Boundary control in transport and diffusion equations

Guillaume Bal

Department of Applied Physics & Applied Mathematics

Columbia University

High Contrast - High Resolution Inverse problems

High *contrast* comes from optical/elastic parameters.

High *resolution* comes from MRI/Ultrasound.

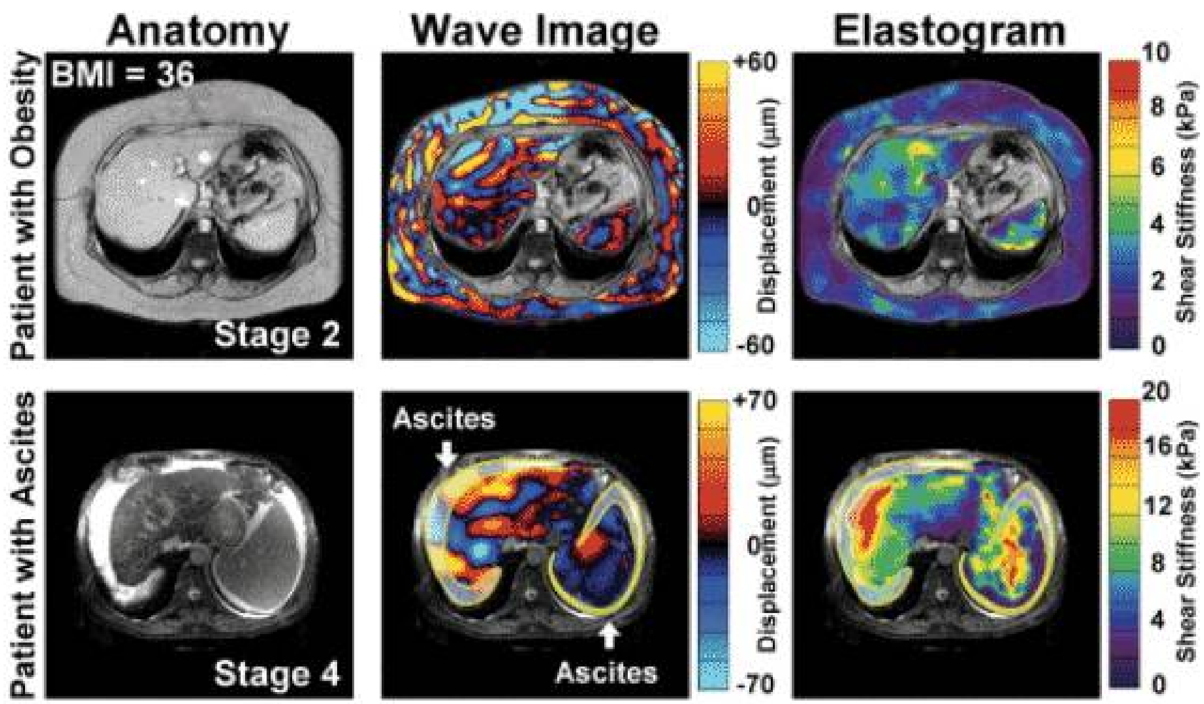
Assume physical coupling between high contrast and high resolution.

Inversion of the high resolution modality (MRI/Ultrasound) provides *internal information* about optical/elastic parameters.

Reconstruction of optical/elastic parameters from *internal information* requires diffusion solutions to satisfy **qualitative properties**.

Boundary controls are sought to ensure **qualitative properties** hold.

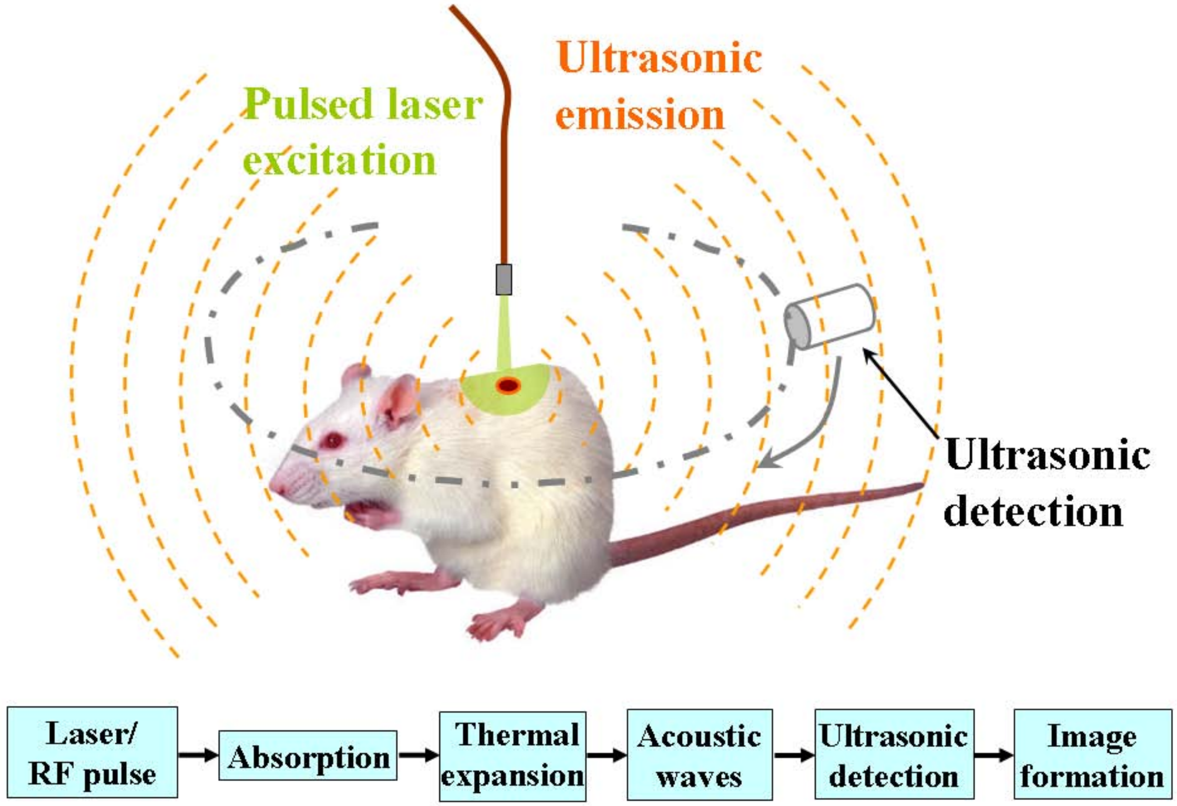
Magnetic Resonance Elastography



Hepatic Fibrosis by Liver Stiffness (Mayo Clinic)

Coupling between Elastic Waves and Magnetic Resonance Imaging.
MRI provides $u =$ **Elastic Displacement.**

The Photo-acoustics Effect



Coupling between Optical Radiation and Ultrasound

Inverse Ultrasound provides $\sigma u =$ **absorption** $\times$ **energy density** .

Quantitative reconstruction

The above and many other examples may be recast as the reconstruction of coefficients in a *second-order elliptic equation* **from knowledge of some of its solutions** $\{u_j\}_{1\le j\le J}$:

$$a_{ij}\partial^2_{ij}u_k + b_i\partial_i u_k + cu_k = 0, \quad x \in X; \qquad u_k = g_k, \quad x \in \partial X.$$

Each u_k provides a *linear constraint* on (a, b, c).

Can we find $\{g_k\}_{1\le k\le J}$ such that $\{(\partial^2_{ij}u_k, \partial_i u_k, u_k)\}_{1\le k\le J}$ are of maximal rank compatible with above constraint in $Y \subset X$?

Locally, maximal rank family may be constructed by plane waves or generalizations of harmonic polynomials. Can such local solutions be *controlled* from the boundary ∂X?

Runge Property and Boundary Control

Approximate boundary control amounts to verifying the **Runge approximation property**: Let $X_1 \subset X_2$ two (simply connected) open domains. Solutions in X_2 restricted to X_1 are *dense* in the set of solutions in X_1. Traces of solutions on the larger domain on ∂X_2 provide appropriate boundary controls for solutions in X_1.

The **Runge approximation** is *equivalent* for second-order elliptic operators to the **Unique Continuation Property** (UCP): two solutions of an elliptic equation on X_2 and equal on an open $X_1 \subset X_2$ have to be equal on X_2. [Lax CPAM 56]

In the context of hybrid inverse problems for scalar equations and systems of second-order equations, see [B. InsideOut 2013]; [B. Uhlmann CPAM 2013]; [B. Monard Uhlmann SIAP 2016].

Local, global, uniform controls

The *Runge Approximation Property* allows us to **control a property of interest** (such as, e.g., linear independence of gradients of solutions) **locally.**

Can we get **global** control (over the whole domain X) ? In general, we do not know how to do so, unless explicit global solutions (harmonic polynomials, CGO solutions) can be constructed.

The control depends on the geometry (coefficients a_{ij}, domain X). Can we get a control (a choice of boundary conditions) **uniform** in the coefficients? **Not in general**: for instance, in dimension $n \geq 3$, we can construct for any given g on ∂X a (scalar) conductivity $a = a(x)$ such that $\nabla u(x_0) = 0$ for some x_0 in the vicinity of a chosen point [Alberti B. Di Cristo, 2017; Capdeboscq 2016].

[B. InsideOut 2013]

Hybrid inverse problem and transport equation

Assume that (optical) radiation is now modeled by a **transport equation** posed on a convex bounded domain $X \subset \mathbb{R}^n$; $V = \mathbb{S}^{n-1}$:

$$v \cdot \nabla u + \sigma(x,v)u = \int_V k(x,v',v)u(x,v')dv', \quad (x,v) \in X \times V$$

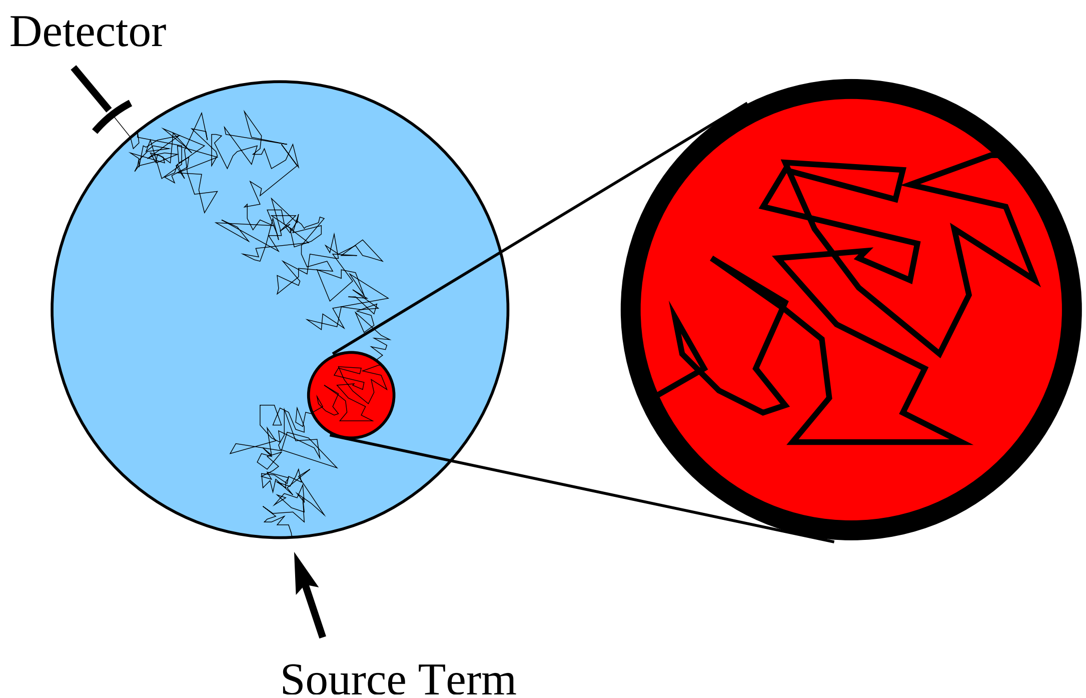
$$u = g, \quad (x,v) \in \Gamma_-; \qquad \Gamma_\pm = \{(x,v) \in \partial X \times V, \pm v \cdot n(x) > 0\}.$$

Assume that optical radiation is coupled to a high resolution modality (ultrasound) that provides $u(x,v)$. [B. Chung Schotland 2015]

Optical coefficients $(\sigma(x,v), k(x,v',v))$ are then fully reconstructed provided $u(x,v)$ can locally (in x) take any **prescribed** $v-$**dependent shape**.

This raises the question of **boundary control of transport solutions**.

Transport versus Diffusion



The solution of the **transport equation** $v \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon}\sigma_s(x)(u_\varepsilon - \int_V u_\varepsilon dv) + \varepsilon\sigma_a(x)u_\varepsilon = 0$ converges as $\varepsilon \to 0$ to the solution of the **diffusion equation** (elliptic second-order equation) : $-\nabla \cdot \frac{1}{n\sigma_s(x)}\nabla U + \sigma_a(x)U = 0$.

Control problems

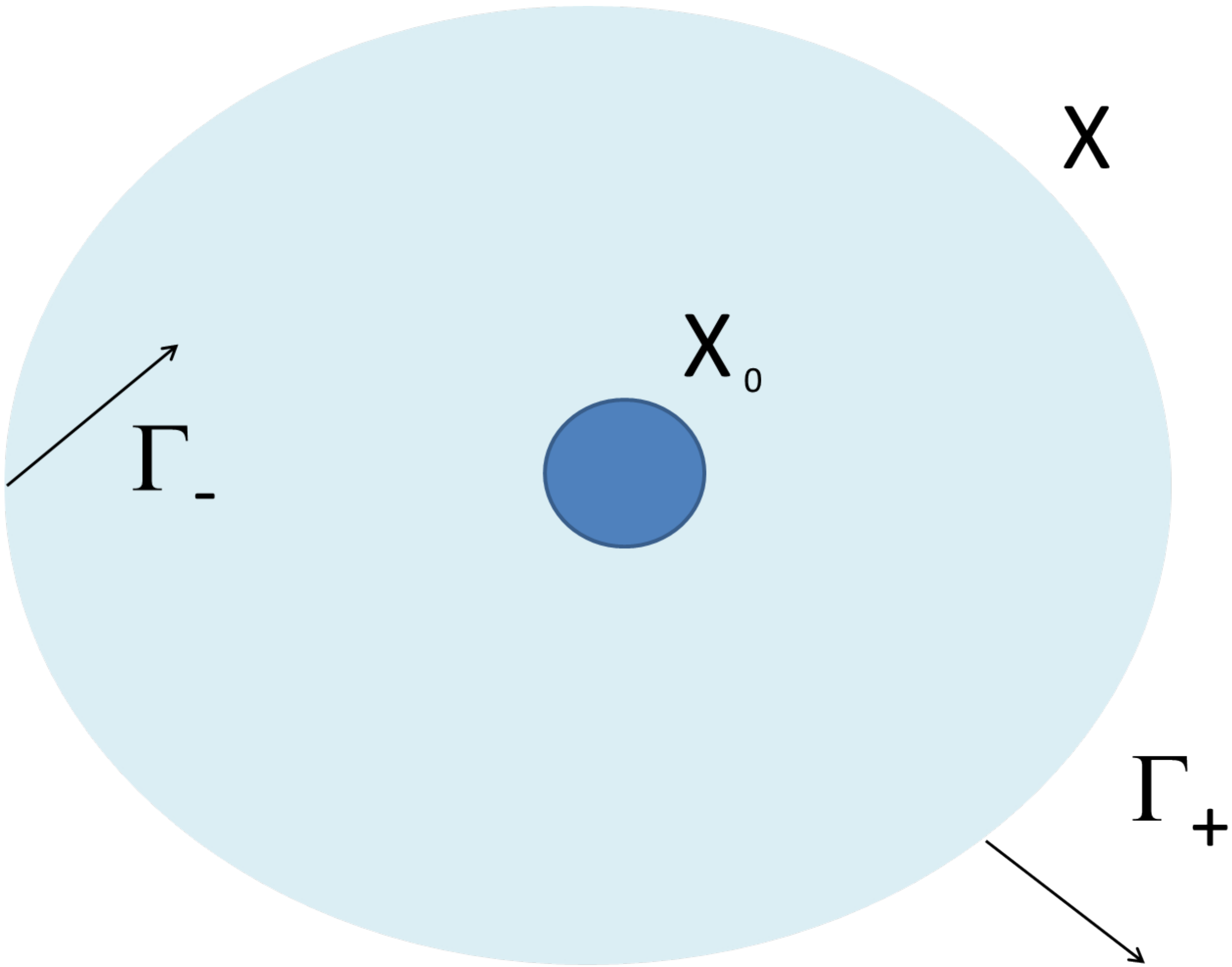
Coupled physics inverse problems require **boundary controls**.

For diffusion equations, this is possible thanks to the unique continuation principle.

Transport solutions in the high scattering (small mean free path) regime are well approximated by diffusion solutions.

- Can we control transport solutions from the boundary?
- Is this related to a unique continuation principle?
- Is the latter even valid?

Geometry of transport problem



Control of Outgoing Transport Solution

$$Tu := v \cdot \nabla u + \sigma(x,v)u = \int_V k(x,v',v)u(x,v')dv' =: Ku, \quad (x,v) \in X \times V$$

$$u = g, \quad (x,v) \in \Gamma_-; \qquad \Gamma_\pm = \{(x,v) \in \partial X \times V, \pm v \cdot n(x) > 0\}.$$

Assume that $0 \leq \int_V k(x,v',v)dv' \leq \sigma(x,v)$ (**sub-criticality** condition).

Then **albedo** $\mathcal{A} : g \mapsto u_{|\Gamma_+}$ **well posed** from $L^2(\Gamma_-, d\xi)$ to $L^2(\Gamma_+, d\xi)$.

Control of outgoing solutions:

Can we find g on Γ_- such that $u_{|\Gamma_+}$ is prescribed?

More precisely, what is the Range of $\mathcal{A}$?

Fredholm theory

Using a *contraction argument* or a *compact perturbation argument*, we obtain that $\mathcal{A}$ is a **Fredholm operator** with vanishing index so that:

$$L^2(\Gamma_+, d\xi) = \mathrm{Ran}(\mathcal{A}) \oplus \mathrm{Ker}(\mathcal{A}^*), \quad L^2(\Gamma_-, d\xi) = \mathrm{Ran}(\mathcal{A}^*) \oplus \mathrm{Ker}(\mathcal{A}).$$

Any g with a non-trivial component in $\mathrm{Ker}(\mathcal{A}^*)$ therefore cannot be *controlled* from Γ_-.

Note that any control g on Γ_- in $\mathrm{Ker}(\mathcal{A})$ leaves no trace on Γ_+ !

Integral formulation and contraction argument

Consider the problem

$$v \cdot \nabla u + \sigma u =: Tu = Ku, \quad (x, v) \in X \times V, \qquad u = g, \quad (x, v) \in \mathcal{C}.$$

Here $\mathcal{C}$ is either $\Gamma_\pm$ or any conditions such that $Tu = 0$ with prescribed (Dirichlet) conditions on $\mathcal{C}$ is invertible. Then the above is equivalent to

$$u = T_{\mathcal{C}}^{-1} K u + L_{\mathcal{C}} g.$$

For k and hence K sufficiently small, the above problem with $\mathcal{C} = \Gamma_+$ admits a unique solution and $\mathcal{A} : L^2(\Gamma_-, d\xi) \to L^2(\Gamma_+, d\xi)$ is invertible.

For sufficiently small scattering k, the boundary control is *exact* with $L^2(\Gamma_+, d\xi) = \mathrm{Ran}(\mathcal{A})$.

Compactness argument

Under mild assumptions on K, we show following [Mikhlin; Mokhtar-Kharroubi; GLPS 88] that $T_{\mathcal{C}}^{-1}K$ is a compact operator. Using $T_{\pm}^{-1}$ for $\mathcal{C} = \Gamma_{\pm}$, let

$$N = \text{Ker}(I - T_{+}^{-1}K) \neq \{0\}, \qquad N_{-} = \{u_{|\Gamma_{-}},\ u \in N\} = \text{Ker}(\mathcal{A}).$$

Note $\text{Ker}(I - T_{-}^{-1}K) = \{0\}$ by sub-criticality and hence $\text{dim}N = \text{dim}N_{-}$.

Then identify compatibility conditions to solve $(I - T_{+}^{-1}K)v = L_{+}g$ with those to solve $\mathcal{A}^{*}h = g$ to obtain that $L^2(\Gamma_{-}, d\xi) = \text{Ran}(\mathcal{A}^{*}) \oplus \text{Ker}(\mathcal{A})$. Pass to adjoints for decomposition of $L^2(\Gamma_{+}, d\xi)$.

It remains to understand whether $\mathcal{N} = \text{dim}N > 0$ is possible.

Non-controllability

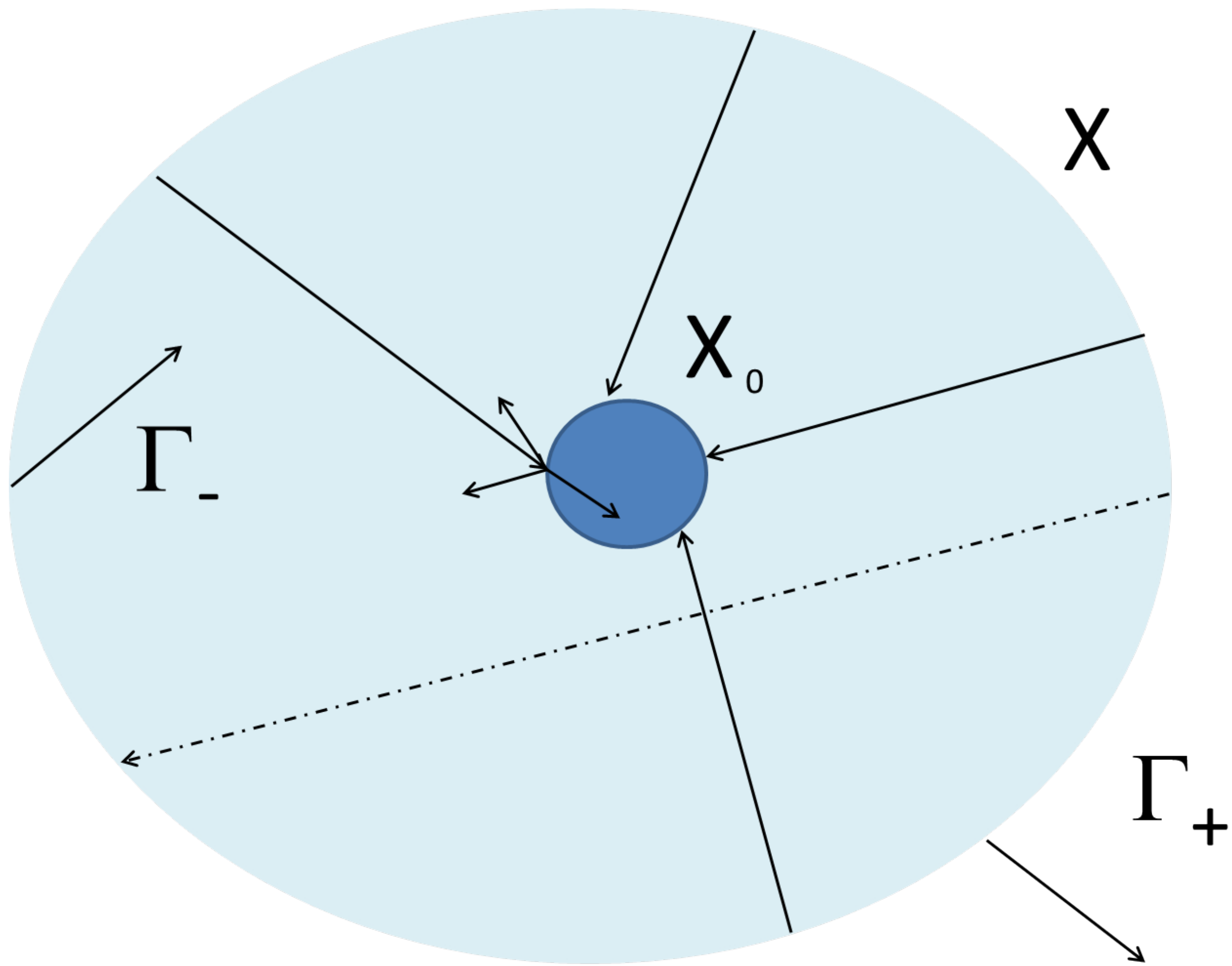
There are coefficients $k(x) \leq \sigma(x)$ such that $\mathcal{N} = \dim \mathrm{Ker}(I - T_+^{-1}K) > 0$.

This implies that $\dim \mathrm{Ker}(\mathcal{A}) = \dim \mathrm{Ker}(\mathcal{A}^*) = \mathcal{N} > 0$.

This implies the existence of $g \neq 0$ on Γ_- such that $\mathcal{A}g = 0$ on Γ_+: **this is a non-radiating boundary condition.**

This implies the existence of $\mathcal{N}$ **compatibility conditions** on h defined on Γ_+ such that h can be *controlled* by some g on Γ_-, i.e., $h = \mathcal{A}g$. **Not every outgoing boundary condition can be controlled.**

Geometry of internal control problem



Internal controls

Let us come back to the **internal control problem** and consider X *strictly convex* and $X_0 \Subset X$ also *strictly convex*. Let u_0 be a solution of

$$v \cdot \nabla u_0 + \sigma u_0 = \int_V k(x, v', v) u_0(x, v') dv' \quad X_0 \times V$$

and let g_0 denote its trace on $\Gamma_+(X_0) \cup \Gamma_-(X_0)$. (essentially $\partial X_0 \times V$.)

Can we find u solution to the following problem?

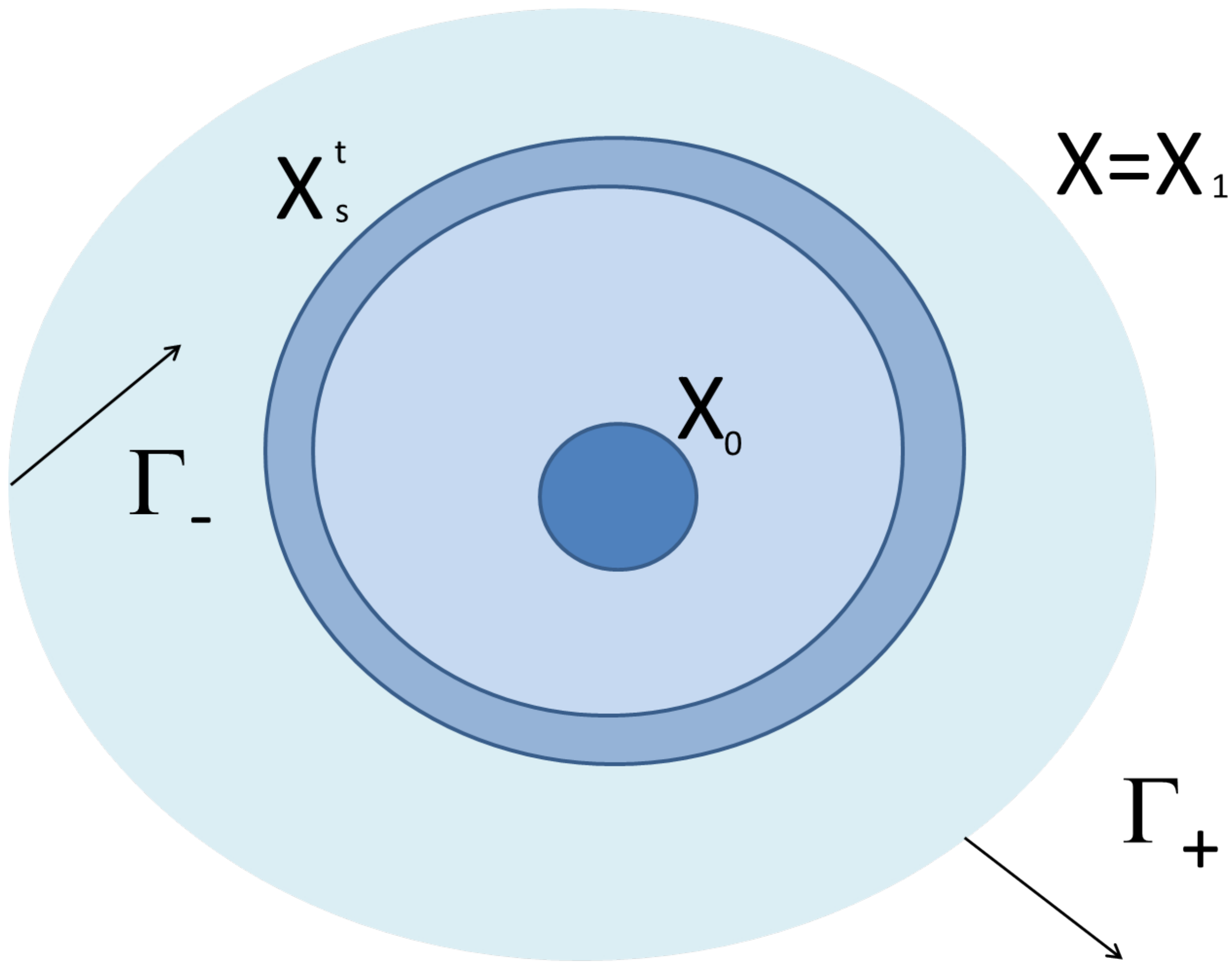
$$\begin{aligned} & v \cdot \nabla u + \sigma u = \int_V k(x, v', v) u(x, v') dv' \quad X_0^1 \times V, \quad X_0^1 = X \backslash \overline{X_0} \\ & u = g_0 \text{ on } \Gamma_+(X_0) \cup \Gamma_-(X_0). \end{aligned}$$

Theorem. **The answer is positive. The solution is not unique.**

Joint with Alexandre Jollivet; see also [B. Chung Schotland 2015].

For $g = u_{\Gamma_-(X)}$, we obtain that $u_{|X_0}$ is equal to the prescribed u_0. We have an **exact internal control**.

Derivation by layer peeling argument



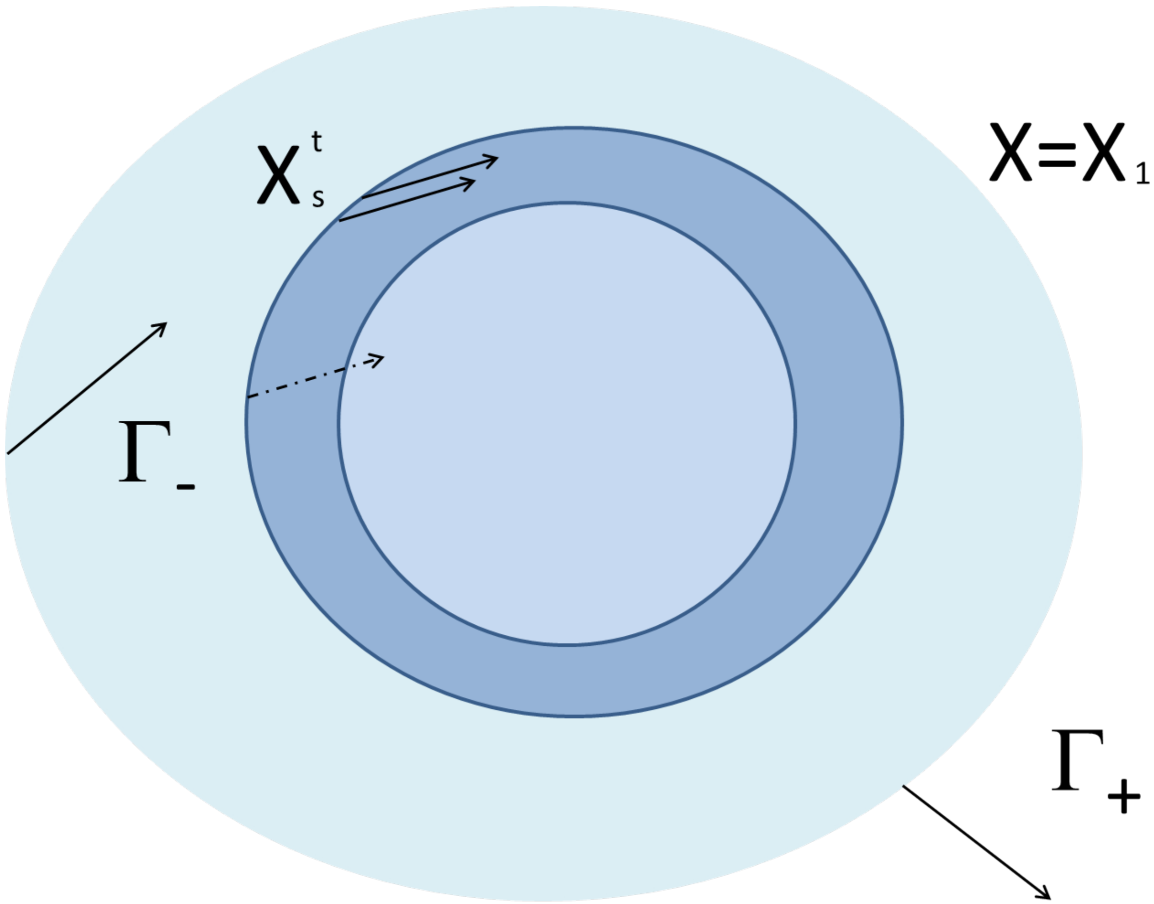
No Unique continuation in transport

Assume u solution on X with $u = 0$ on X_0. Then is $u = 0$ on X?

Result. The answer is negative: **UCP does not hold for transport**.

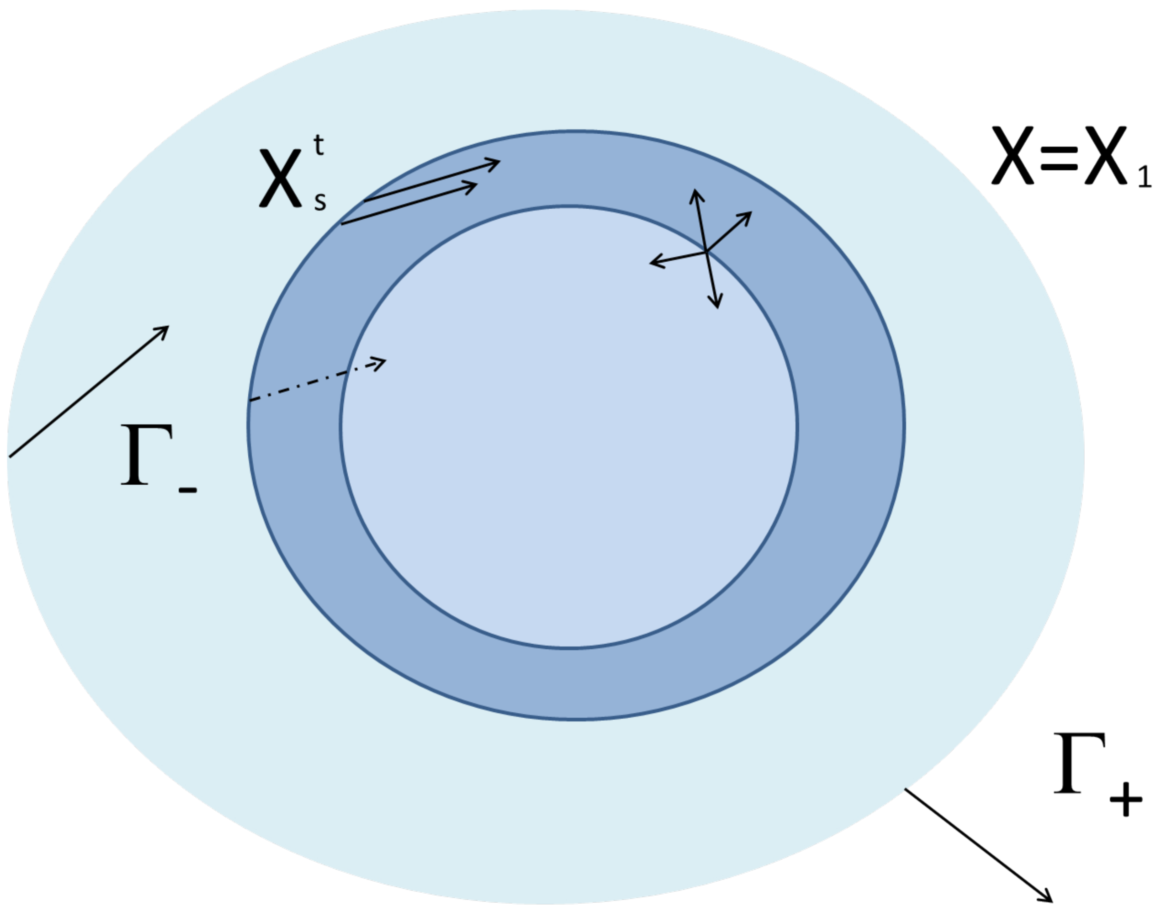
Consider the last layer $X_{s_{J-1}}^{s_J}$ and consider $h \neq 0$ with support in $\Gamma_-(X)$ on the set of incoming conditions that do not meet $Y_{s_{J-1}}$. Let u_h be the solution on X_0^1 with such a boundary condition and vanishing incoming conditions on Y_0. Let g_0 be the trace of u_h on ∂X_0. Let u be constructed as in the preceding result so that in particular $u = 0$ on the support of h. Consider $v = u - u_h$. Then $v = 0$ on ∂X_0 and is extended by 0 on X_0. v solves the transport equation on X and $v = -h$ on the support of h in $\Gamma_-(X)$. As a consequence, the transport solution v cannot vanish on the whole of X even though it vanishes on $X_0 \times V$.

No Unique Continuation Property



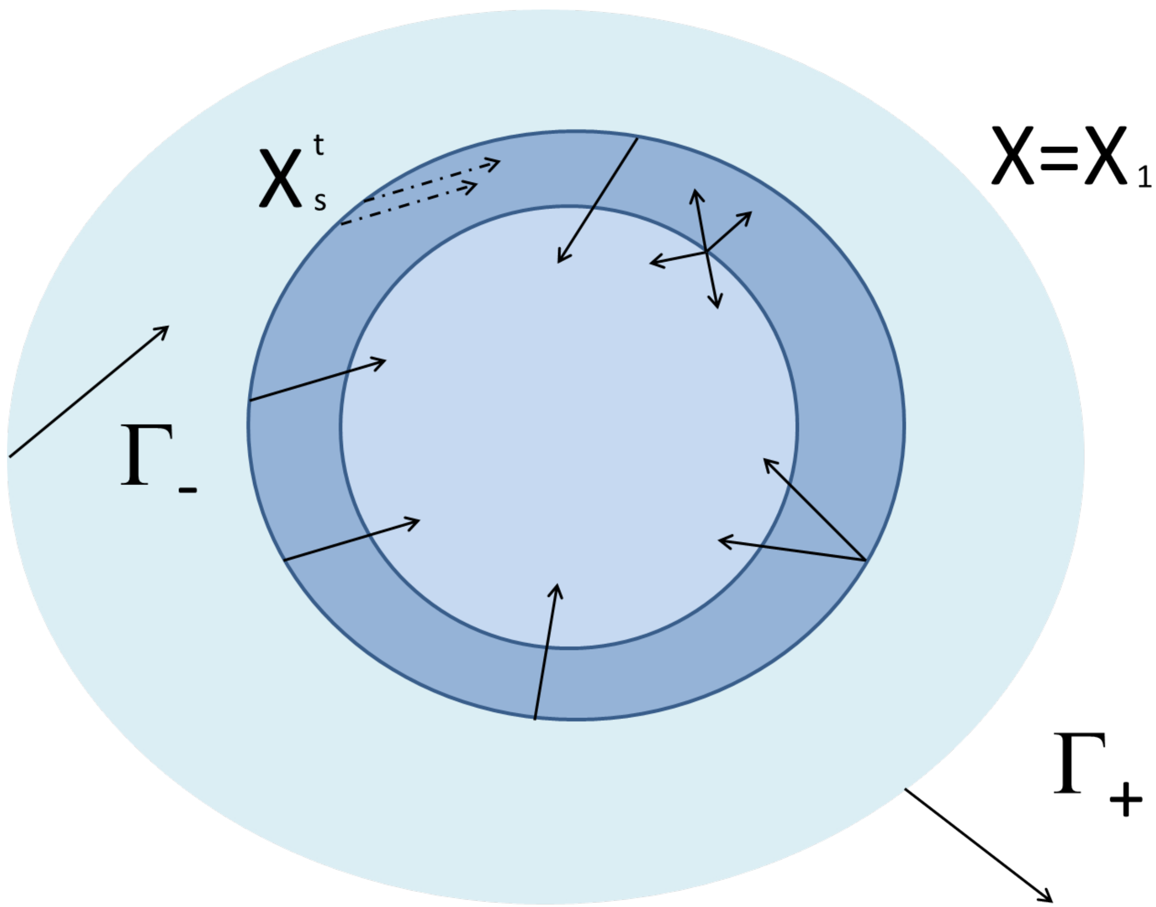
Consider an incoming condition on $\Gamma_-(X_t)$ on lines not crossing X_s

No Unique Continuation Property



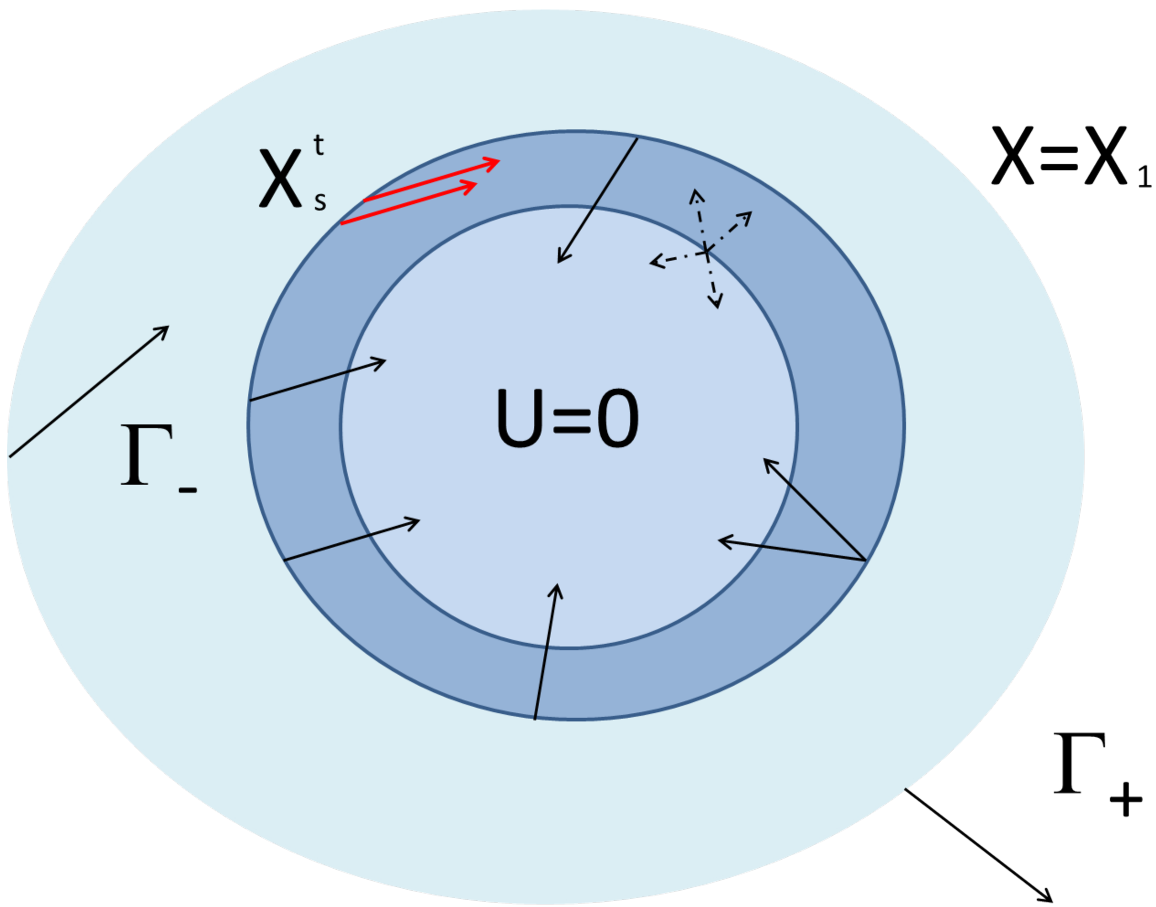
Solve Transport on large domain X_t and identify trace on $\partial X_s \times V$.

No Unique Continuation Property



Control trace on $\partial X_s \times V$ from boundary ∂X_t: lines not crossing X_s not necessary.

No Unique Continuation Property



Add negative of initial radiation condition on ∂X_t to violate UCP.

Conclusions

Heuristically, in an inverse problems/imaging method, we wish the *probing signal* to be as "varied" and "informative" as possible.

Realizing this variability from the available part of the boundary amounts to a **boundary control** problem.

For diffusion equations, **approximate** boundary control is possible as an application of a **unique continuation property.** Such a control is an *unstable* process.

For transport equations, **exact** boundary control is feasible even though **UCP does not hold**. However, the stability degrades as the mean free path tends to 0.