Collapses and nonlinear laser beam combining

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- Explosive instability (blow-up)
  – formation of singularity in a finite time

- Collapse – blow-up with the contraction of the spatial extent of solution to zero
Self-focusing (collapse) of laser beam

Nonlinear medium

Laser beam

Singularity point

- 2D Nonlinear Schrödinger Equation

\[ i \frac{\partial}{\partial z} \psi + \nabla^2 \psi + |\psi|^2 \psi = 0 \]

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

\[ \psi \] - amplitude of light
Self-focusing of optical bullet

Laser beam

Nonlinear medium

Singularity point

\[ i \frac{\partial \psi}{\partial z} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \beta \frac{\partial^2}{\partial t^2} \right) \psi + |\psi|^2 \psi = 0 \]

- 3D Nonlinear Schrödinger Equation (NLSE)
Hamiltonian:

\[ i \frac{\partial}{\partial t} \psi + \Delta \psi + |\psi|^2 \psi = 0 \]

Conserved Integrals:

Nonlinear Schrödinger Equation (NLSE)

\[ H = \int \left( |\nabla \psi|^2 - \frac{1}{2} |\psi|^4 \right) d^D r \]

- Hamiltonian: \[ i \psi_t = \frac{\delta H}{\delta \psi^*} \]

\[ N = \int |\psi|^2 d^D r \]

- optical power (in optics) or number of particles (in quantum mechanics) or wave action in oceanology

Conserved Integrals: \[ \frac{d}{dt} N = \frac{d}{dt} H = 0 \]
Mean square width: \( A \equiv \int |r|^2 |\psi|^2 d^D r \)

\[ D = 2 \]

Virial theorem\(^1\): \( A_{tt} = 8H \)

\[ \Rightarrow A = 4Ht^2 + c_1 t + c_2 \]

Singularity formation:

\[ H < 0 \quad \Rightarrow \quad A \bigg|_{t\to t_0} \to 0 \quad \Rightarrow \quad \max_{t\to t_0} |\psi| \to \infty \]

\(^1\)S.N.Vlasov, V.A Petrishchev, and V.I. Talanov (1971).
V.E. Zakharov JETP, (1972)
Critical collapse of 2D Nonlinear Schrödinger Equation: Self-similar solution near singularity

\[
i \frac{\partial}{\partial t} \psi + \Delta \psi + |\psi|^2 \psi = 0
\]

\[
\Psi(x, t) = \Psi(r, t), \quad r = (x^2 + y^2)^{1/2}
\]

\[
\Psi(r, t) \simeq \frac{1}{L} V(\rho) e^{i \tau + i L L_t \rho^2/4}, \quad L \to 0,
\]

Soliton solution of NLSE:

LogLog law\(^1\):

\[
\Delta V - V + |V|^2 V = 0
\]

\[
L = \left( 2\pi \frac{t_c - t}{\ln |\ln (t_c - t)|} \right)^{1/2}
\]

Strong vs. weak collapse of NLSE

\[ i \frac{\partial \psi(r)}{\partial t} + \nabla^2 \psi(r) + |\psi(r)|^2 \psi(r) = 0 \]

**D=2:** Strong critical collapse (mass critical) as above

**D=3:** Weak supercritical collapse

\[ |\psi_{c,\text{weak}}(r, t)| \approx \frac{1}{L(t)} \eta \left( \frac{r}{L(t)} \right), \quad L(t) \to 0 \quad \text{for} \ t \to t_0 \]

Self-similar variable \( \xi \equiv r/L(t) \)

Number of particles in collapsing region

\[ N_{\text{collapse,weak}} \approx \int_{|r| < \xi_c L(t)} |\psi_{c,\text{weak}}(r, t)|^2 d^3r \]

\[ = L(t) \int_{|\xi| < \xi_c} \eta^2(\xi) d^3\xi \sim L(t) \to 0 \quad \text{for} \ t \to t_0 \]
Collapses at the National Ignition Facility
Target Chamber’s Dedication Marks a Giant Milestone

Energy Secretary Bill Richardson addresses the crowd that attended the June 11, 1999, dedication of the National Ignition Facility target chamber.
Target

- Gold hohlraum, temperature about 18 kelvins
- 1-micrometer-thick polyimide window
- Solid D-T fuel layer, thickness about 80 micrometers
- He + H₂ fill, about 1 milligram per cubic centimeter
- Cooling ring
- Sapphire cooling rods
- 2-millimeter-diameter capsule, beryllium or polymer

Dimensions:
- 9.5 millimeters height
- 5.5 millimeters width
Thermonuclear burn

\( \text{D} + \text{T} = ^4\text{He} \ (3.5 \ \text{Mev}) + \text{n} \ (14.1 \ \text{Mev}) \)

**Required temperature:** 10 KeV = 100 millions Celsius

\( \text{D} + ^3\text{He} = ^4\text{He} \ (3.7 \ \text{Mev}) + \text{p} \ (14.7 \ \text{Mev}) \)

**Required temperature:** 100 KeV = 1000 millions Celsius
Goal: propagation of laser light in plasma with minimal distortion

Difficulties: collapses from self-focusing of light and Langmuir wave collapses

Strong beam spray

No spray

Laser propagation in plasma
Main laser-plasma interaction effects at The National Ignition Facility

1. Stimulated Brillouin Scattering (SBS)

Electromagnetic wave (from laser)

$\mathbf{k}_1$

Light

$\mathbf{k}_2$

Ion acoustic wave

Scattered electromagnetic wave

A particular version of SBS: cross-beam energy transfer (CBET) with both electromagnetic waves corresponding to two laser beams

2. Stimulated Raman Scattering (SRS)

Electromagnetic wave (from laser)

$\mathbf{k}_1$

Light

$\mathbf{k}_2$

Electron plasma wave (Langmuir wave)

$k_1 - k_2, \omega \approx \omega_p$
Collapse from Forward Stimulated Brillouin Scattering

\[ \mathcal{E} = E(\mathbf{r}, z, t) e^{i k_0 z - i\omega_0 t} + c.c. \quad - \text{amplitude of light} \]

\[ r = (x, y) \]

\[ i \frac{\partial}{\partial z} E + \nabla^2 E = \rho E, \quad \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \]

\[ \left( \frac{\partial^2}{\partial t^2} + 2\tilde{\nu} \frac{\partial}{\partial t} - c_s^2 \nabla^2 \right) \rho = c_s^2 \nabla^2 \left[ |E|^2 + \frac{\delta T_e}{T_e} \right] \]

\[ \rho - \text{low frequency plasma density fluctuation} \]

\[ \tilde{\nu} = \nu k c_s \quad - \text{Landau damping} \]

\[ c_s \quad - \text{speed of sound} \]

\[ \rho = -|E|^2 \quad \Rightarrow \quad i \frac{\partial E}{\partial z} + \nabla_\perp^2 E + |E|^2 E = 0 \]

\[ \frac{P}{P_{cr}} = 10^5 \]
Stimulated Brillouin Scattering (SBS) is strongly affected by $Z$ number both for forward SBS\textsuperscript{1,2} (at the nonlinear stage results in multiple collapses\textsuperscript{3,4}) and backward SBS\textsuperscript{5}.

E.g. gold plasma is subject to strong SBS for $\sim$10 times smaller laser intensities than low $Z$ plasma.

Stimulated Raman Scattering (SRS)

In the kinetic regime:

\[ 0.25 < k \lambda_D < 0.45 \]

\( \lambda_D \) – Debye length

Found that SRS can be reduced by the filamentation from the collapse of Langmuir waves
Collapse of Langmuir waves: Generalized Zakharov Eq.

\[
 i \left[ \frac{\partial}{\partial t} + v_{\text{group}} \frac{\partial}{\partial x_p} + v_{\text{Landau}}(k,|\psi|) \right] \psi = \left[ -D_\perp \Delta_\perp + \Delta \omega_{\text{trapped}} + \frac{1}{2} \frac{\delta n}{n} \omega_{\text{pe}} \right] \psi
\]

\[
 \left( \frac{1}{c_{ia}^2} \frac{\partial^2}{\partial t^2} - \Delta_\perp \right) \frac{\delta n}{n} = \frac{1}{4} \left( k \lambda_D \right)^2 \Delta_\perp \left| \frac{e \psi}{T_e} \right|^2
\]

\( \psi \) - amplitude of Langmuir wave

\( v_{\text{Landau}}(k,|\psi|) \) - nonlinear Landau damping

\( \Delta \omega_{\text{trapped}} \) - nonlinear frequency shift

\( c_{ia} \) - speed of ion-acoustic waves

\( \delta n \) - density of low frequency fluctuations

\[
 \delta n \propto -|\psi|^2 \implies i \frac{\partial \psi}{\partial z} + \nabla_\perp^2 \psi + |\psi|^2 \psi = 0
\]
Kinetic effects in Generalized Zakharov Eq:

\[ \nu_{\text{Landau}}(k,|\psi|) \quad - \quad \text{nonlinear Landau damping} \]

\[ \Delta \omega_{\text{trapped}} \quad - \quad \text{nonlinear frequency shift} \]

\[
i \left[ \frac{\partial}{\partial t} + v_{\text{group}} \frac{\partial}{\partial x_p} + \nu_{\text{Landau}}(k,|\psi|) \right] \psi = \left[ -D_{\perp} \Delta_{\perp} + \Delta \omega_{\text{trapped}} + \frac{1}{2} \frac{\delta n}{n} \omega_{\text{pe}} \right] \psi
\]

\[
\left( \frac{1}{c_{ia}^2} \frac{\partial^2}{\partial t^2} - \Delta_{\perp} \right) \frac{\delta n}{n} = \frac{1}{4} \left( k \lambda_D \right)^2 \Delta_{\perp} \left| \frac{e \psi}{T_e} \right|^2
\]
Kinetic effects require to solve 3+3 Vlasov equation (3 velocity dimensions and 3 spatial dimensions) for the phase space distribution function $f(r, v, t)$

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla + E \cdot \frac{\partial}{\partial v}\right)f = 0$$

$E = -\nabla \phi \quad \Delta \phi = -\rho = -\int f \, dv_x \, dv_y \, dv_z$

Scaled units: electron thermal units

Another example of collapse:

If quadratic optical nonlinearity is added to Kerr nonlinearity then NLSE is replaced by Davey-Stewartson (Benney-Roskes) equation\textsuperscript{1,2}

\[ i\psi_t + (\partial_x^2 + \partial_y^2)\psi + |\psi|^2\psi - \rho \psi \partial_x \phi = 0, \]
\[ \partial_x^2 \phi + \nu \partial_y^2 \phi = \partial_x (|\psi|^2), \quad \nu > 0, \quad \rho < 0 \]

Collapse of Davey-Stewartson (Benney-Roskes) equation

\[ i\psi_t + (\partial_x^2 + \partial_y^2)\psi + |\psi|^2\psi - \rho\psi \partial_x \phi = 0, \]
\[ \partial_x^2 \phi + \nu \partial_y^2 \phi = \partial_x (|\psi|^2), \quad \nu > 0, \quad \rho < 0 \]

The Hamiltonian

\[ H = \int |\nabla \psi|^2 d\mathbf{r} - \frac{1}{2} \int |\psi|^4 d\mathbf{r} + \frac{\rho}{2} \int (\phi_x^2 + \nu \phi_y^2) d\mathbf{r} \]

Virial theorem

\[ \frac{d^2}{dt^2} \int (x^2 + y^2)|\psi|^2 d\mathbf{r} = 8H \quad \Rightarrow \quad \text{collapse}^1 \]

\[ ^1 \text{G.C. Papanicolaou, C. Sulem, P.L. Sulem, X.P. Wang, Physica D, 72, 61 (1994)} \]
Critical NLSE collapse

$$|\psi| \sim \frac{1}{L(t)} R \left( \frac{r}{L(t)} \right), \quad L(t) \propto (t_c - t)^{1/2}$$

ground state soliton of NLSE

Critical collapse in Davey-Stewartson Eq (DSE):

$$|\psi| \sim \frac{1}{L(t)} R \left( \frac{x}{L(t)}, \frac{y}{L(t)} \right), \quad L(t) \propto (t_c - t)^{1/2}$$

ground state soliton of DSE

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Critical collapse of 2D
Nonlinear Schrödinger Equation:
Self-similar solution near singularity

\[ i \frac{\partial}{\partial t} \psi + \Delta \psi + |\psi|^2 \psi = 0 \]

\[ \Psi(x, t) = \Psi(r, t), \quad r = (x^2 + y^2)^{1/2} \]

\[ \Psi(r, t) \approx \frac{1}{L} V(\rho) e^{i \tau + i LL_t \rho^2/4}, \quad L \to 0, \]

\[ \rho = \frac{r}{L}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')}, \]

\[ \Delta V - V + |V|^2 V = 0 \]

Soliton solution of NLSE:

LogLog law\(^1\):

\[ L = \left( 2\pi \frac{t_c - t}{\ln |\ln (t_c - t)|} \right)^{1/2} \]

But simulations failed to confirm log-log law in a convincing way\textsuperscript{1} although the exact proof of the existence of log-log scaling was given\textsuperscript{2}

Example of NLS simulations:

$L(t)$ depends on initial conditions

\textsuperscript{2}F. Merle and P. Raphael (2006).
A little of history of 2D NLS collapse

- **1962** G.A. Askaryan: Self-focusing of laser beam
- **1970** V.I. Talanov: lens transform
- **1971** S.N. Vlasov, V.A. Petrishchev, and V.I. Talanov: Virial theorem and exact proof of collapse formation
- **1985** G. Fraiman: “almost” log-log scaling of collapse
- **1987** M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem: log-log scaling of collapse
- **1993** V.M. Malkin: collapse in terms of the excess of number of particles above critical
- **2006** F. Merle and P. Raphael: exact proof of existence of log-log scaling
Working on NLSE and DSE collapse in parallel

NLSE blow-up variables

\[ \rho = \frac{r}{L}, \quad \tau = \int_{t'}^{t} \frac{dt'}{L^2(t')} \]

and lens transform

\[ \psi(r, t) = \frac{1}{L} V(\rho, \tau) e^{i\tau + iLL_t \rho^2 / 4} \]

\[ \Rightarrow \quad iV_\tau + \nabla^2 V - V + |V|^2 V + \frac{\beta}{4} \rho^2 V = 0, \]

where \( \beta = -L^3 L_{tt} \) - adiabatically slow small parameter \( \beta \ll 1 \)

Looking for solution in the form

\[ V = V_0 + V_1 + \ldots \]

\[ \nabla^2 V_0 - V_0 + |V_0|^2 V_0 + \frac{\beta}{4} \rho^2 V_0 = 0 \]
Looking for the DSE collapsing solution as

\[ \psi(r, t) = \frac{1}{L(t)} V(p, q, \tau) e^{i\xi \tau + i L(t) t L(t)(p^2 + q^2)/4}, \quad \xi = \text{const}, \]

For blow up variables

\[ p \equiv \frac{x}{L(t)}, \quad q \equiv \frac{y}{L(t)}, \quad \tau = \int_0^t \frac{dt'}{L(t')^2} \]

Collapse \( L(t) \rightarrow 0 \) as \( t \rightarrow t_c \) and \( \tau \rightarrow \infty \) as \( t \rightarrow t_c \)

\[ \Rightarrow \quad \text{For Davey-Stewartson Eq. transforms into} \]

\[ i \partial_\tau V + \nabla_{p,q}^2 V - \xi V + |V|^2 V + \frac{\beta}{4} (p^2 + q^2) V - \mu V \frac{\partial^2}{\partial_p^2 + \nu \partial_q^2} |V|^2 = 0, \]

where \( \beta = -L^3 L_{tt} \) - adiabatically slow small parameter \( \beta \ll 1 \)

and \( \nabla_{p,q}^2 \equiv \partial_p^2 + \partial_q^2 \)
Looking for solution in the form

\[ V = V_0 + V_1 + \ldots \]

In adiabatic approximation of slow \( \beta \)

\[
\nabla^2_{p,q} V_0 - \xi V_0 + |V_0|V_0 + \frac{\beta}{4}(p^2 + q^2)V_0 - \mu V_0 \frac{\partial^2 p}{\partial^2 p + \nu \partial^2 q} |V_0|^2 = 0
\]
Tail minimization principle: during collapse dynamics system dynamically select collapsing solution with minimal tail amplitude

Then we look for $V_0$ with the minimal tail
NLSE: In adiabatic approximation of slow \( \beta \) minimizing tails by shooting method:

\[
\nabla^2 V_0 - V_0 + |V_0|^2 V_0 + \frac{\beta}{4} \rho^2 V_0 = 0
\]

\( \beta = 0.2 \)
Approximation through ground state soliton $R(\rho)$

$$V_0 = R(\rho) + \beta \frac{\partial V_0}{\partial \beta} \bigg|_{\beta=0} + O(\beta^2)$$

$$-R + \nabla^2 R + R^3 = 0$$
NLSE: Full solution $V$ match the envelope of $V_0$ of in the tail:

$V_0 \simeq |V|$ to the left from $\rho_b$

$\rho_b \simeq 7.4$

$\beta = 0.073$
DSE: In adiabatic approximation of slow $\beta$ minimizing tails by Newton-conjugate-gradient method (J. Yang, 2009) combined with the correct choice of the asymptotic at infinity:

$$\nabla^2_{p,q} V_0 - \xi V_0 + |V_0| V_0 + \frac{\beta}{4} (p^2 + q^2) V_0 - \mu V_0 \frac{\partial^2_{p} \partial^2_{n} + \nu \partial^2_{q} |V_0|^2 = 0}$$
Approximation through DSE ground state soliton \( R(p, q) \)

\[
V_0 = R(p, q) + \beta \frac{\partial}{\partial \beta} V_0 \big|_{\beta=0} + O(\beta^2)
\]

Ground state soliton solution of Davey-Stewartson Eq.

\[
i\psi_t + (\partial_x^2 + \partial_y^2)\psi + |\psi|^2\psi - \rho \psi \partial_x \phi = 0,
\]

\[
\partial_x^2 \phi + \nu \partial_y^2 \phi = \partial_x (|\psi|^2), \quad \nu > 0, \quad \rho < 0
\]

\[
\psi(x, y, t) = R(x, y)e^{i\lambda t}
\]

\[
\phi(x, y, t) = G(x, y)
\]

\[
\Rightarrow \quad -\lambda R + (\partial_x^2 + \partial_y^2)R + R^3 - \mu R \partial_x G = 0,
\]

\[
\partial_x^2 G + \nu \partial^2 G = \partial_x (R^2)
\]
NLSE: Full solution $V$ match the envelope of $V_0$ of in the tails along both spatial directions
NLSE: How to extract $L(t)$ and $\beta$ from simulations:

The analysis of Taylor series solution of

\[ iV_\tau + \nabla^2 V - V + |V|^2 V + \frac{\beta}{4} \rho^2 V = 0, \quad \text{at} \quad \rho \ll 1 \]

\[ \Rightarrow \quad L = \frac{1}{|\psi|} \left( 1 + 2 \frac{|\psi|_{rr}}{|\psi|^3} \right)^{-1/2} |r=0 \]

Then $\beta$ is found from the implicit equation

\[ \psi(r = 0, t) = \frac{1}{L(t)} V_0(\beta, \rho = 0) \]

for each given $L(t)$ and $\psi(r = 0, t)$

DSE: qualitatively similar procedure
But simulations failed to confirm log-log law in a convincing way\textsuperscript{1} although the exact proof of the existence of log-log scaling was given\textsuperscript{2}

Example of NLS simulations:

$L(t)$ depends on initial conditions


\textsuperscript{2}F. Merle and P. Raphael (2006).
NLSE: Modifying the standard theory

$L(t)$ is not universal but $\beta_\tau(\beta)$ is universal:

$$\beta_\tau = -\tilde{M} \exp \left[ -\frac{\pi}{\beta^{1/2}} \right]$$
DSE: universality of $\beta_\tau(\beta)$
Focus on NLSE collapse:

Recall that we are looking for solution in the form

$$V = V_0 + V_1 + \ldots$$

- has the imaginary part because of slow dependence of $\beta$ on $\tau$:

$$i \frac{\partial V_0}{\partial \tau} = i \beta \frac{\partial V_0}{\partial \beta}$$

Also we need to make sure that $V$ has only outgoing waves for $\rho \to \infty$

In analogy with Gamov $\alpha$-decay theory introduce nonself-adjoint problem

$$\nabla^2 \tilde{V}_0 - \tilde{V}_0 + |\tilde{V}_0|^2 \tilde{V}_0 + \frac{\beta}{4} \rho^2 \tilde{V}_0 - i \nu(\beta) \tilde{V}_0 = 0,$$

where $\nu$ can be determined from the balance of the norm of $V$ as

$$\nu \sim e^{-\pi/\sqrt{\beta}}$$
In other words, we look at
\[
\nabla^2 \tilde{V}_0 - \tilde{V}_0 + |\tilde{V}_0|^2 \tilde{V}_0 + \frac{\beta}{4} \rho^2 \tilde{V}_0 - i \nu(\beta) \tilde{V}_0 = 0,
\]
as the Schrödinger equation with the effective potential \( U \):
\[
U(\rho) = -|\tilde{V}_0|^2 + \frac{\beta}{4} \rho^2
\]
and complex eigenvalue \( E \):
\[
E = -1 - i \nu(\beta)
\]

\( \Rightarrow \) 2 turning points \( \rho_a \) and \( \rho_b \) of WKB:
\[
\rho_a \sim 1
\]
\[
\rho_b \sim \frac{2}{\beta^{1/2}}
\]
Solution near $\rho_b$

$\rho_b \simeq 7.4$

$\beta = 0.073$

$V_0 \simeq |V|$ to the left from $\rho_b$

$|V|$ - from numerics
$V_0$ – soliton with $\beta$
$R$ – ground state
soliton with $\beta = 0$
Oscillating tail is given by the linear combination of confluent hypergometric functions of the first and second kinds:

\[
c_1 e^{-\frac{i}{4} \sqrt{\beta} \rho^2} {}_1 F_1 \left( \frac{1}{2} + i \frac{1}{2 \sqrt{\beta}} ; 1 ; i \sqrt{\beta} \rho^2 \right) + c_2 e^{-\frac{i}{4} \sqrt{\beta} \rho^2} U \left( \frac{1}{2} + i \frac{1}{2 \sqrt{\beta}} ; 1 ; i \sqrt{\beta} \rho^2 \right).
\]

Matching asymptotics and using WKB give

\[
V_0(\beta, \rho) = \frac{2^{1/2} A_R}{\beta^{1/4}} e^{-\frac{\pi}{2 \beta^{1/2}}} \frac{1}{\rho} \cos \left( \frac{\beta^{1/2}}{4} \rho^2 - \beta^{-1/2} \ln \rho + \phi_0 \right), \quad \rho \gg \rho_b
\]

Here \( A_R \equiv 3.52 \) is determined by the asymptotic of ground state soliton \( R_0(\rho) = \frac{A_R}{\rho^{1/2}} e^{-\rho}, \quad \rho \gg 1 \)

\[\Rightarrow\] Asymptotics of complex solution

\[
V(\beta, \rho) = \frac{2^{1/2} A_R}{-\beta^{1/4}} e^{-\frac{\pi}{2 \beta^{1/2}}} \frac{1}{\rho} \exp \left( \frac{i \beta^{1/2}}{4} \rho^2 - i \beta^{-1/2} \ln \rho - i \phi_0 \right), \quad \rho \gg \rho_b.
\]
Introducing the number of particles to the left of the second turning point

\[ N_b = \int_{r<\rho_b L} |\psi|^2 \, dr = 2\pi \int_{\rho<\rho_b} |V|^2 \rho \, d\rho. \]

and balancing the flux of particles through that point

\[ \frac{dN_b}{d\tau} = \rho \left[ iV^* V_\rho + c.c. \right] \bigg|_{\rho=\rho_b}, \quad \frac{dN_b}{d\tau} = \beta_\tau \frac{dN_b}{d\beta} \]

\[ \Rightarrow \quad \text{New basic ODE system} \]

\[ \begin{cases} 
\beta_\tau = -\tilde{M} \left[ 1 + c_1 \beta + c_2 \beta^2 + c_3 \beta^3 + c_4 \beta^4 + c_5 \beta^5 + O(\beta^6) \right]^{-1} \exp \left[ -\frac{\pi}{\beta^{1/2}} \right], \\
L^3 L_{tt} = -\beta, \\
\tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases} \]

Here

\[ \frac{dN_b}{d\beta} = M \left[ 1 + c_1 \beta + c_2 \beta^2 + c_3 \beta^3 + c_4 \beta^4 + c_5 \beta^5 + O(\beta^6) \right] \]

\[ c_1 = 4.793, \quad c_2 = 52.37, \quad c_3 = 296.99, \quad c_4 = -4660.87, \quad c_5 = 10540.4 \]
Compare with old basic ODE system of the standard theory

\[
\begin{align*}
\beta_\tau &= -\tilde{M} \exp \left[-\frac{\pi}{\beta^{1/2}}\right], \quad \tilde{M} = 45.056 \ldots, \\
L^3 L_{tt} &= -\beta, \\
\tau &= \int_0^t \frac{dt'}{L^2(t')}
\end{align*}
\]

Asymptotic solution near collapse time \(t_c\):

\[
L = \left(2\pi \frac{t_c - t}{\ln |\ln (t_c - t)|}\right)^{1/2}
\]

\[\text{References:}\]
$L(t)$ is not universal but $\beta_\tau(\beta)$ is universal:

$$\beta_\tau = -\tilde{M} \exp \left[ -\frac{\pi}{\beta^{1/2}} \right]$$

$$\beta_\tau = -\tilde{M} \left[ 1 + c_1 \beta + c_2 \beta^2 + c_3 \beta^3 + c_4 \beta^4 + c_5 \beta^5 + O(\beta^6) \right]^{-1} \exp \left[ -\frac{\pi}{\beta^{1/2}} \right]$$
Finding asymptotic of a new basic ODE system

\[
\beta_t = -\tilde{M} \left[ 1 + c_1 \beta + c_2 \beta^2 + c_3 \beta^3 + c_4 \beta^4 + c_5 \beta^5 + O(\beta^6) \right]^{-1} \exp \left[ -\frac{\pi}{\beta^{1/2}} \right],
\]

\[
L^3 L_{tt} = -\beta,
\]

\[
\tau = \int_0^t \frac{dt'}{L^2(t')},
\]

\[-\ln \frac{L}{L_0} = \frac{2\pi^3 e^x}{\tilde{M}} \left[ \frac{1}{x^4} + \frac{4}{x^5} + \frac{20 + \pi^2 c_1}{x^6} + \frac{120 + 6\pi^2 c_1}{x^7} + \frac{840 + 42\pi^2 c_1 + \pi^4 c_2}{x^8} + \frac{6720 + 336\pi^2 c_1 + 8\pi^4 c_2}{x^9} \right. \\
+ \left. \frac{60480 + 3024\pi^2 c_1 + 72\pi^4 c_2 + \pi^6 c_3}{x^{10}} + \frac{604800 + 30240\pi^2 c_1 + 720\pi^4 c_2 + 10\pi^6 c_3}{x^{11}} \right. \\
+ \left. \frac{6652800 + 332640\pi^2 c_1 + 7920\pi^4 c_2 + 110\pi^6 c_3 + \pi^8 c_4}{x^{12}} + \frac{79833600 + 3991680\pi^2 c_1 + 95040\pi^4 c_2 + 1320\pi^6 c_3 + 12\pi^8 c_4}{x^{13}} \right. \\
+ \left. \frac{1037836800 + 51891840\pi^2 c_1 + 1235520\pi^4 c_2 + 17160\pi^6 c_3 + 156\pi^8 c_4 + \pi^{10} c_5}{x^{14}} + O \left( \frac{1}{x^{15}} \right) \right]
\]

\[x = \frac{\pi}{\beta^{1/2}}\]
\[
\tau = \int_0^t \frac{dt'}{L^2(t')} \quad \Rightarrow
\]

\[
t_c - t = \int_t^{t_c} dt = \int_\tau^\infty L^2 d\tau = \int_\beta^0 L^2 \frac{d\tau}{d\beta} d\beta
\]

\[
= -\int_\beta^0 L^2 \frac{1}{\tilde{M}} \left[ 1 + c_1 \beta + c_2 \beta^2 + c_3 \beta^3 + c_4 \beta^4 + c_5 \beta^5 + O(\beta^6) \right] \exp \left[ \frac{\pi}{\beta^{1/2}} \right] d\beta
\]

Using \( \beta(L) \) from the inversion of previous expression and inverting that equation

\[
\Rightarrow
\]
Asymptotic of new basic ODE system

\[
\begin{align*}
\beta_\tau &= -\tilde{M} \left[ 1 + c_1 \beta + c_2 \beta^2 + c_3 \beta^3 + c_4 \beta^4 + c_5 \beta^5 + O(\beta^6) \right]^{-1} \exp \left[ -\frac{\pi}{\beta^{1/2}} \right], \\
L^3 L_{tt} &= -\beta, \\
\tau &= \int_0^t \frac{dt'}{L^2(t')}
\end{align*}
\]

\[
L = \left( \frac{2\pi(t_c - t)}{\ln A - 4 \ln 3 + 4 \ln \ln A} \right)^{1/2} \left[ 1 + \frac{2(1 + 4 \ln 3 - 4 \ln \ln A)}{(\ln A)^2} \right.
\]

\[+ \frac{14 - 48 \ln \ln A + 48(\ln \ln A)^2 + 48 \ln 3 - 96(\ln A)(\ln 3) + 48(\ln 3)^2 + \frac{1}{2} \pi^2 c_1}{(\ln A)^3} \left. + O \left( \frac{(\ln \ln A)^3}{(\ln A)^4} \right) \right]
\]

\[
A = -3^4 \frac{\tilde{M}}{2\pi^3} \ln \left[ 2\pi(t_c - t) \right]^{1/2} e^{-a_0} \left( \frac{L(z_0)}{L_{(z_0)}} \right), \quad \tilde{M} = 44.773 \ldots, \quad \beta_0 = \beta(t_0), \quad c_1 = 4.793 \ldots, c_2 = 52.37 \ldots
\]

\[
a_0 = \frac{e^{\sqrt{\beta_0}}}{\tilde{M}} \left( \frac{2\beta_0^2}{\pi} + \frac{8\beta_0^{5/2}}{\pi^2} + \frac{2\beta_0^3(20 + \pi^2 c_1)}{\pi^3} + \frac{12\beta_0^{7/2}(20\pi^3 + \pi^5 c_1)}{\pi^7} + \frac{2\beta_0^4(840\pi^3 + 42\pi^5 c_1 + \pi^7 c_2)}{\pi^8} \right)
\]
Simulations vs. analytic

$L = \left( \frac{2\pi(t_c - t)}{\ln A - 4 \ln 3 + 4 \ln \ln A} \right)^{1/2}$
Simulations vs next order analytic numerics

\[ L = \left( \frac{2\pi(t_c - t)}{\ln A - 4 \ln 3 + 4 \ln \ln A} \right)^{1/2} \left[ 1 + \frac{2(1 + 4 \ln 3 - 4 \ln \ln A)}{(\ln A)^2} \right. \\
\left. + \frac{14 - 48 \ln \ln A + 48(\ln \ln A)^2 + 48 \ln 3 - 96(\ln A)(\ln 3) + 48(\ln 3)^2 + \frac{1}{2}\pi^2 c_1}{(\ln A)^3} \right] + O \left( \frac{(\ln \ln A)^3}{(\ln A)^4} \right) \]

\[ A = -3^4 \frac{\tilde{M}}{2\pi^3} \ln \left[ 2\pi(t_c - t) \right]^{1/2} \frac{e^{-a_0}}{L(z_0)} \]

Solid – numerics
Dashed - analytics
Simulations vs. analytic – larger interval starting from the initial Gaussian

\[ L(t) \]

Solid – numerics
Dashed - analytics
In comparison, the standard log-log scaling dominates only for amplitudes above $100$

$10 = 10$ = Googol

$10 = 10$ = Googolplex

$L = \left(2\pi \frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}$

---

Evolution of the average output power of nearly diffraction limited fiber lasers (emitting either a continuous wave or ultrashort pulses) \(^1\)

\[\text{Average power (W)}\]

\[\text{Year}\]

---

\(^1\) C. Jauregui, J. Limpert, and A. Tunnermann, Nature Photonics 7, 861 (2013)
The signal is coupled in the fiber core, which contains the active material, whereas the pump is coupled in the fiber cladding. This structure allows the pump to be progressively absorbed by the active material in the core as the pump propagates along the fiber. This absorbed pumped energy is used to amplify the signal.
Advantages of fiber lasers

- Alignment-free laser systems
- High efficiency (50-80%)
- Compact design
- Maintenance-free operation

Disadvantage of fiber lasers

- Mode instabilities limiting average power
Commercially available IPG Photonics Fiber lasers up to 50kW¹

¹www.ipgphotonics.com
Overcoming power limitations: Laser beam combining

Standard schemes\(^1\)

\(^1\)http://www.laserfocusworld.com
Coherent beam combining:

Combine several laser beams such that the phase of each laser beam is controlled to ideally produce the combined beam with the coherent phase.

Example\(^1\): five 500W laser beams into 1.9kW Gaussian beam with a good beam quality \(M^2 = 1.1\).

**Difficulties in coherent beam combining:**
- Complicated adaptive optics scheme
- Bad scaling with power due to nonlinearity

---

**New proposal**$^{1,2}$:

Use nonlinearity to our advantage to achieve combining of multiple laser beams into a diffraction-limited beam by the strong self-focusing in a waveguide with the Kerr nonlinearity.

Nonlinear laser beam combining

(1) Simpler limit: combining of a few laser beams into a single diffraction-limited beam

FIG. 1: Schematics of beam combining setup.
Propagation and combining of 3 beams along $z$

Cross sections at different $z$

$z = 0$

$z = 0.6$

$z = 2.1$
Propagation and combining of 7 beams along $z$.

Cross sections at different $z$: $z = 0$, $z = 1.7$, $z = 5.0$. 

Color bar: amplitude range from 0 to 1.
Laser beam quality $M^2$ after exit of the collapsed beam from Kerr media based on least square fit of beam waste $w$ on the propagation distance $z$ \(^1\)

$$w^2(z) = w_0^2 + \left(\frac{2M^2}{k_0 w_0}\right)^2 (z - z_0)^2$$

---

\(^1\) T. Sean Ross, Laser Beam Quality Metrics, SPIE Press, 204 pages (2013)
(2) Nonlinear laser beam combining of multiple beams

Beams from fiber lasers

Side-by-side combining

Nonlinear propagation of multiple beams

Output coherent beam

Laser amplitudes and phases in fiber cross sections:
z-dependence of the maximum light amplitude at the cross-section vs. \( z \):

\[
i \partial_z \psi + \nabla^2 \psi + |\psi|^2 \psi - a_1 |\psi|^4 \psi = 0, \quad 0 < a_1 \ll 1
\]

Regularization of collapse
Probability density function (PDF) of the collapse distance

N=6N_c

N=5N_c

N=4N_c
Inverse cascade before collapse
Physical units: \[ i \partial_z \psi + \frac{1}{2k} \nabla^2 \psi + \frac{kn_2}{n_0} |\psi|^2 \psi = 0, \]

\[ k = \frac{2\pi n_0}{\lambda_0} \quad \lambda_0 \text{-wavelength in vacuum} \]

\[ n_0 \text{- linear index of refraction} \]

\[ n_2 \text{- nonlinear Kerr index with} \quad n = n_0 + n_2 I \]

\[ I = |\psi|^2 \text{- laser intensity} \]

\[ n_0 = 1.4496, \quad n_2 = 2.46 \cdot 10^{-16} \text{cm}^2/\text{W} \text{ for } \lambda_0 = 1070\text{nm} \]

Critical power: \[ P_c = \frac{N_c \lambda_0^2}{8\pi^2 n_2 n_0} \approx \frac{11.70 \lambda_0^2}{8\pi^2 n_2 n_0} \approx 4.7\text{MW} \]

\[ N_c \equiv 2\pi \int R^2 r dr = 11.7008965\ldots \]

\[ \psi = e^{iz} R(r) \quad -R + \nabla^2 R + R^3 = 0 \quad \text{- ground state soliton} \]
Optical fiber and laser intensity parameters for the nonlinear beam combining in fused silica:

\[ I_0 = 10^9 W/cm^2 \] - Laser intensity for the continuous wave operations

\[ \Rightarrow \] Optical fiber length \( \sim 4 \text{m} \)

Optical fiber diameter \( \sim 2 \text{mm} \)

Combined beam power \( P_c = 4.7 \text{MW} \)

\[ \Rightarrow \] Requires to combine several hundreds of the commercially available fiber lasers. The proposed scheme does not high quality beams for combining.
Short pulse operations for the nonlinear beam combining in fused silica:

Fused silica optical damage threshold:

\[ I_{\text{thresh}} \sim 5 \cdot 10^{11} \text{W/cm}^2 \text{ for } 8 \text{ ns pulses} \]
\[ I_{\text{thresh}} \sim 1.5 \cdot 10^{12} \text{W/cm}^2 \text{ for } 14 \text{ ps pulses} \]

\[\Rightarrow\] Fiber length and cross section can be scaled down by a factor \( \sim 10^{-3} \) compare with the continuous wave operations

\[1^{\text{P.M. Lushnikov and N. Vladimirova, Optics Letters 39, 3429-3432 (2014).}}\]
\[2^{\text{P.M. Lushnikov and N. Vladimirova, Optics Express 23, 31120-31125 (2015).}}\]
Conclusion

- Suggest that optical collapses will be around for a very long time

- Propose to achieve a nonlinear beam combining by propagating multiple laser beams in the waveguide with the Kerr nonlinearity.

- Large fluctuations during propagation seed the collapse event resulting in the formation of near diffraction-limited beam.

- Optical fiber length $\sim 4\text{m}$ with optical fiber diameter $\sim 2\text{mm}$ is sufficient to achieve the combined beam power

\[ P_c = 4.7\text{MW} \]