a(x/\epsilon)$, Neumann transmission conditions are not satisfied by u_0.
- This is further disturbed by the the presence of $v^{(1)}$.
- To compensate need a boundary corrector.

First order boundary correction

The first order boundary corrector function is

$$
\begin{aligned}
\nabla \cdot a(x/\epsilon)\nabla\theta_\epsilon + k^2 n(x/\epsilon)\theta_\epsilon &= 0 \quad \text{in} \quad D \\
\Delta\theta_\epsilon + k^2\theta_\epsilon &= 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\
\theta_\epsilon^+ - \theta_\epsilon^- &= u^{(1)} \quad \text{on} \quad \partial D \\
(\nabla\theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\theta_\epsilon \cdot \nu)^- &= \left(\frac{v_0 - \overline{v}_0}{\epsilon} + v^{(1)}\right) \cdot \nu \quad \text{on} \quad \partial D
\end{aligned} \tag{24}
$$

with Sommerfeld radiation conditions at infinity.

Error estimates

Theorem

Let u_ϵ be the solution to (6), u_0 the solution to (7), and the bulk and boundary corrections $u^{(1)}$ and θ_ϵ given by (16) and (24) respectively. Then for any ball B_R of radius $R > 0$ which contains D,

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\theta_\epsilon)\|_{L^2(B_R)} \leq C_R \epsilon^2 \|u_0\|_{H^4(D)}$$

and

$$\|u_\epsilon - (u_0 + \epsilon u^{(1)} + \epsilon\theta_\epsilon)\|_{H^1(D)} + \|u_\epsilon - (u_0 + \epsilon\theta_\epsilon)\|_{H^1(B_R \setminus D)} \leq C_R \epsilon \|u_0\|_{H^4(D)}$$

where the constant C_R is independent of ϵ and u_0.

Remarks-bulk mean field

- Note that the above result shows (as was done for Dirichlet and Neumann problems $n = 0$ Santosa, Vogelius '93 Vogelius, M. '97) that the only bulk effect at first order is $u^{(1)} = -\chi^j(y)\frac{\partial u_0}{\partial x_j}$.

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- We can show this with the formal asymptotics directly- assume $u^{(1)} = -\chi^j(y)\frac{\partial u_0}{\partial x_j} + \hat{u}(x)$

Remarks-bulk mean field

- One finds that $\hat{u}$ solves

$$\nabla \cdot A \nabla \hat{u} + k^2 \overline{n} \hat{u} =$$

$$- \left(- \overline{a_{ki} \chi^j} + \overline{a_{kl} \frac{\partial \chi^{ij}}{\partial y_l}} \right) \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} - k^2 (\overline{a_{ki} \frac{\partial \beta}{\partial y_i}} - \overline{n \chi^k}) \frac{\partial u_0}{\partial x_k}.$$

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- So there is no mean field at order ϵ.

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- They used Bloch decomposition to derive dispersion equations for periodic media in time domain
- They showed dispersion will appear for large times, numerically demonstrate effects at $t \sim 1/\epsilon^2$

Remarks-bulk mean field

- Here we find that if we set

$$u^{(2)} = \chi^{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j} + k^2\beta(y)u_0 + \hat{u}^{(2)}(x) \tag{25}$$

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- We get

$$A:\nabla\nabla\hat{u}^{(2)} + k^2\overline{n}\,\hat{u}^{(2)} \;=\; -\Big(\mathcal{A}:\nabla\nabla\nabla\nabla u_0 + k^2\mathcal{N}:\nabla\nabla u_0\Big), \tag{26}$$

where $\mathcal{A}$ and $\mathcal{N}$ are respectively fourth- and second-order constant tensors and

$$\mathcal{A}:\nabla\nabla\nabla\nabla u_0 \;=\; \mathcal{A}_{ijkl}\,\frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l},$$

exhibiting dispersive effects.

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- This makes is useless as a numerical corrector.
- However, the presence of θ_ϵ or something close to it is *necessary* to obtain a higher order approximation.
- We must understand its behavior to go past first order.

Boundary corrector for Dirichlet problems

- Presented in BLP
- Behavior first analyzed in Santosa Vogelius '93 and further Vogelius, M '97. Found limit as $\epsilon \to 0$ not necessarily unique.
- Tails analyzed in Allaire, Amar '99

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- If boundary is flat at a rational angle with respect to the periodic structure, precise value of ϵ dictates what portion of the medium the boundary "sees".
- Gérard-Varet, Masmoudi '12 show that boundary layer limit exists and is unique for smooth domains with no flat parts.

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- Half space for irrational normals Gérard-Varet, Masmoudi '12

Boundary behavior for transmission problems

- Recent results for transmission problem in halfspace- Claeys, Fliss, Vinoles '2016.
- Use matched asymptotics to develop expansion which captures boundary behavior
- Computable expansion gives high order approximation.

The transmission boundary corrector

The first order boundary corrector function is

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with Sommerfeld radiation conditions at infinity.

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- Data is sum of oscillating functions for $j = 1, 2$ and consider each separately

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\theta_\epsilon^+ - \theta_\epsilon^- &= \chi^1(x/\epsilon)\frac{\partial u_0}{\partial x_1} \quad \text{on } \Gamma \\
\theta_\epsilon^+ - \theta_\epsilon^- &= 0 \quad \text{on} \quad \partial D \setminus \Gamma \\
(\nabla\theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\theta_\epsilon \cdot \nu)^- &= \frac{1}{\epsilon} g_1(x/\epsilon)\frac{\partial u_0}{\partial x_1} + \overline{v^{(1)}}^{\partial\Omega} \quad \text{on } \Gamma \\
(\nabla\theta_\epsilon \cdot \nu)^+ - (a(x/\epsilon)\nabla\theta_\epsilon \cdot \nu)^- &= 0 \quad \text{on} \quad \partial D \setminus \Gamma
\end{aligned} \tag{27}
$$

together with the Sommerfeld radiation condition (5) at infinity, where

$$
g_1(x/\epsilon) = a_{11}(x/\epsilon) - a_{1k}(x/\epsilon)\frac{\partial \chi^1}{\partial y_k}(x/\epsilon) - A_{11}. \tag{28}
$$

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- It is for this reason that one can expect different limits of the boundary layer function for different sequences of ϵ going to zero.
- Suppose that the fractional part of $1/\epsilon_k$ is constant, i.e.

$$\frac{1}{\epsilon_k} - \lfloor \frac{1}{\epsilon_k} \rfloor = \delta,$$

An example of a boundary layer limit

- Set the oscillatory boundary functions to their restrictions:

$$\chi^1(y_2) = \chi^1(\delta, y_2); \quad g_1(y_2) = g_1(\delta, y_2). \tag{29}$$

- Scale up boundary cell to strip,

$$G = \{-\infty < y_1 < \infty; y_2 \in [0, 1]\}$$

with its two halves

$$G^+ = \{y_1 > 0; y_2 \in [0, 1]\}$$

and

$$G^- = \{y_1 < 0; y_2 \in [0, 1]\}.$$

An example of a boundary layer limit

Let $\hat{w}(y_1, y_2)$ solve

$$\begin{aligned}
\nabla_y \cdot a(y_1 + \delta, y_2)\nabla \hat{w} &= 0 \quad \text{in} \quad G^- \\
\Delta_y \hat{w} &= 0 \quad \text{in} \quad G^+ \\
\hat{w}(0, y_2)^+ - \hat{w}(0, y_2)^- &= \chi^1(y_2) \\
\partial_{y_1} \hat{w}(0, y_2)^+ - a_{1i}(\delta, y_2)\partial_{y_i} \hat{w}(0, y_2)^- &= g_1(y_2) \\
\hat{w} \quad [0, 1] - \text{periodic in} \quad & \quad y_2 \\
\text{There exists } \gamma > 0 \text{ such that} \quad e^{\gamma y_1}\nabla \hat{w} &\in L^2(G^+) \\
\text{and} \quad e^{-\gamma y_1}\nabla \hat{w} &\in L^2(G^-).
\end{aligned} \tag{30}$$

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- We can now define our limiting boundary value
$$\chi_1^* = d^+ - d^- \tag{31}$$

An example of a boundary layer limit

Then, we can define $w(y_1, y_2)$ similarly

$$\begin{aligned}
\nabla_y \cdot a(y_1+\delta, y_2)\nabla w &= 0 \quad \text{in} \quad G^- \\
\Delta_y w &= 0 \quad \text{in} \quad G^+ \\
w(0,y_2)^+ - w(0,y_2)^- &= \chi^1(y_2) - \chi_1^* \\
\partial_{y_1} w(0,y_2)^+ - a_{1i}(\delta, y_2)\partial_{y_i} w(0,y_2)^- &= g_1(y_2) \\
w \quad [0,1] - \text{periodic in} &\quad y_2 \\
\text{There exists } \gamma > 0 \text{ such that} \quad e^{\gamma y_1}\nabla w &\in L^2(G^+) \\
\text{and} \quad e^{-\gamma y_1}\nabla w &\in L^2(G^-).
\end{aligned} \tag{32}$$

Now w itself also decays to zero exponentially as $|y_1| \to \infty$.

Theorem

Let ϵ_k be a sequence approaching zero such that $\frac{1}{\epsilon_k} - \lfloor \frac{1}{\epsilon_k} \rfloor = \delta$ for all k. Then if θ_{ϵ_k} solves (27) for $\epsilon = \epsilon_k$, we have that $\theta_{\epsilon_k} \to \theta^*$ strongly in $L^2_{loc}(\mathbb{R}^2)$ where θ^* solves

$$\begin{aligned}
\nabla \cdot A \nabla \theta^* + k^2 \overline{n} \theta^* &= 0 \quad in \quad D \\
\Delta \theta^* + k^2 \theta^* &= 0 \quad in \quad \mathbb{R}^2 \setminus \overline{D} \\
(\theta^*)^+ - (\theta^*)^- &= \chi_1^* \frac{\partial u_0}{\partial x_1} \quad on\ \Gamma \\
(\theta^*)^+ - (\theta^*)^- &= 0 \quad on \quad \partial D \setminus \Gamma \\
(\nabla \theta^* \cdot \nu)^+ - (A \nabla \theta^* \cdot \nu)^- &= \overline{a_{12}(\delta, y_2) w(0, y_2)^-} \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \overline{v^{(1)}}^{\partial \Omega} \quad on\ \Gamma \\
(\nabla \theta^* \cdot \nu)^+ - (A \nabla \theta^* \cdot \nu)^- &= 0 \quad on \quad \partial D \setminus \Gamma
\end{aligned}$$

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- Boundary layer limits difficult to characterize explicitly in general
- If a is constant (periodicity only in lower order term n), we can say more, work in progress.
- For a square- its limit is not zero in general- so its effects are higher order than dispersion !