# Absence of bound states for waveguides in 2D periodic structures

Maria Radosz
Rice University

(Joint work with Vu Hoang)

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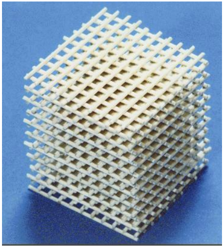
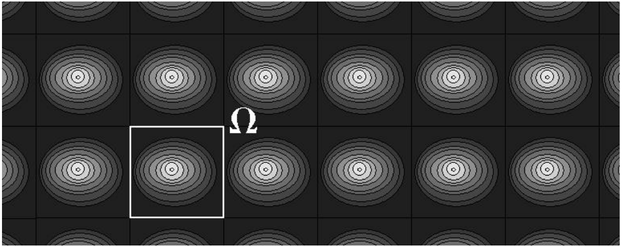
- Introduction: Photonic Crystals and Periodic media
- Introduction: Floquet theory/Absolute continuity of spectra of periodic operators
- The waveguide problem in 2D periodic structure and main result
- Reformulation of the problem
- Analytic continuation of resolvent operators
- Proof of main result

# Photonic Crystals and Periodic media



- Photonic crystal: artificial dielectric material.
- Propagation of e.m. waves, Maxwell equations
- Two-dimensional situation: Helmholtz equation (polarized waves)

# Photonic Crystals and Periodic media




$\Omega = (0, 1)^2$

# Mathematical modeling

- Start from Maxwell's equations:

$$\operatorname{curl} E = -\mu_0 \frac{\partial}{\partial t} H$$
$$\operatorname{curl} H = \varepsilon(x) \frac{\partial}{\partial t} E$$
$$\operatorname{div} H = 0, \operatorname{div} \varepsilon E = 0$$

  where $\varepsilon(x)$ describes the material configuration.

- Assume $E = (0, 0, u)$. This leads to

$$\varepsilon(x) \frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

- Look for time-harmonic solutions $u = e^{i\omega t} v$. This gives

$$-\varepsilon(x)\omega^2 v - \Delta v = 0.$$

# Mathematical problem

- In the following, let $\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be a periodic function:

$$\varepsilon(\mathbf{x}+\mathbf{m}) = \varepsilon(\mathbf{x}) \quad (\mathbf{m} \in \mathbb{Z}^2)$$

  s.t. $\varepsilon \in L^\infty(\mathbb{R}^2),\ 0 < c \leq \varepsilon$.

- Time-independent problem:

$$-\Delta u + \varepsilon(x)\lambda u = 0, \qquad \lambda = \omega^2$$

- Spectral problem: study the spectrum of Helmholtz operator

$$-\frac{1}{\varepsilon}\Delta$$

  where $D(-\frac{1}{\varepsilon}\Delta) = H^2(\mathbb{R}^2)$. Self-adjoint in $\varepsilon$-weighted $L^2$-space.

# Spectrum of periodic operators

- Spectrum has band structure:

$$\operatorname{spec}\left(-\frac{1}{\varepsilon}\Delta\right) = \bigcup_{s\in\mathbb{N}} \lambda_s((-\pi,\pi]^d)$$

  where $\lambda_s(k)$ are band functions, $k \in (-\pi,\pi]^d$ is the *quasi momentum* (wave vector).
- We expect that $\operatorname{spec}\left(-\frac{1}{\varepsilon}\Delta\right)$ has no *point eigenvalues*.
- An eigenvalue implies existence of $u \in L^2(\mathbb{R}^d) \setminus \{0\}$ (bound state) s.t.

$$(-\Delta - \omega^2\varepsilon)u = 0.$$

- Hence, $e^{i\omega t}u$ solves the wave equation $\Rightarrow$ "wave gets stuck"

# Absolute continuity of spectra of periodic operators

Let

$$L = -q^{-1}\nabla \cdot \mathbf{A}\nabla + V$$

be an s.a. operator with periodic coefficients, $D(L) \subset L^2(\mathbb{R}^d)$. Want to prove: $L$ has *absolutely continuous* spectrum.

- Difficult: exclude point spectrum, i.e. prove that $(L-\lambda)u = 0$ has no nontrivial solutions.
- Schrödinger/Helmholtz [Thomas, '73], Magnetic Schrödinger operator [Birman-Suslina, '00], [Sobolev, '02], Divergence form operator with symmetry [Friedlander '03]
- Problem in full generality still unsolved.

# Floquet-Bloch transformation

Recall *constant* coefficient operators are invariant

$$L[u(\cdot + \mathbf{s})] = L[u](\cdot + \mathbf{s}) \quad (\mathbf{s} \in \mathbb{R}^d)$$

w.r.t. arbitrary shifts.

Periodic operators are invariant

$$L[u(\cdot + \mathbf{n})] = L[u](\cdot + \mathbf{n}) \quad (\mathbf{n} \in \mathbb{Z}^d)$$

w.r.t. integer shifts.

Can we construct an analogue of the *Fourier transform*?

# Floquet-Bloch transformation

Gel'fand introduced the following transform ($B = (-\pi, \pi)^d$ = Brillouin zone)

$$(Vf)(\mathbf{x}, \mathbf{k}) := \frac{1}{\sqrt{|B|}} \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{i\mathbf{k}\cdot(\mathbf{n}-\mathbf{x})} f(\mathbf{x} - \mathbf{n})$$

**Theorem**

$V : L^2(\mathbb{R}^d) \to L^2(\Omega \times B) = L^2((0,1)^d \times (-\pi, \pi)^d)$ *is an isometric isomorphism*

$$\| Vf \|_{L^2(\Omega \times B)} = \|f\|_{L^2(\mathbb{R}^d)}.$$

Alternatively: $V : L^2(\mathbb{R}^d) \to L^2(B, L^2(\Omega)), (Vf)(\mathbf{k})(\cdot) = Vf(\cdot, \mathbf{k})$.

# Floquet-Bloch transformation

It is important to understand the action of $V$ on $H^s(\mathbb{R}^d)$. Define

$$H^s_{\mathrm{per}}(\Omega) := \{u \in H^s(\Omega) : Eu \in H^s_{\mathrm{loc}}(\mathbb{R}^d)\},$$

the space of periodic $H^s$-functions. $E$ is the periodic extension operator

$$(Eu)(\mathbf{x}+\mathbf{n}) = u(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^d, \mathbf{n} \in \mathbb{Z}^d).$$

*Example:* $u \in H^1_{\mathrm{per}}(\Omega)$ implies e.g. that $u|_{\{x_1=0\}} = u|_{\{x_1=1\}}$ in the trace sense.

# Floquet-Bloch transformation

**Theorem**

*$V : H^s(\mathbb{R}^d) \to L^2(B, H^s_{per}(\Omega))$ is an topological isomorphism. The inverse transform is*

$$(V^{-1}g)(x) = \frac{1}{\sqrt{|B|}} \int_B g(\mathbf{x}, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}$$

Let a family of cell operators be defined by

$$L(\mathbf{k}) = -\frac{1}{\varepsilon}(\nabla + i\mathbf{k}) \cdot (\nabla + i\mathbf{k}) = -\frac{1}{\varepsilon}\Delta_{\mathbf{k}}$$

acting on $H^2_{\text{per}}(\Omega)$.

# Floquet-Bloch transformation

The *key fact* is: the operator $L$ is decomposed into operators $L(\mathbf{k})$ under the transform $V$:

$$V(Lu)(\mathbf{x},\mathbf{k}) = L(\mathbf{k})[\,Vu(\mathbf{x},\mathbf{k})] \quad (u \in D(L)).$$

or

$$Lu = V^{-1}[L(\mathbf{k})\,Vu(\mathbf{x},\mathbf{k})].$$

Sometimes this is written symbolically as

$$L = \int_B^{\oplus} L(\mathbf{k})\,d\mathbf{k}$$

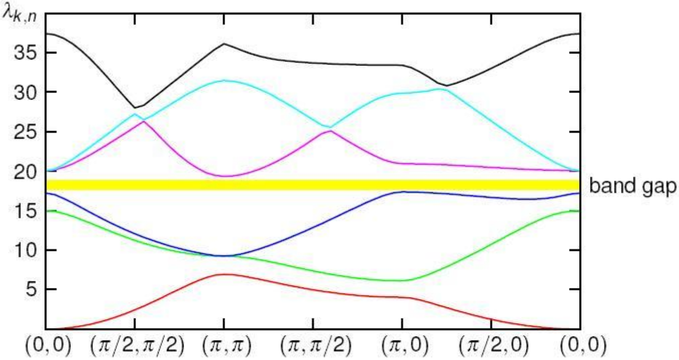
(direct integral of operators).

## Floquet-Bloch transformation

Further consequences:

- $$((L-\lambda)^{-1}f)(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d}} \int_B (L(\mathbf{k})-\lambda)^{-1}\, Vf(\cdot,\mathbf{k})\, d\mathbf{k}$$
- Write $\mathrm{spec}(L(\mathbf{k}))$ as
  $$\lambda_1(\mathbf{k}) \le \lambda_2(\mathbf{k}) \le \ldots \le \lambda_s(\mathbf{k}) \le \ldots.$$
  Then
  $$\mathrm{spec}(L) = \bigcup_{\mathbf{k}\in B} \mathrm{spec}(L(\mathbf{k})) = \bigcup_{n\in\mathbb{N}} \lambda_s(B).$$

# Band structure



# Thomas argument

Consider the problem of excluding point spectrum for $-\varepsilon^{-1}\Delta$, $\varepsilon, \frac{1}{\varepsilon} \in L^\infty(\mathbb{R}^d)$:

- Existence of a $u \in H^2(\mathbb{R}^d)$ solving $(-\Delta - \lambda\varepsilon)u = 0$ implies:

$$(-\Delta_{\mathbf{k}} - \lambda\varepsilon)v = 0 \qquad \text{has a nontrivial solution } v \in H^1_{\text{per}}((0,1)^d)$$
$$\text{for a positive measure set of } \mathbf{k} \in [-\pi, \pi]^d.$$

- Extension into the complex plane: analytic Fredholm theory ($(0,1)^d$ bounded) implies:

$$(-\Delta_{\mathbf{k}} - \lambda\varepsilon)v = 0 \qquad \text{has nontrivial solution for all } \mathbf{k} \text{ of the form}$$
$$\mathbf{k} = (k, 0, \dots, 0), \quad k \in \mathbb{C}.$$

- Study $-\Delta_{\mathbf{k}}$ using Fourier series.

## Thomas argument (continued)

- Expand $v \in H^2((0,1)^d)$ into Fourier series $v = \sum_{\mathbf{m}\in 2\pi\mathbb{Z}^d} c_{\mathbf{m}} e^{i\mathbf{m}x}$, then

$$-\Delta_{\mathbf{k}} v = \sum_{\mathbf{m}} (\mathbf{m}+\mathbf{k})\cdot(\mathbf{m}+\mathbf{k}) c_{\mathbf{m}} e^{i\mathbf{m}x}.$$

- Let $\mathbf{k} = (\pi + i\tau, 0, \ldots, 0)$ and compute

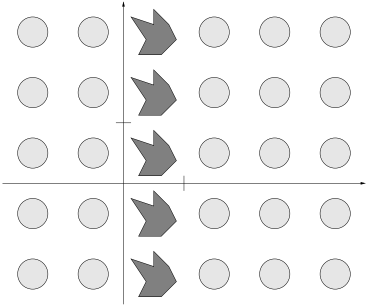
$$\begin{aligned} |\mathrm{Im}(\mathbf{m}+\mathbf{k})^2| &= |\mathrm{Im}[(\mathbf{m}+\pi\mathbf{e}_1)^2 + 2i(m_1+\pi)\tau - \tau^2]| \\ &= 2|m_1+\pi||\tau| \geq 2c_0|\tau| \end{aligned}$$

since $|m_1+\pi| \geq c_0 > 0$ for all $m_1 \in 2\pi\mathbb{Z}$.

$$\Rightarrow \quad \|-\Delta^{-1}_{(\pi+i\tau,0,\ldots,0)}\| \leq C/|\tau|$$

- $(-\Delta_{(\pi+i\tau,0,\ldots,0)} - \lambda\varepsilon)^{-1}$ exists for $\tau$ sufficiently large (Neumann series). Contradiction!

# The waveguide problem in 2D periodic structure



- strip $S := \mathbb{R} \times (0,1)$, unit cell $\Omega = (0,1)^2$, $\varepsilon = \varepsilon_0 + \varepsilon_1$
- $\varepsilon_0, \varepsilon_1 \in L^\infty(\mathbb{R}^2, \mathbb{R})$, $\varepsilon_0$ periodic with respect to $\mathbb{Z}^2$,

$$\varepsilon_1(x_1, x_2 + m) = \varepsilon_1(x_1, x_2) \quad (m \in \mathbb{Z})$$

$\operatorname{supp} \varepsilon_1 \subset (0,1) \times \mathbb{R}$, $\inf_{\mathcal{M}} |\varepsilon_1| > 0$ on some open set $\mathcal{M}$

# Guided modes vs Bound states

Perturbations create extra spectrum.

- guided modes $\psi$: Exist and corresponding new spectrum is continuous

$$(-\Delta - \lambda\varepsilon)\psi = 0, \quad 0 \neq \psi \in H^2(S)$$

  But $\psi(x_1, x_2 + m) = e^{i\beta m}\psi(x_1, x_2)$ (not decaying in $x_2$ direction).

- bound states: localized standing waves, corresponding spectrum is point spectrum

$$(-\Delta - \lambda\varepsilon)\psi = 0, \quad 0 \neq \psi \in H^2(\mathbb{R}^2).$$

  We show that this is impossible.

# Guided modes and Leaky waveguides

Existence of guided mode spectrum:

- strong defects: Kuchment-Ong ('03,'10)
- weak defects: Parzygnat-Avniel-Lee-Johnson ('10), Brown-Hoang-Plum-(R.)-Wood ('15,'16)

Absence of bound states:

- "hard-wall"waveguides: Sobolev-Walthoe ('02), Friedlander ('03), Suslina-Shterenberg ('03)
- "soft-wall"waveguides with asymptotically constant background: Filonov-Klopp ('04,'05), Exner-Frank ('07)
- "soft-wall"waveguides with periodic background: Hoang-Radosz ('14)

# Main result

Consider a Helmholtz-type spectral problem on $\mathbb{R}^2$ of the form

$$-\Delta u = \lambda \varepsilon u. \qquad (*)$$

**Definition**

$\sigma(-\frac{1}{\varepsilon}\Delta) \cap (\mathbb{R} \setminus \sigma(-\frac{1}{\varepsilon_0}\Delta))$ is called guided mode spectrum.

Question: does the guided mode spectrum really correspond to truly guided modes? Are there possibly localized standing waves?

**Theorem (V.H., M.R.)**

*Let $\lambda \in \mathbb{R}$. In $H^2(\mathbb{R}^2)$, the equation $(*)$ has only the trivial solution.*
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# Reformulation of the problem, Floquet-Bloch reduction to $S$

Recall the partial Floquet-Bloch transform in $x_2$ direction

$$(Vf)(x_1, x_2, k_2) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{ik_2(n-x_2)} f(x_1, x_2 - n).$$

- $V : L^2(\mathbb{R}^2) \to L^2(S \times (-\pi, \pi))$ is isometry
- $-\frac{1}{\varepsilon}\Delta = \int_{[-\pi,\pi)}^{\oplus} -\frac{1}{\varepsilon}\Delta_{k_2}\ dk_2$
- $\sigma\left(-\frac{1}{\varepsilon}\Delta\right) = \overline{\bigcup_{k_2 \in [-\pi,\pi]} \sigma\left(-\frac{1}{\varepsilon}\Delta_{k_2}\right)}.$

where

- $-\Delta_{k_2} := -(\nabla + i(0,k_2)) \cdot (\nabla + i(0,k_2))$ with domain $H^2_{\mathrm{per}}(S)$

# Reformulation of the problem, Floquet-Bloch reduction to $S$

- The problem $-\Delta u = \lambda \varepsilon u$ has a nontrivial solution in $H^2(\mathbb{R}^2)$

$$\overset{\text{Floquet-Bloch reduction in } x_2\text{-direction.}}{\Longleftrightarrow}$$

  The problem $(-\Delta_{k_2} - \lambda\varepsilon)u = 0,\ u \in H^2_{\text{per}}(S)$, has a nontrivial solution for $k_2$ in a set of positive measure $\mathcal{P}$.

- **However**, standard Thomas approach not possible, since $S$ not bounded!

# Derivation of a Fredholm-type problem on $\Omega$

Fix $\lambda \notin \sigma\left(-\frac{1}{\varepsilon_0}\Delta\right)$, let $k_2 \in \mathcal{P}$. Define $G(k_2) : L^2(\Omega) \to L^2(\Omega)$ by

$$G(k_2)v := \varepsilon_1(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\widetilde{v}$$

where $\widetilde{v} = v$ on $\Omega$ and $\widetilde{v} = 0$ outside.

**Lemma**

*If $k_2 \in \mathbb{R}$ and $u \in H^2_{per}(S)$, $u \neq 0$ solves*

$$(-\Delta_{k_2} - \lambda\varepsilon)u = 0,$$

*then $v \in L^2(\Omega)$ defined by $v = \varepsilon_1 u$ solves*

$$v + \lambda G(k_2)v = 0 \quad \textit{on} \quad \Omega$$

*and $v \neq 0$.*

# Derivation of a Fredholm-type problem on $\Omega$

Proof

- $v \equiv 0$ *gives a contradiction to* $u \neq 0$ *by a unique continuation principle.*
- $\lambda \notin \sigma\left(-\frac{1}{\varepsilon_0}\Delta\right) \Rightarrow (-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}$ *exists as a bounded operator in* $L^2(S)$.
- $0 = (-\Delta_{k_2} - \lambda(\varepsilon_0 + \varepsilon_1))u \quad \Longrightarrow$
  $0 = u + \lambda(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\varepsilon_1 u.$
  $\Longrightarrow \quad 0 = \varepsilon_1 u + \lambda\varepsilon_1(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\varepsilon_1 u.$

# Analytic continuation of resolvent operators

For $k_2$ close to the real axis, by Floquet-Bloch reduction in $x_1$-direction

$$\begin{aligned} G(k_2)f(\mathbf{x}) &= \varepsilon_1((-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\widetilde{f})(\mathbf{x}) \\ &= \varepsilon_1 \int_{-\pi}^{\pi} e^{ik_1 x_1}(T(k_1,k_2)e^{-ik_1\cdot}f)(\mathbf{x})\ dk_1 \\ &= \varepsilon_1 \int_{-\pi}^{\pi} (H(k_1,k_2)f)(\mathbf{x})\ dk_1 \qquad (\mathbf{x} \in \Omega) \end{aligned}$$

where

- $T(\mathbf{k}) = T(k_1,k_2) := \frac{1}{2\pi}(-\Delta_{\mathbf{k}} - \lambda\varepsilon_0)^{-1} \qquad (\mathbf{k} \in \mathbb{C}^2)$
- $T(\mathbf{k}) : L^2(\Omega) \to L^2(\Omega)$
- $H(k_1,k_2)r := e^{ik_1x_1}\, T(k_1,k_2)[e^{-ik_1\cdot}\, r] \qquad k_1,k_2 \in \mathbb{C}$

# Analytic continuation of resolvent operators

$$G(k_2)f(\mathbf{x}) \quad = \quad \varepsilon_1 \int_{-\pi}^{\pi} (H(k_1, k_2)f)(\mathbf{x}) \; dk_1.$$

Properties of $H$ (e.g. Steinberg ['68], Kato ['76]):

- $k_1 \mapsto H(k_1, k_2)$ is meromorphic
- $H(k_1 + 2\pi m, k_2) = H(k_1, k_2)$
- isolated poles of $H(\cdot, k_2)$ are analytic in $k_2$
- In general, the poles $q_j(k_2)$ of are algebraic functions of $k_2$, i.e. $\exists$ analytic $\mathfrak{g}$

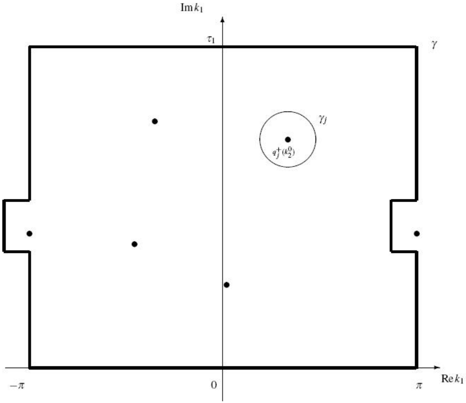
  $$\{q_j(k_2)\} = \mathfrak{g}\left(\sqrt[p]{k_2 - k_2^0}\right)$$

  (multivalued complex function)

# Analytic continuation of resolvent operators

$$G(k_2)f(\mathbf{x}) \quad = \quad \varepsilon_1 \int_{-\pi}^{\pi} (H(k_1, k_2)f)(\mathbf{x}) \; dk_1.$$

Deformation of the integral in $k_1$-plane: ($D$=the enclosed region)



# Analytic continuation of resolvent operators

Thus for $k_2$ close to the real axis,

$$G(k_2)f = \varepsilon_1 \int_{[-\pi,\pi]+i\tau_1} H(k_1,k_2)f \;dk_1 + 2\pi i \sum_{j=1}^{N^+} \varepsilon_1 \operatorname{res}(H(\cdot,k_2)f, q_j^+(k_2)) \quad (**)$$

- $q_j^+(k_2)$ are those poles of $H(\cdot,k_2)$ which lie in the upper half-plane when $k_2 \in [\pi-\delta_2, \pi+\delta_2]$ for some small $\delta_2 > 0$
- Idea: to construct analytic continuation of $G$, use the rhs of $(**)$ as a definition!
- Problem: since $q_j^+(k_2)$ are algebraic in $k_2$ (root-like singularities), there exists no direct continuation of the rhs for all $k_2$, $\operatorname{Im} k_2 > 0$.

# Analytic continuation of resolvent operators

**Lemma**

*There exist a continuous path* $\Gamma : [0,\infty) \to \mathbb{C}$ *satisfying*

(i) $\Gamma(0) \in \mathbb{R}$, $(-\Delta_{k_2} - \lambda\varepsilon)u = 0$ *has a nontrivial solution for* $k_2$ *in a small ball around* $\Gamma(0)$.

(ii) $t \mapsto \operatorname{Im}\Gamma(t)$ *is nondecreasing,*

(iii) $\operatorname{Im}\Gamma(t) \to +\infty$ *for* $t \to \infty$,

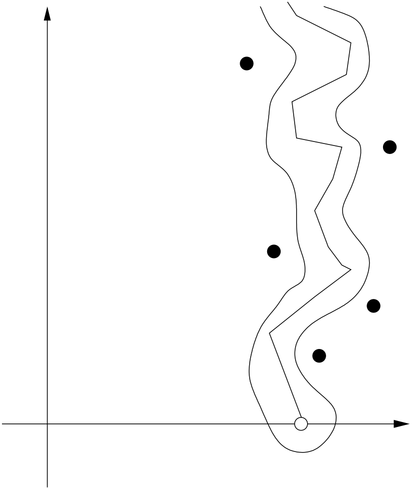
*with the property that there exists a neighborhood*

$$\mathcal{N}(\Gamma) := \mathcal{N}(\Gamma([0,\infty)))$$

*of the path* $\Gamma$ *and a* $N \in \mathbb{N}$ *such that the number of poles of* $T(\cdot, k_2)$ *in* $D$ *is equal to* $N$ *for all* $k_2 \in \mathcal{N}(\Gamma)$.

# Analytic continuation of resolvent operators

The path $\Gamma$ and a neighborhood $\mathcal{N}(\Gamma)$ on which there exists an analytic continuation of $G(k_2)$. Picture in $k_2$ plane:



# Analytic continuation of resolvent operators

**Definition**

For $k_2 \in \mathcal{N}(\Gamma)$ let

$$q_j^+(k_2) \qquad (j = 1, \dots, N^+)$$

denote the poles of $H(\cdot, k_2)$ in $D$ with the property that $\operatorname{Im} q_j^+(\Gamma(0)) > 0$, i.e. those poles which initially lie in the upper half-plane. For any $k_2 \in \mathcal{N}(\Gamma)$ define

$$A(k_2)r \quad := \quad \varepsilon_1 \int_{[-\pi,\pi]+i\tau_1} H(k_1, k_2)r \; dk_1 + 2\pi i \sum_{j=1}^{N^+} \varepsilon_1 \operatorname{res}(H(\cdot, k_2)r, q_j^+(k_2))$$

for all $r \in L^2(\Omega)$.

# Analytic continuation of resolvent operators

Properties of $A(k_2)$:

- $k_2 \mapsto A(k_2)$ is analytic on $\mathcal{N}(\Gamma)$
- for all $k_2 \in \mathcal{N}(\Gamma)$, $A(k_2)$ is compact
- A careful study of the poles $q_j^+(k_2)$ as $\operatorname{Im} k_2 \to \infty$ reveals (hard!)

**Theorem**

*There exist constants $C = C(\delta, \tau_1, \lambda) > 0, M = M(\delta, \tau_1, \lambda) > 0$ such that for $k_2 \in \mathcal{N}(\Gamma)$ of the form $k_2 = \operatorname{Re} k_2 + i(\frac{\pi}{2} + \ell)$ with $\ell \in 2\pi\mathbb{N}, \ell > M$,*

$$\|A(k_2)\| \leq C\ell^{-1}.$$

# The key estimate

The technical heart of the whole construction is (need 2D)

**Theorem**

*For* $s(\mathbf{m},\mathbf{k}) = (\mathbf{m}+\mathbf{k})^2$, $\boldsymbol{\xi} = (\xi_1,\xi_2)$, $\boldsymbol{\eta} = (\eta_1,\eta_2) \in \mathbb{R}^2$, *the following estimates hold:*

$$|s(\mathbf{m},\boldsymbol{\xi}+i\boldsymbol{\eta})|^2 \geq [(m_2+\xi_2)^2-\eta_1^2]^2+[(m_1+\xi_1)^2-\eta_2^2]^2$$

$$|s(\mathbf{m},\boldsymbol{\xi}+i\boldsymbol{\eta})|^2 \geq 2[(m_1+\xi_1)\eta_1+(m_2+\xi_2)\eta_2]^2$$

Proof:

$$\begin{aligned} |s(\mathbf{m},\boldsymbol{\xi}+i\boldsymbol{\eta})|^2 &= [(m_2+\xi_2)^2-\eta_1^2]^2+[(m_1+\xi_1)^2-\eta_2^2]^2 \\ &\quad +2[(m_1+\xi_1)\eta_1+(m_2+\xi_2)\eta_2]^2 \\ &\quad +2[(m_2+\xi_2)(m_1+\xi_1)+\eta_1\eta_2]^2. \end{aligned}$$

# Proof of the main result

$\|A(\operatorname{Re} k_2 + i(\frac{\pi}{2} + \ell))\| \leq C\ell^{-1}$

$\Downarrow$

$v + \lambda A(k_2)v = 0$ only has
the trivial solution for $\ell$ large

$\Downarrow$

$v + \lambda A(k_2)v = 0$ has nontrivial
solution only for a discrete
set of $k_2 \in [-\pi, \pi]$

$-\Delta u = \lambda \varepsilon u = 0$
for some $u \in H^2(\mathbb{R}^2) \setminus \{0\}$

$\Downarrow$

$(-\Delta_{k_2} - \lambda\varepsilon)w = 0$ has
nontrivial solution
for $k_2 \in \mathcal{P}$

$\Downarrow$

$v + \lambda G(k_2)v = 0$ has
nontrivial solution
for almost all $k_2 \in \mathcal{P}$

$\leftrightsquigarrow$

$\uparrow$

Contradiction!
since $A(k_2) = G(k_2)$
for $k_2 \in \mathcal{P}$

Thank you for your attention!