Boundary control in transport and diffusion equations

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**High Contrast - High Resolution Inverse problems**

High *contrast* comes from optical/elastic parameters.

High *resolution* comes from MRI/Ultrasound.

Assume *physical coupling* between high contrast and high resolution.

Inversion of the *high resolution* modality (MRI/Ultrasound) provides *internal information* about optical/elastic parameters.

Reconstruction of optical/elastic parameters from *internal information* requires diffusion solutions to satisfy *qualitative properties*.

Boundary controls are sought to ensure *qualitative properties* hold.

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Magnetic Resonance Elastography

Hepatic Fibrosis by Liver Stiffness (Mayo Clinic)

**Coupling** between Elastic Waves and Magnetic Resonance Imaging.

MRI provides $u = \text{Elastic Displacement}$. 

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The Photo-acoustics Effect

Coupling between Optical Radiation and Ultrasound
Inverse Ultrasound provides $\sigma u = \text{absorption} \times \text{energy density}$.
Quantitative reconstruction

The above and many other examples may be recast as the reconstruction of coefficients in a second-order elliptic equation from knowledge of some of its solutions $\{u_j\}_{1 \leq j \leq J}$:

$$a_{ij} \partial_{i,j}^2 u_k + b_i \partial_i u_k + cu_k = 0, \quad x \in X; \quad u_k = g_k, \quad x \in \partial X.$$ 

Each $u_k$ provides a linear constraint on $(a, b, c)$.

Can we find $\{g_k\}_{1 \leq k \leq J}$ such that $\{(\partial_{i,j}^2 u_k, \partial_i u_k, u_k)\}_{1 \leq k \leq J}$ are of maximal rank compatible with above constraint in $Y \subset X$?

**Locally**, maximal rank family may be constructed by plane waves or generalizations of harmonic polynomials. Can such local solutions be controlled from the boundary $\partial X$?
Runge Property and Boundary Control

Approximate boundary control amounts to verifying the Runge approximation property: Let $X_1 \subset X_2$ two (simply connected) open domains. Solutions in $X_2$ restricted to $X_1$ are dense in the set of solutions in $X_1$. Traces of solutions on the larger domain on $\partial X_2$ provide appropriate boundary controls for solutions in $X_1$.

The Runge approximation is equivalent for second-order elliptic operators to the Unique Continuation Property (UCP): two solutions of an elliptic equation on $X_2$ and equal on an open $X_1 \subset X_2$ have to be equal on $X_2$. [Lax CPAM 56]

In the context of hybrid inverse problems for scalar equations and systems of second-order equations, see [B. InsideOut 2013]; [B. Uhlmann CPAM 2013]; [B. Monard Uhlmann SIAP 2016].
Local, global, uniform controls

The Runge Approximation Property allows us to control a property of interest (such as, e.g., linear independence of gradients of solutions) locally.

Can we get global control (over the whole domain $X$)? In general, we do not know how to do so, unless explicit global solutions (harmonic polynomials, CGO solutions) can be constructed.

The control depends on the geometry (coefficients $a_{ij}$, domain $X$). Can we get a control (a choice of boundary conditions) uniform in the coefficients? **Not in general:** for instance, in dimension $n \geq 3$, we can construct for any given $g$ on $\partial X$ a (scalar) conductivity $a = a(x)$ such that $\nabla u(x_0) = 0$ for some $x_0$ in the vicinity of a chosen point [Alberti B. Di Cristo, 2017; Capdeboscq 2016].

[B. InsideOut 2013]
Hybrid inverse problem and transport equation

Assume that (optical) radiation is now modeled by a transport equation posed on a convex bounded domain $X \subset \mathbb{R}^n$; $V = S^{n-1}$:

$$v \cdot \nabla u + \sigma(x, v)u = \int_V k(x, v', v)u(x, v')dv', \quad (x, v) \in X \times V$$

$$u = g, \quad (x, v) \in \Gamma_-; \quad \Gamma_{\pm} = \{(x, v) \in \partial X \times V, \pm v \cdot n(x) > 0\}.$$

Assume that optical radiation is coupled to a high resolution modality (ultrasound) that provides $u(x, v)$. [B. Chung Schotland 2015]

Optical coefficients $(\sigma(x, v), k(x, v', v))$ are then fully reconstructed provided $u(x, v)$ can locally (in $x$) take any prescribed $v$–dependent shape. This raises the question of boundary control of transport solutions.

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Boundary controls
The solution of the transport equation
\[ v \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon} \sigma_s(x)(u_\varepsilon - \int_V u_\varepsilon dv) + \varepsilon \sigma_a(x)u_\varepsilon = 0 \]
converges as \( \varepsilon \to 0 \) to the solution of the diffusion equation (elliptic second-order equation):
\[ -\nabla \cdot \frac{1}{n\sigma_s(x)} \nabla U + \sigma_a(x)U = 0. \]
Control problems

Coupled physics inverse problems require boundary controls.

For diffusion equations, this is possible thanks to the unique continuation principle.

Transport solutions in the high scattering (small mean free path) regime are well approximated by diffusion solutions.

• Can we control transport solutions from the boundary?
• Is this related to a unique continuation principle?
• Is the latter even valid?
Geometry of transport problem

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Boundary controls
Control of Outgoing Transport Solution

\[ Tu := v \cdot \nabla u + \sigma(x,v)u = \int_V k(x,v',v)u(x,v')dv' =: Ku, \quad (x,v) \in X \times V \]
\[ u = g, \quad (x,v) \in \Gamma_--; \quad \Gamma_\pm = \{(x,v) \in \partial X \times V, \pm v \cdot n(x) > 0\}. \]

Assume that \( 0 \leq \int_V k(x,v',v)dv' \leq \sigma(x,v) \) (sub-criticality condition).

Then albedo \( A : g \mapsto u|_{\Gamma_+} \) well posed from \( L^2(\Gamma_-,d\xi) \) to \( L^2(\Gamma_+,d\xi) \).

Control of outgoing solutions:

Can we find \( g \) on \( \Gamma_- \) such that \( u|_{\Gamma_+} \) is prescribed?

More precisely, what is the Range of \( A \)?
Fredholm theory

Using a *contraction argument* or a *compact perturbation argument*, we obtain that $\mathcal{A}$ is a **Fredholm operator** with vanishing index so that:

\[ L^2(\Gamma_+, d\xi) = \text{Ran}(\mathcal{A}) \oplus \text{Ker}(\mathcal{A}^*), \quad L^2(\Gamma_-, d\xi) = \text{Ran}(\mathcal{A}^*) \oplus \text{Ker}(\mathcal{A}). \]

Any $g$ with a non-trivial component in $\text{Ker}(\mathcal{A}^*)$ therefore cannot be *controlled* from $\Gamma_-$. Note that any control $g$ on $\Gamma_-$ in $\text{Ker}(\mathcal{A})$ leaves no trace on $\Gamma_+$!
Integral formulation and contraction argument

Consider the problem

\[ v \cdot \nabla u + \sigma u =: Tu = Ku, \quad (x, v) \in X \times V, \quad u = g, \quad (x, v) \in C. \]

Here \( C \) is either \( \Gamma_{\pm} \) or any conditions such that \( Tu = 0 \) with prescribed (Dirichlet) conditions on \( C \) is invertible. Then the above is equivalent to

\[ u = T_C^{-1}Ku + L_Cg. \]

For \( k \) and hence \( K \) sufficiently small, the above problem with \( C = \Gamma_{+} \) admits a unique solution and \( A : L^2(\Gamma_-, d\xi) \to L^2(\Gamma_+, d\xi) \) is invertible.

For sufficiently small scattering \( k \), the boundary control is exact with \( L^2(\Gamma_+, d\xi) = \text{Ran}(A) \).
Compactness argument

Under mild assumptions on $K$, we show following [Mikhlin; Mokhtar-Kharroubi; GLPS 88] that $T^{-1}_C K$ is a compact operator. Using $T^{-1}_\pm$ for $C = \Gamma_\pm$, let

$$N = \text{Ker}(I - T^{-1}_+ K) \neq \{0\}, \quad N_- = \{u|_{\Gamma_-}, \ u \in N\} = \text{Ker}(A).$$

Note $\text{Ker}(I - T^{-1}_- K) = \{0\}$ by sub-criticality and hence $\dim N = \dim N_-$. Then identify compatibility conditions to solve $(I - T^{-1}_+ K)v = L_+ g$ with those to solve $A^* h = g$ to obtain that $L^2(\Gamma_-, d\xi) = \text{Ran}(A^*) \oplus \text{Ker}(A)$. Pass to adjoints for decomposition of $L^2(\Gamma_+, d\xi)$.

It remains to understand whether $N = \dim N > 0$ is possible.

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Non-controllability

There are coefficients $k(x) \leq \sigma(x)$ such that $\mathcal{N} = \dim \ker(I - T_+^{-1} K) > 0$.

This implies that $\dim \ker(A) = \dim \ker(A^*) = \mathcal{N} > 0$.

This implies the existence of $g \neq 0$ on $\Gamma_-$ such that $Ag = 0$ on $\Gamma_+$: this is a non-radiating boundary condition.

This implies the existence of $\mathcal{N}$ compatibility conditions on $h$ defined on $\Gamma_+$ such that $h$ can be controlled by some $g$ on $\Gamma_-$, i.e., $h = Ag$. Not every outgoing boundary condition can be controlled.
Geometry of internal control problem

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Boundary controls
Internal controls

Let us come back to the **internal control problem** and consider $X$ strictly convex and $X_0 \in X$ also strictly convex. Let $u_0$ be a solution of

$$v \cdot \nabla u_0 + \sigma u_0 = \int_{V} k(x, v', v) u_0(x, v') dv' \quad X_0 \times V$$

and let $g_0$ denote its trace on $\Gamma_+(X_0) \cup \Gamma_-(X_0)$. (essentially $\partial X_0 \times V$.) Can we find $u$ solution to the following problem?

$$v \cdot \nabla u + \sigma u = \int_{V} k(x, v', v) u(x, v') dv' \quad X_1^1 \times V, \quad X_0^1 = X \setminus \bar{X_0}$$

$$u = g_0 \text{ on } \Gamma_+(X_0) \cup \Gamma_-(X_0).$$

**Theorem.** The answer is positive. The solution is not unique.

Joint with Alexandre Jollivet; see also [B. Chung Schotland 2015].

For $g = u|_{\Gamma_-}(X)$, we obtain that $u|_{X_0}$ is equal to the prescribed $u_0$. We have an **exact internal control**.
Derivation by layer peeling argument

\[ X_s^t \quad X = X_1 \]

\[ \Gamma_- \quad \Gamma_+ \]
No Unique continuation in transport

Assume $u$ solution on $X$ with $u = 0$ on $X_0$. Then is $u = 0$ on $X$?

Result. The answer is negative: **UCP does not hold for transport.**

Consider the last layer $X^1_{s_{s_{j-1}}}$ and consider $h \neq 0$ with support in $\Gamma_-(X)$ on the set of incoming conditions that do not meet $Y_{s_{j-1}}$. Let $u_h$ be the solution on $X^1_0$ with such a boundary condition and vanishing incoming conditions on $Y_0$. Let $g_0$ be the trace of $u_h$ on $\partial X_0$. Let $u$ be constructed as in the preceding result so that in particular $u = 0$ on the support of $h$. Consider $v = u - u_h$. Then $v = 0$ on $\partial X_0$ and is extended by 0 on $X_0$. $v$ solves the transport equation on $X$ and $v = -h$ on the support of $h$ in $\Gamma_-(X)$. As a consequence, the transport solution $v$ cannot vanish on the whole of $X$ even though it vanishes on $X_0 \times V$. 

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No Unique Continuation Property

Consider an incoming condition on $\Gamma_-(X_t)$ on lines not crossing $X_s$.
No Unique Continuation Property

Solve Transport on large domain $X_t$ and identify trace on $\partial X_s \times V$.

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Boundary controls
No Unique Continuation Property

Control trace on $\partial X_s \times V$ from boundary $\partial X_t$: lines not crossing $X_s$ not necessary.
No Unique Continuation Property

Add negative of initial radiation condition on $\partial X_t$ to violate UCP.
Conclusions

Heuristically, in an inverse problems/imaging method, we wish the *probing signal* to be as "varied" and "informative" as possible.

Realizing this variability from the available part of the boundary amounts to a **boundary control** problem.

For diffusion equations, **approximate** boundary control is possible as an application of a **unique continuation property**. Such a control is an **unstable** process.

For transport equations, **exact** boundary control is feasible even though **UCP does not hold**. However, the stability degrades as the mean free path tends to 0.