STEKLOFF EIGENVALUES AND INVERSE SCATTERING THEORY

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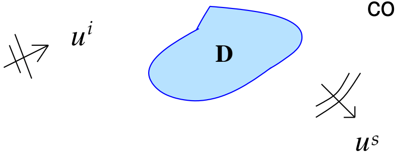
Scattering by an Inhomogeneous Medium



The direct scattering problem under consideration is to find u such that

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3$$
$$u = u^s + u^i$$
$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $u^i(x) = \mathrm{e}^{ikd\cdot x}$, $|d| = 1$, $n(x)$ is piecewise continuous and $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus \overline{D}$. There exists a unique solution u of the direct scattering problem for which u^s has the asymptotic behavior

$$u^s(x,d) = \frac{e^{ikr}}{r} u_\infty(\hat{x},d) + O\left(\frac{1}{r^2}\right)$$

where $\hat{x} = x/|x|$. u_∞ is the far field pattern of u^s.

Transmission Eigenvalues

The far field operator $F : L^2(S^2) \to L^2(S^2)$ where $S^2 := \{x : |x| = 1\}$ is defined by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d) ds_d$$

and is injective unless k is a transmission eigenvalue, i.e. a value of k such that there exists a nontrivial solution to

$$\begin{aligned}
\Delta w + k^2 n(x) w = 0 \quad & \text{in} \quad D \\
\Delta v + k^2 v = 0 \quad & \text{in} \quad D \\
w = v \quad & \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D
\end{aligned}$$

where ν is the unit outward normal to ∂D.

Transmission eigenvalues can be determined using the operator F and carry information about $n(x)$.

Transmission Eigenvalues

Let k_1 be the first transmission eigenvalue and suppose $n(x) > 1$ for $x \in \overline{D}$ or $n(x) < 1$ for $x \in \overline{D}$. Then, given k_1 and a knowledge of D, a constant n_0 can be determined such that the scattering problem for $n(x) = n_0$ also has k_1 as its first transmission eigenvalue. Then

$$\min_{\overline{D}} n(x) \leq n_0 \leq \max_{\overline{D}} n(x).$$

Flaws or voids in D can be detected by changes in k_1 and hence n_0.

In our previous research initiative with Wright-Patterson AFB the above ideas were applied to anisotropic materials interrogated by microwaves, in which case the changes in the material structure is characterized by changes in the eigenvalues of the permittivity tensor. Two problems were considered:

1. Scattering by anisotropic dielectrics in free space,
2. Scattering by isotropic dielectrics on a perfectly conducting plane.

See RIMSS Task Order 0005, Final Report, August 2013.

Pros and Cons of Using Transmission Eigenvalues

Pros:

1. A number n_0 is obtained which for isotropic media is close to an average value of $n(x)$ and for anisotropic media is close to the arithmetic mean of the eigenvalues.
2. Transmission eigenvalues are a physical characterization of the media which corresponds to the non-scattering of special incident fields.

Cons:

1. The first transmission eigenvalue is determined by the material properties of the scatterer, i.e. one can not choose the interrogating frequency.
2. The method of transmission eigenvalues only applies to dielectrics or materials with very small absorption.

Modified Far Field Operator

An alternative approach to using transmission eigenvalues is to modify the far field operator and make use of Stekloff eigenvalues. In particular let ν be the unit outward normal to ∂D and h denote the solution of the exterior impedance problem (where $\Im(\lambda) \geq 0$):

$$\Delta h + k^2 h = 0, \qquad x \in \mathbb{R}^3 \setminus \overline{D}$$

$$h(x) = \mathrm{e}^{ikd\cdot x} + h^s(x)$$

$$\frac{\partial h}{\partial \nu} + \lambda h = 0 \qquad \text{on } \partial D$$

$$\lim_{r\to\infty} r\left(\frac{\partial h^s}{\partial r} - ikh^s\right) = 0.$$

Modified Far Field Operator

If we now replace the far field operator F by the modified far field operator $\mathcal{F} : L^2(S^2) \to L^2(S^2)$ defined by

$$(\mathcal{F}g)(\hat{x}) := \int_{S^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]\, g(d)\, ds(d) \qquad (*)$$

where h_∞ is the far field pattern of h then $\mathcal{F}$ is injective unless λ is a Stekloff eigenvalue i.e. a value of λ such that there exists a nontrivial solution of

$$\begin{aligned} \Delta w + k^2 n(x) w &= 0 \quad \text{in } D \\ \frac{\partial w}{\partial \nu} + \lambda w &= 0 \text{ on } \partial D. \end{aligned}$$

Modified Far Field Operator

Now for $z \in D$ let

$$\Phi(x,z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}$$

and let Φ_∞ be the far field pattern for $\Phi(x,z)$, i.e.

$$\Phi_\infty(\hat{x},z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}, \qquad \hat{x} = \frac{x}{|x|}.$$

Consider the modified equation

$$\mathcal{F}g = \Phi_\infty(\hat{x},z)$$

where $\mathcal{F}$ is the modified far field operator defined by $(*)$ and for $g \in L^2(S^2)$ define the Heglotz wave function with kernel g by

$$v_g(x) := \int_{S^2} exp(ikx\cdot d)g(d)ds(d).$$

Modified Far Field Operator

Theorem

If λ is not a Stekloff eigenvalue then for every $\epsilon > 0$ there exists $g_\epsilon^z \in L^2(S^2)$ such that for $z \in D$

$$\|\mathcal{F} g_\epsilon^z - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} < \epsilon. \qquad (**)$$

If w is the solution to

$$\begin{aligned} \Delta w + k^2 n(x) w &= 0 \quad \text{in } D \\ \frac{\partial w}{\partial \nu} + \lambda w &= \frac{\partial \Phi(\cdot, z)}{\partial \nu} + \lambda \Phi(\cdot, z) \quad \text{on } \partial D. \end{aligned}$$

then w can be uniquely decomposed as $w = w^i + w^s$ where $w^i \in H^1(D)$ satisfies $\Delta w + k^2 w = 0$ in D and $w^s \in H^1_{loc}(\mathbb{R}^3)$ satisfies the radiation condition. The function g_ϵ^z in $(**)$ is such that $v_{g_\epsilon^z}$ satisfies

$$\|w^i - v_{g_\epsilon^z}\|_{H^1(D)} = O(\epsilon).$$

Modified Far Field Operator

Theorem

1. Assume that λ is not a Stekloff eigenvalue and let g_ϵ^z satisfy $(**)$. Then if $v_{g_\epsilon^z}$ is the Herglotz wave function with kernel g_ϵ^z for every $z \in D$ we have that $\|v_{g_\epsilon^z}\|_{H^1(D)}$ is bounded as $\epsilon \to 0$.
2. Assume that λ is a Stekloff eigenvalue and let $g_\epsilon^z \in L^2(S^2)$ satisfy $(**)$. Then for all $z \in D$, except for possibly a nowhere dense subset, we have that $\|v_{g_\epsilon^z}\|_{H^1(D)}$ cannot be bounded as $\epsilon \to 0$.

Remark If λ is a Stekloff eigenvalue then $(**)$ is valid unless the Stekloff eigenfunction can be uniquely continued as a solution of $\Delta w + k^2 n(x) w = 0$ into all of $\mathbb{R}^3$.

Stekloff Eigenvalues

Theorem

Assume that $n(x)$ is real valued. Then Stekloff eigenvalues exist, are real and are discrete.

The case when $n(x)$ is complex valued (i.e. the scattering object is absorbing) is more difficult since then the eigenvalue problem is no longer self-adjoint.

Theorem

Assume that $n(x) = n_1(x) + i\frac{n_2(x)}{k}$ when $n_1 > 0$ and $n_2 > 0$. Then

1. There exist infinitely many Stekloff eigenvalues in the complex plane and they form a discrete set without finite accumulation points.
2. Except for a finite number of eigenvalues, all the Stekloff eigenvalues lie in a wedge of arbitrarily small angle with lower edge on the negative x-axis.

Stekloff Eigenvalues

The advantage of using Stekloff eigenvalues as a target signature are the following:

1. The interrogation frequency can be chosen arbitrarily.
2. The method of Stekloff eigenvalues in principle applies to both absorbing and non-absorbing media.

Remark: The Stekloff eigenvalue problem for Maxwell's equations is an open problem. In particular, when D is a ball it can be shown that if λ is a Maxwell's Stekloff eigenvalue than so is $1/\lambda$ i.e. the use of compact operators to establish the existence and discreteness of Stekloff eigenvalues is problematic.

Stekloff Eigenvalues and Nondestructive Testing

In nondestructive testing one is interested in small changes in the inhomogeneity $n(x)$. In particular, suppose $n(x)$ is perturbed by δn giving rise to a change in the Stekloff eigenfunction $w \in H^1(D)$ by δw and Stekloff eigenvalue by $\delta\lambda$. Define

$$(f, g) := \int_D f\,\overline{g}\,dx, \qquad \langle f, g\rangle := \int_{\partial D} f\,\overline{g}\,ds.$$

Then, neglecting quadratic terms, we have that

$$\delta\lambda \approx \frac{k^2\,(\delta n w, w)}{\langle w, w\rangle}.$$

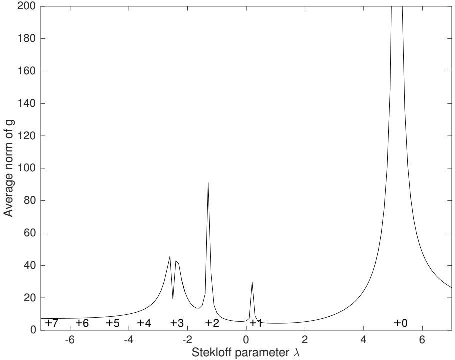
We will illustrate the applicability of this formula in the numerical examples that follows.

Remarks on Numerical Examples

- The preceding formula suggests that some eigenvalues are more susceptible to changes in $n(x)$ than others.
- The first two or three Stekloff eigenvalues (ordered by absolulte value) are the ones best approximated using the far field data for the wave number we have used.
- Examples are given using far field data. However, near field data can also be used just as easily.

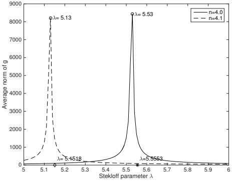
Stekloff Eigenvalues: Unit Disk

- D is the unit disk
- $n(x) = 4$, $k = 1$
- Arbitrary 51 incoming waves
- No extra noise on the data
- Eigenvalues are exact and shown by $+$ in the graph.

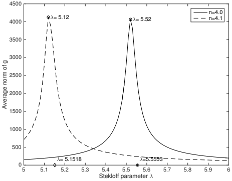


Sensitivity to Noise

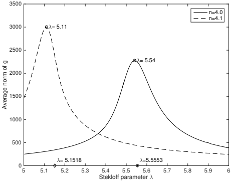
- Same example as before but $n(x) = 4$ or $n(x) = 4.1$
- Noise added pointwise
- Percentage is the relative ℓ^2 norm



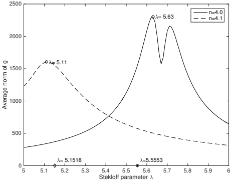
$\epsilon = 0(0\%)$



$\epsilon = 0.01(.57\%)$



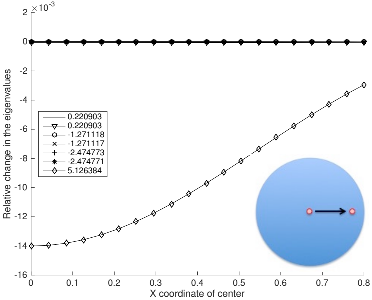
$\epsilon = 0.05(2.9\%)$



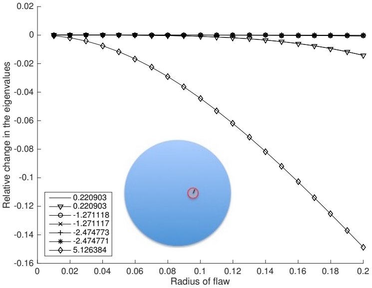
$\epsilon = 0.15(8.6\%)$

Sensitivity of Eigenvalues: Unit Disk with Flaw

The "flaw" is a circular region of radius r_c centered at $(x_c, 0)$ with $n(x) = 1$ inside the flaw. Noise $\epsilon = 0.01$. Wavenumber $k = 1$.



Changing x_c, $r_c = 0.05$

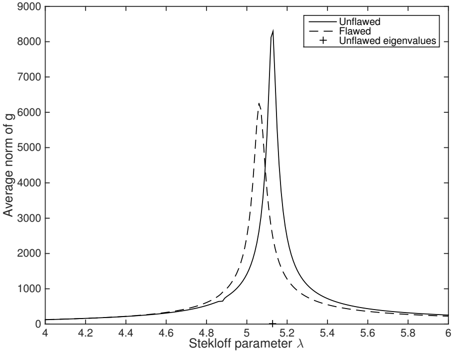


Changing r_c, $x_c = 0.3$.

Plot $(\lambda^c_{j*} - \lambda_j)/|\lambda_j|$, $j = 1, \cdots, 7$ against geometric parameters.

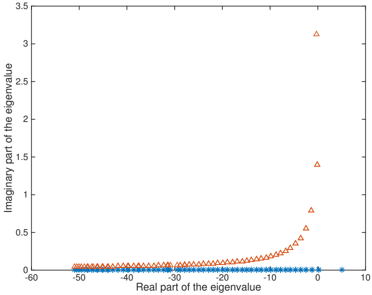
Changes in Eigenvalues: Unit Disk with Flaw

Flaw is radius $r_c = 0.05$ centered at $(0.3, 0)$. All parameters as in the previous examples.

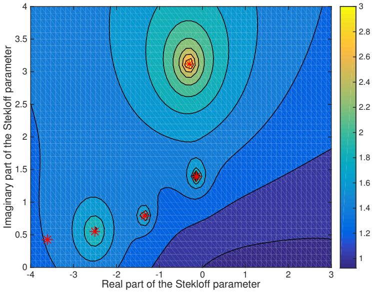


Complex Eigenvalues: Unit Disk $n(x) = 4 + 4i$

Complex eigenvalues can be detected by the same procedure as before but now searching in a region in the complex plane.



Comparison of eigenvalues for $n(x) = 4$ (blue) and $n(x) = 4 + 4i$ (red)



Contours of $\log_{10}(\|g\|)$. Exact Stekloff eigenvalues are shown as $*$.

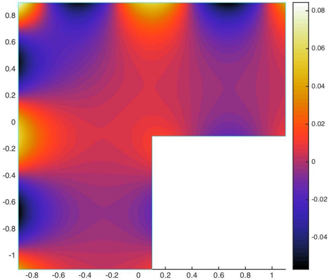
Replacing D by a Ball

The impedance boundary value problem for h can be replaced by an impedance boundary value problem where the boundary condition is prescribed on the boundary of a ball B centered at the origin and containing D in its interior instead of on the boundary of D.

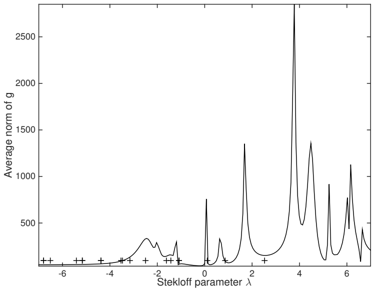
- The price for doing this is that the Stekloff eigenvalues are less sensitive to changes in the refractive index than if the boundary condition in h is prescribed on the boundary of D.
- The advantage of doing this is that for complex geometries the Stekloff eigenvalues are more accurately computed from the modified far field equation.

Replacing D by a Ball

D is L-shaped domain with $n(x) = 4$.



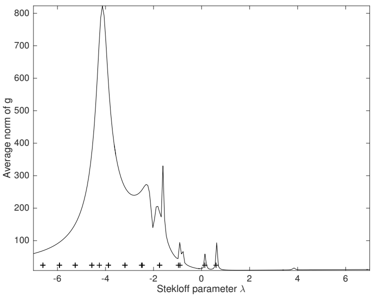
The L-shaped domain



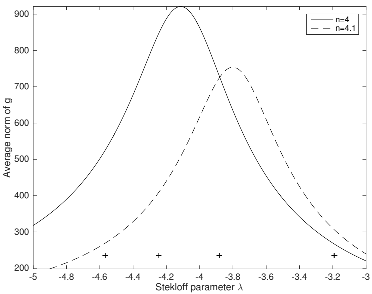
Plot of eigenvalues for h prescribed on ∂D computed using modified far field field operator. Noise $\epsilon = 0$

Replacing D by a Ball

B is a disk of radius 1.5 centered at the origin
D is again L-shaped domain with $n(x) = 4$.



Plot of eigenvalues for h prescribed on ∂B computed using modified far field operator. Noise $\epsilon = 0$



Change in main peak at left when $n(x)$ changes from 4 to 4.1. Noise $\epsilon = 0.01$