Long-time dynamics of NLS with harmonic trapping

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Outline

1. Three NLS systems with confinement

2. Analysis and effective dynamics of the three systems
   - System I: Cubic NLS on a periodic box
   - System II: NLS with harmonic trapping
   - NLS with partial trapping

3. A surprising connection between the effective dynamics

System I: Cubic NLS on a periodic box

$$\begin{cases} i\partial_t v(t,x) + \Delta v(t,x) & = \pm |v(t,x)|^2 v(t,x), \\ v(0,x) & = v_0(x), \end{cases} \tag{S-I}$$

- **Confinement:** $x \in \mathbb{T}_L^d = [0, L]^d$ with periodic boundary conditions, $d \geqslant 2$.
- **Effective dynamics:** We shall derive effective dynamics in the weak nonlinearity ($\Leftrightarrow$ small data of characteristic size $\epsilon \ll 1$) and large box limit ($L \gg 1$). This will hold up to time scales $\sim \frac{L^2}{\epsilon^2}$.

[Faou-Germain-H., JAMS 2015], [Buckmaster, Germain, H., Shatah; preprint 2016].

System II: Cubic NLS with harmonic trapping

$$\begin{cases} (i\partial_t + H)v(t,y) & = \pm|v(t,y)|^2 v(t,y), \\ H & = -\Delta_{\mathbb{R}^2} + |y|^2; \qquad y \in \mathbb{R}^2 \\ v(0,y) & = v_0(y), \end{cases} \tag{S-II}$$

- **Confinement:** Full harmonic trapping, $y \in \mathbb{R}^2$.
- **Effective dynamics:** We shall derive effective dynamics in the weak nonlinearity ($\Leftrightarrow$ small data of characteristic size $\epsilon \ll 1$) limit. This will hold up to time-scales $\sim \frac{1}{\epsilon^2}$.

[Germain, H. , Thomann; JMPA 2015]

System III: Cubic NLS with partial harmonic trapping

$$\begin{cases} i\partial_t v(t,x) + H_{pt} & = \pm|v(t,x)|^2 v(t,x), \\ H_{pt} & = -\Delta_{\mathbb{R}^3} + |y|^2, \qquad x = (x_1, y) \in \mathbb{R} \times \mathbb{R}^2 \\ v(0,x) & = v_0(x), \end{cases} \tag{S-III}$$

- **Partial Confinement:** Partial harmonic trapping, $x = (x_1, y) \in \mathbb{R}^3$.
- **Effective dynamics:** We shall derive effective dynamics asymptotically in time (as $t \to \infty$).

[H.-Thomann, CPAM 2016].

System I: Cubic NLS on a periodic box

$$\begin{cases} i\partial_t v(t,x) + \Delta v(t,x) &= \pm|v(t,x)|^2 v(t,x), \\ v(0,x) &= v_0(x), \quad x \in \mathbb{T}^d_L = [0,L]^d \circlearrowright \end{cases} \tag{S-I}$$

- Ansatz $v(t,x) = \epsilon u(t,x)$ with $\|u_0\|_{L^2(\mathbb{T}^2_L)} \sim 1 \rightsquigarrow$ Weak nonlinearity.

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) &= \pm\epsilon^2 |u(t,x)|^2 u(t,x) \\ u(0) &= u_0, \end{cases} \tag{S-I}$$

$$E[u(t)] := \frac{1}{2}\int_{\mathbb{T}^d_L} |\nabla u(t)|^2 dx \pm \frac{\epsilon^2}{4}\int_{\mathbb{T}^d_L} |u(t)|^4 dx = E[u(0)]$$

.

- Sign of nonlinearity is not important for us today, partly because ϵ is small. The energy controls the H^1 norm for any ϵ in the defocusing case, and if ϵ isn't too large in the focusing case (use the Gagliardo-Nirenberg-Sobolev inequality $\|u\|_{L^4} \lesssim \|u\|_{L^2}^{1-\frac{d}{4}} \|\nabla u\|_{L^2}^{\frac{d}{4}}, d \leqslant 4$). From now on $\pm \leftrightarrow +$.

Fourier picture

- $u(t,x) = \frac{1}{L^d} \sum_{K \in \mathbb{Z}^2/L} \widehat{u}_K(t) e^{2\pi i K.x}$; $\widehat{u}_K(t) := \int_{\mathbb{T}_L^d} u(t,x) e^{-2\pi i K.x}\, dx$.
- The equation satisfied by $\widehat{u}_K(t)$ is equivalent to $(a_K(t) := e^{4\pi^2 i|K|^2 t} \widehat{u}_K(t))$,

$$\begin{cases} i\partial_t a_K(t) = \frac{\epsilon^2}{L^{2d}} \sum_{(K_1,K_2,K_3)\in\mathcal{S}_K} a_{K_1}(t)\overline{a_{K_2}(t)} a_{K_3}(t) e^{-4\pi^2 i\Omega t} \\ \Omega = |K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2. \end{cases} \tag{S-I}$$

where $\mathcal{S}_K = \{(K_1,K_2,K_3) \in (\mathbb{Z}_L^d)^3 \ : \ K_1 - K_2 + K_3 = K\}$.

- **Resonant interactions**: $\mathcal{R}(K) = \mathcal{S}(K) \cap \{\Omega = 0\}$ are most important.

$$i\partial_t r_K = \frac{\epsilon^2}{L^{2d}} \sum_{(K_1,K_2,K_3)\in\mathcal{R}_K} r_{K_1}(t)\overline{r_{K_2}(t)} r_{K_3}(t) \tag{RS-I}$$

Effective dynamics on large domains

- **Aim:** Understand the effective dynamics in the regime of small $\epsilon \ll 1$ (weak nonlinearity), and large $L \gg 1$ (thermodynamic limit).
- The motivation for doing this was two-fold:
  1. Wave turbulence theory: This is the theory of nonequilibrium statistical physics of dispersive waves. The main theme is that the long-time behavior of dispersive PDE is described from a statistical point of view by a kinetic equation (*Wave Kinetic equation*) analogous to Boltzmann's equation of particle collisions. Understanding those two limits is part of an attempt to justify the wave turbulence closures.
  2. Effective dynamics on large domains: Given a dispersive system that is naturally posed on a large domains (e.g. water waves equation). What is the right "infinite volume approximation"? Time-scales are important!

The $\epsilon \to 0$ limit

- Approximate the NLS dynamic with that of (RNLS).
- Normal form transformation: This allows to replace to write the equation as

$$i\partial_t a_K = \frac{\epsilon^2}{L^{2d}} \sum_{(K_1,K_2,K_3)\in\mathcal{R}_K} a_{K_1}(t)\overline{a_{K_2}(t)}a_{K_3}(t) + \text{“}O(\epsilon^4)\text{”}$$

$$\mathcal{R}(K) = \{(K_1, K_2, K_3) \in \mathbb{Z}_L^2 : K_1 - K_2 + K_3 = K, \quad \Omega := |K_1|^2 - |K_2|^2 + |K_3|^2 - |K|^2 = 0\}.$$

- Doing one order of normal forms leads to a rather restrictive condition on ϵ (namely $\epsilon < L^{-1}$, [FGH '15]) So, we apply a very large number of normal forms to improve on this [BGHS '16].
- Upshot is that if $\epsilon < L^{-\gamma}$ for any $\gamma > 0$, we can pretend that we are working with (RNLS).

The large-box limit

- $L \to \infty$ limit: This amounts to thinking of the resonant sum on the R.H.S. as a Reimann sum (formally),

$$\sum_{(K_1,K_2,K_3)\in\mathcal{R}_K} a_{K_1}(t)\overline{a_{K_2}(t)}a_{K_3}(t) = C(L)\int_{(\xi_1,\xi_2,\xi_3)\in\mathcal{R}(K)} a(\xi_1)\overline{a(\xi_2)}a(\xi_3)d\sigma(\vec{\xi})$$

$$+ \text{ error terms}$$

1) What is $C(L)$?, 2) What is $d\sigma$?, 3) How small is the error?

- Answering this question falls in the realm of analytic number theory. In 2D, one can get an answer by elementary number theoretic techniques [F-G-H 2015].
- For general dimension (and degree of nonlinearity), the key method is the Hardy-Littlewood circle method, particularly a new form of it due to Heath-Brown.
- **End Result**: $d\sigma$ is the Dirac measure on the variety $\mathcal{R}(K)$, and

$$C(L) = \begin{cases} \frac{\zeta(d-1)}{\zeta(d} L^{2d-2} & \text{if } d \geqslant 3 \\ \frac{1}{\zeta(2)} L^2 \log L & \text{if } d \geqslant 2 \end{cases}, \quad \zeta(\cdot) \text{ is the Riemann zeta fn.}$$

Continuum limit

- In the upshot, we get (first at a formal level)

$$i\partial_t a(K,t) = \frac{\epsilon^2}{L^{2d}} \sum_{(K_1,K_2,K_3)\in\mathcal{R}_K} a_{K_1}(t)\overline{a_{K_2}(t)}a_{K_3}(t)$$

$$= \frac{1}{T^*} \int_{\substack{K_1,K_2,K_3\in\mathbb{R}^d \\ \cap\mathcal{R}(K)}} a(K_1,t)\bar{a}(K_2,t)a(K_3)\,d\sigma(K_1,K_2,K_3) + \textit{errors}$$

where $T_R \stackrel{def}{=} \frac{C(L)}{L^{2d}} \sim \frac{L^2}{\epsilon^2}$ (up to $\log L$ for $d=2$).

- Reparametrizing time $t = T_R\tau$, we get formally that

$$i\partial_\tau a(K,\tau) = \int_{\substack{K_1,K_2,K_3\in\mathbb{R}^d \\ \cap\mathcal{R}(K)}} a(K_1,\tau)\bar{a}(K_2,\tau)a(K_3,\tau)\,d\sigma(K_1,K_2,K_3).$$

Effective dynamics on large domains

$$
\begin{aligned}
i\partial_t g(\xi,t) =& \mathcal{T}(g,g,g)(\xi,t); \qquad \xi\in\mathbb{R}^d\\
\mathcal{T}(g,g,g)(\xi,t) =& \int_{\mathcal{R}(\xi)} g(\xi_1,t)\overline{g}(\xi_2,t)g(\xi_3)\,d\sigma(\xi_1,\xi_2,\xi_3).\\
\mathcal{R}(\xi) =& \{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^d : \xi_1-\xi_2+\xi_3=\xi,\\
& \Omega := |\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi|^2=0\}
\end{aligned}
\tag{CR}
$$

- Here $g:\mathbb{R}_t\times\mathbb{R}^d_\xi\to\mathbb{C}$, and $d\sigma$ is the Dirac measure on the variety $\mathcal{R}(\xi)$. The equation seems to be new.
- It is Hamiltonian (like NLS) with Hamiltonian functioal:

$$
\mathcal{H}(g)=\frac{(2\pi)^{d-1}}{2}\int_{\mathbb{R}_s}\int_{\mathbb{R}^d_x}|e^{is\Delta_{\mathbb{R}^2}}\widehat{g}(x)|^4 dsdx \quad \to L^4_{t,x} \text{ Strichartz norm!}
$$

Theorem (FGH '15, BGHS '16)

Let $d \geqslant 2$ and $\gamma > 0$ be arbitrary. Suppose that $g(t, \xi)$ is a sufficiently "nice" solution[a] of the (CR) equation on an interval $[0, M]$(M arbitrary). Suppose we start with an NLS solution such that $a_K(0) = g_0(K)$. If L is large enough, and if $\epsilon < L^{-\gamma}$, then

$$\|a_K(t) - g(\frac{t}{T_R}, K)\|_{L^2 \cap L^\infty} = o(1)_{L \to \infty}, \qquad \text{i.e. it } \to 0 \text{ as } L \to \infty$$

for all $0 \leqslant t \leqslant MT_R$ *where* $T_R = \frac{\zeta(d)}{\zeta(d-1)} \left(\frac{L^{2d}}{Z(L)\epsilon^2} \right)$ "$\sim$" $\frac{L^2}{\epsilon^2}$.

[a] (CR) is globally regular.

- The $o(1)$ rate is a negative power of L for $d \geqslant 3$.
- The $d = 2$ result was first worked out in [F-G-H., JAMS 2015]. We refine it in [B-G-H-S] by improving the on the $\epsilon - L$ relationship and the polynomial decay rate which requires isolating a logarithmic correction term.

Properties of the (CR) equation

- The equation enjoys many remarkable properties, particularly when $d = 2$ (also for $d = 1$ in the quintic case). For example,

Theorem (Invariance under Fourier transform, $d = 2$)

If $g(t)$ is a solution of (CR), the so is $\widehat{g}(t) := \mathcal{F}g(t)$. Moreover,

$$\mathcal{H}(f) = \mathcal{H}(\widehat{f}) \qquad \text{for any function } f \in L^2.$$

- It also interacts nicely with the harmonic oscillator operator: $H = -\Delta + |x|^2$ which admits an orthonormal basis of eigenvectors for $L^2(\mathbb{R}^2)$.

Theorem

Let E_k be the eigenspace of H corresponding to the eigenvalue $2k$ $(k = 1, 2, \ldots)$. The spaces E_k are invariant by the nonlinear flow of (CR), i.e. if $g_0 \in E_k$, then $g(t) \in E_k$ for all $t \in \mathbb{R}$.

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Solutions

- (CR) is globally regular and has many explicit stationary solutions.
- Gaussian family: For any $\alpha > 0$,

$$g(t,z) = e^{i\frac{\pi^2}{2}t} e^{-\frac{\pi}{2}|z|^2} \quad \text{for any } \alpha > 0, z \in \mathbb{R}^2$$

$$g(t,z) = e^{i\frac{\pi^2}{2}\frac{(2n)!}{4^n(n!)^2}t} z^n e^{-\frac{\pi}{2}|z|^2}, z \in \mathbb{R}^2 = \mathbb{C}, n \geqslant 0.$$

- Many other dynamics were later discovered and studied offering insight about that of NLS. See recent works [Germain, H., Thomann, 2015] for a deterministic and probabilistic analysis.
- "Raleigh-Jeans" solution

$$g(t,\xi) = \frac{e^{ict}}{|\xi|} \quad \text{solves} \quad (\text{CR}) \qquad \text{corresponds to } n(\xi) = |\xi|^{-2} \text{of (WKE)}.$$

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System II: Cubic NLS with harmonic trapping

$$\begin{cases} (i\partial_t + H)v(t,y) &= |v(t,y)|^2 v(t,y), \\ H &= -\Delta_{\mathbb{R}^2} + |y|^2 \\ v(0,y) &= v_0(y), \qquad y \in \mathbb{R}^2, \end{cases} \tag{S-II}$$

- To find the effective dynamics for small solutions. Again we write $v = \epsilon u$, then
$$(i\partial_t + H)u = \epsilon^2 |u|^2 u.$$
- Interaction representation: Let $f(t) = e^{-itH}u$,
$$i\partial_t f = \epsilon^2 e^{-itH}\left(|e^{itH}f|^2 e^{itH} f\right).$$
- Freuqency Formulation: Expand $f = \sum_p f_p$ where $f_p = \Pi_p f$ is the projection on the $p-$th eigenspace of H: With $\lambda_p = 2(p+1)$

$$i\partial_t f_p = \epsilon^2 \Pi_p \sum_{p_1,p_2,p_3 \in \mathbb{N}} e^{i(\lambda_{p_1} - \lambda_{p_2} + \lambda_{p_3} - \lambda_p)t} f_{p_1} \overline{f_{p_2}} f_{p_3} \qquad \Leftrightarrow \text{S-II}.$$

Resonant system

- Resonant interactions correspond to $\lambda_{p_1} - \lambda_{p_2} + \lambda_{p_3} = \lambda_p$.

$$
\begin{aligned}
i\partial_t f =& \epsilon^2 \mathcal{R}(f,f,f) \\
\Pi_p \mathcal{R}(f,f,f) =& \sum_{\lambda_{p_1} - \lambda_{p_2} + \lambda_{p_3} = \lambda_p} f_{p_1} \overline{f_{p_2}} f_{p_3}.
\end{aligned} \tag{RS-II}
$$

- In order to observe the Hamiltonian structure, this can be written as

$$
\begin{aligned}
\mathcal{R}(f,f,f) =& \frac{2}{\pi} \sum_{p_1,p_2,p_3,p \in \mathbb{N}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{i \overbrace{(\lambda_{p_1} - \lambda_{p_2} + \lambda_{p_3} - \lambda_p)}^{multiple\ of\ 4} \tau} \Pi_p \left(f_{p_1} \overline{f_{p_2}} f_{p_3} \right) \, d\tau \\
=& \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-i\tau H} \left(e^{i\tau H} f \overline{e^{i\tau H} f} e^{i\tau H} f \right) \, d\tau = \frac{\partial \mathcal{Q}}{\partial \bar{f}}
\end{aligned}
$$

- Hamiltonian is

$$
\mathcal{Q} = \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\mathbb{R}^2} \left| e^{i\tau H} f \right|^4 d\tau \, dy
$$

Resonant approximation

Theorem

Let $u(t)$ be a solution of (S-II) and $f(t)$ a solution to (RS-II) starting with the same initial data with characteristic size ϵ. Then for all $0 \leqslant t \leqslant C\epsilon^{-2}\log(\epsilon^{-1})$, one has that

$$\|e^{-itH}u(t) - f(t)\|_{\mathcal{H}^s} \leqslant C\epsilon^{5/2}.$$

Here, $\mathcal{H}^s$ is the Hermite Sobolev space for any $s > 1$.

Standard normal forms argument. See for example [Germain, H., Thomann '15].

System III: Cubic NLS with partial harmonic trapping

$$\begin{cases} (i\partial_t + H_{pt})v(t,x_1,y) & = \pm|v(t,x_1,y)|^2 v(t,x_1,y), \\ H_{pt} & = -\Delta_{\mathbb{R}^3} + |y|^2, \qquad x = (x_1,y) \in \mathbb{R}\times\mathbb{R}^2. \end{cases} \tag{S-III}$$

- The resonant system is a simple lift of (RS-II) to $\mathbb{R}\times\mathbb{R}^2$. However, the justification of this is a bit more involved to derive. Denoting by $\widehat{f}(t,\xi,y)$ the Fourier transform in the x_1 variable, the resonant system is:

$$i\partial_t \widehat{f}(t,\xi,\cdot) = \mathcal{R}[\widehat{f}(t,\xi,\cdot),\widehat{f}(t,\xi,\cdot),\widehat{f}(t,\xi,\cdot)]. \tag{RS-III}$$

For each fixed ξ, this is the same as RS-II. The appearance of ξ is parametric.

Resonant **Asymptotics**

Theorem (H.-Thomann, CPAM 2016)

Suppose that the initial data u_0 is small in some Banach space S^+ of functions. Let $u(t)$ be the solution of (S-III) with initial data u_0.
Then $u(t)$ exhibits **modified scattering** *to the resonant dynamics of (RS-III) in the following sense: there exists a solution $g(t)$ of (RS-III) such that*

$$\left\| e^{-itH_{pt}} u(t) - g(\pi \log t) \right\|_S \to 0 \text{ as } t \to +\infty.$$

- Recall that scattering means that $e^{-itH_{pt}} u(t)$ converges to a time-independent g_∞ as $t \to \infty$. Also, compare to modified scattering by phase correction.
- The proof follows a similar result of H.-Pausader-Tzvetkov-Visciglia for NLS on waveguide domains.

Explaining the logarithmic time-scale

- The idea of the proof is to look at the equation of $f(t) := e^{-itH_{pt}}u(t)$, and try to write it as

$$i\partial_t f(t) = \text{effective part} + \text{integrable part}.$$

- This requires a rather delicate energy-decay bootstrap scheme to propagate infinite-time estimates. The effective part turns out to be (in Fourier space $\mathcal{F}_{x_1}$)

$$i\partial_t \widehat{f}(t,\xi,\cdot) = \frac{\pi}{t}\mathcal{R}[\widehat{f}(t,\xi,\cdot),\widehat{f}(t,\xi,\cdot),\widehat{f}(t,\xi,\cdot)] + O(t^{-1-\delta}).$$

- Changing to the timescale $\tau = \pi \log t$, one obtains exactly (RS-III).

Conclusion up till now

- We considered three systems (S-1), (S-II), and (S-III) and derived effective equations for them as follows:
  1. **System S-I:** This was NLS on a box of size L with periodic bc. We derived effective dynamics for small $\epsilon \ll 1$ and large L. This was given by the (CR) system.
  2. **System S-II:** This was NLS with full trapping on $\mathbb{R}^2$. We derived effective dynamics for small $\epsilon \ll 1$. This was given by (RS-II).
  3. **System S-III:** This was NLS with partial trapping on $\mathbb{R}^3$. We derived effective dynamics for large time. This was given by (RS-III).
- We saw that (RS-III) is essentially (RS-II) lifted trivially one dimension up.
- **Surprise:** In fact, (CR) system is exactly (RS-II) in disguise.

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Compare the Hamiltonians

- To see the equivalence of the two systems, we compare the two Hamiltonians

$$\mathcal{H}_{(\mathrm{CR})} = \int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{-is\Delta} f|^4 \, ds \, dx \quad \text{and} \quad \mathcal{Q} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\mathbb{R}^2} \left| e^{i\tau H} f \right|^4 d\tau \, dx$$

- Lens transform: If $u(t,x) = e^{-it\Delta} f$ and $v(t,x) = e^{itH} f$, then

$$u(t,x) = \frac{1}{\sqrt{1+4t^2}} v\Big(\frac{\arctan 2t}{2}, \frac{x}{\sqrt{1+4t^2}}\Big) e^{i\frac{x^2 t}{1+4t^2}}$$

- This gives that

$$\mathcal{Q} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\mathbb{R}^2} \left| e^{i\tau H} f \right|^4 d\tau \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{-is\Delta} f|^4 \, ds \, dx = \mathcal{H}_{(\mathrm{CR})}!!$$

Theorem (Germain, H., Thomann 2014)

The (CR) equation derived as the large box limit of the resonant system of the homogeneous (NLS) is exactly the resonant systems with harmonic trapping.

Applications

- Therefore, we have $\boxed{(CR) = (RS - II)``="(RS - III)}$.
- (CR) has many special symmetries and solutions that give us deep insight into its long-time dynamics. Let's see what they imply for the the NLS with full and partial harmonic trapping.
  - A) Dynamical justification of physical approximations.

    [P. Kevrekidis, D. Frantzeskakis and R. Carretero-Gonzàlez, *Emergent nonlinear phenomena in Bose-Einstein condensates*, Springer (2008)].
  - B) New insight on the long time dynamics (NLS) in partial and full harmonic traps.

A) Quasi-1D dynamics

- One ansatz commonly used in BEC is the quasi-1D dynamics:

$$u(t,x,y) \sim \psi(t,x)e^{2it}e^{-|y|^2/2}; \qquad y = (y_1, y_2),$$

where $\psi(t,x)$ is solution of the 1d NLS

$$(1dNLS) \qquad (i\partial_t - \partial_x^2)\psi = \frac{1}{2}|\psi|^2\psi, \quad (t,x) \in \mathbb{R}\times\mathbb{R},$$

Interest :

  - The dynamics in (x,y) is decoupled
  - The long time behaviour if $(1dNLS)$ is well-understood

- Using that $g(t,z) = e^{i\frac{\pi^2}{2}t}e^{-\frac{\pi}{2}|z|^2}(z = y_1 + iy_2 \in \mathbb{C})$ is a stationary solution of (CR), this can be dynamically justified as follows:

Theorem (H.-Thomann, CPAM 2016)

Let $\psi(t,x_1)$ be a solution of (1dNLS). Then there exists a solution $u(t,x_1,y) \in \mathcal{C}\big([0,+\infty); L^2(\mathbb{R}\times\mathbb{R}^2)\big)$ of (S-III) such that

$$\big\|u(t,x_1,y) - \psi(t,x_1)e^{2it}e^{-\frac{1}{2}|y|^2}\big\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \longrightarrow 0 \quad as \quad t \longrightarrow +\infty.$$

Vortex solutions and their dynamics

- Another interesting set of stationary solutions of (CR) are the so-called vortex solutions

$$g(t,z) = e^{i\frac{\pi^2}{2}\frac{(2n)!}{4^n(n!)^2}t} z^n e^{-\frac{\pi}{2}|z|^2}, z \in \mathbb{R}^2 = \mathbb{C}, n \geqslant 0$$

- Similar to above, one can find solutions to (NLS) with harmonic trapping that "shadow" such vortex solutions of (CR).
- More importantly, (CR) allows to understand the dynamics of a non-stationary state that is formed of a superposition of several vortices.
- Let $\varphi_n(z) = \frac{1}{\sqrt{\pi n!}} z^n e^{-\frac{\pi}{2}|z|^2}$, and take your initial data

$$g_0(z) = \sum_{n\geqslant 0} a_{n,0}\varphi_n(z) \in L^2(\mathbb{R}^2) \leftarrow \quad \text{Bargmann-Fock space.}$$

Theorem (Germain, H., Thomann, JMPA 2015)

The Bargmann-Fock space is an invariant subspace for the (CR) dynamics. If

$$g_0(z) = \sum_{n\geqslant 0} a_{n,0}\varphi_n(z)$$

then the solution $g(t,z)$ of (CR) can be written as: $g(t,z) = \sum_{n\geqslant 0} a_n(t)\varphi_n(z)$, where $a_n(t)$ satisfies the following system of ODE

$$i\partial_t a_n(t) = \sum_{n_1-n_2+n_3=n} \Gamma^n_{n_1 n_2 n_3} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3(t)}; \qquad \Gamma = \frac{\pi}{2}\frac{(n_1+n_3)!}{2^{(n_1+n_3)}\sqrt{n_1!n_2!n_3!n!}}.$$

- This allows studying the dynamics of states with many vortices excited at $t = 0$. Again, this dynamics is shadowed by solutions of NLS with harmonic trapping.
- This is the same as the so-called LLL equation (Lowest Landau Level equation) which is used to model fastly rotating BEC.

Thanks!

Thanks for your attention!

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