Necklace solitary waves on bounded domains

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• with Dima Shpigelman
Physical setup

- Intense laser beam that propagates in a transparent medium (air, water, glass, ...)
- Direction of propagation is $z$
NLS in nonlinear optics

\[ i\psi_z(z, x, y) + \Delta_{x,y} \psi + |\psi|^2 \psi = 0, \]

- Competition between diffraction \( \Delta \psi = \psi_{xx} + \psi_{yy} \) and focusing nonlinearity
- \( z \) “=” \( t \) (evolution variable)
- Initial value problem in \( z \)

\[ \psi(z = 0, x, y) = \psi_0(x, y) \]
Optical collapse

- Experiments in the 1960s showed that intense laser beams become narrower with propagation (self-focusing)
- Sufficiently powerful beams undergo optical collapse

- Kelley (1965): Solutions of 2D cubic NLS

\[ i\psi_t(t, x, y) + \Delta \psi + |\psi|^2 \psi = 0, \quad \psi(0, x, y) = \psi_0(x, y) \]

can become singular (blowup) in finite time (distance) \( T_c \)
Gadi Fibich

The Nonlinear Schrödinger Equation
Singular Solutions and Optical Collapse

This book is an interdisciplinary introduction to optical collapse of laser beams, which is modelled by singular (blow-up) solutions of the nonlinear Schrödinger equation. With great care and detail, it develops the subject including the mathematical and physical background and the history of the subject. It combines rigorous analysis, asymptotic analysis, informal arguments, numerical simulations, physical modelling, and physical experiments. It repeatedly emphasizes the relations between these approaches, and the intuition behind the results.

The Nonlinear Schrödinger Equation will be useful to graduate students and researchers in applied mathematics who are interested in singular solutions of partial differential equations, nonlinear optics and nonlinear waves, and to graduate students and researchers in physics and engineering who are interested in nonlinear optics and Bose-Einstein condensates. It can be used for courses on partial differential equations, nonlinear waves, and nonlinear optics.

Gadi Fibich is a Professor of Applied Mathematics at Tel Aviv University.

“This book provides a clear presentation of the nonlinear Schrödinger equation and its applications from various perspectives (rigorous analysis, informal analysis, and physics). It will be extremely useful for students and researchers who enter this field.”

Frank Merle, Université de Cergy-Pontoise and Institut des Hautes Études Scientifiques, France
Solitary waves

- NLS solutions of the form

\[ \psi_{\text{solitary}} = e^{i\mu t} R_\mu(x), \quad x = (x, y), \]

where

\[ \Delta R_\mu(x) - \mu R_\mu + R_\mu^3 = 0 \]

- Dependence on \( \mu \) through the **dilation symmetry**

\[ R_\mu(x) = \sqrt{\mu} R(\sqrt{\mu} x), \quad R := R_{\mu=1} \]

- **Power** (L2 norm) of \( R_\mu \) is **independent of** \( \mu \):

\[ P(R_\mu) \equiv P(R), \quad P(R_\mu) := \left\| R_\mu \right\|_2^2 = \int \left| R_\mu \right|^2 dx \, dy \]
Infinite number of radial solutions $R^{(n)}(r)$

Of most interest is the ground state $R^{(1)}$ (Townes profile)

- Positive, monotonically decreasing
- Has exactly the critical power for collapse $P_{cr} = \| R^{(1)} \|_2^2$
- Single peak at the origin

\[ \Delta R(x) - R + R^3 = 0, \quad x = (x, y) \]
Stability of solitary waves

\[ i\psi_t(t, x, y) + \Delta \psi + |\psi|^2 \psi = 0, \quad \psi(0, x, y) = \psi_0(x, y) \]

**Lemma** Let \( \psi_0 = c R^{(1)}_\mu(x) \)
- If \( c > 1 \), then \( \psi \) **blows up** in finite time
- If \( 0 < c < 1 \), then \( \psi \) **scatters** as \( t \to \infty \)

- Ground-state solitary waves have a **dual instability**
Challenge

How can we “overcome” the dual instability and propagate localized laser beams in a Kerr medium over long distances?

Important for applications
Necklace beams

Soljacic, Sears, Segev (1998): Use *necklace beams*
e.g., consider the input beam

\[
\psi_0 = c \cdot \text{sech}(r - r_{\text{max}}) \cos(m\theta)
\]

- Consists of 2m identical beams (``pearls''), located
  at equal distances along a circle of radius \(r_{\text{max}}\)
  
- Adjacent input beams have opposite signs (phases)
Necklace beams

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  - Hence, \( \Psi_0 = 0 \) on the rays separating the beams
  - By uniqueness, \( \Psi = 0 \) on these rays for \( t > 0 \)
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  - These rays thus act as reflecting boundaries
    - Beams do not interact
    - Necklace structure preserved for \( t > 0 \)
Necklace beams

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  - By uniqueness, \( \Psi = 0 \) on these rays for \( t > 0 \)
- These rays thus act as reflecting boundaries
  - Beams do not interact
  - Necklace structure preserved for \( t > 0 \)
- Beams with opposite phases repel each other
Necklace beams

\[ \psi_0 = c \cdot \text{sech}(r - r_{\text{max}}) \cos(m\theta) \]

**Idea:**
- Let each beam (``pearl'') have power below \( P_{\text{cr}} \)
  - Nonlinearity weaker than diffraction, so beam expands
- Out-of-phase adjacent beams repel each other
  - Slows down the expansion of each beam
Simulation
Simulation
Necklace expansion much slower than for a single beam

Ultimately, necklace either scatters as $t \to \infty$, or collapses in finite time (distance), simultaneously at 2m points
Experiments

- Grow et al. (07): First observation of a necklace beam in a Kerr medium

![Input beam](image1.png) ![After propagating 30cm in glass](image2.png)
Recent application

- Jhajj et al (14): used a necklace beam configuration to set up a thermal waveguide in air
Comparison with vortex beams

Necklace beams

\[ \psi_0 = c \cdot \text{sech}(r - r_{\text{max}}) \cos(m\theta) \]

Vortex beams

\[ \psi_0 = c \cdot \text{sech}(r - r_{\text{max}}) \exp(i m\theta) \]
Necklace solitary waves?

**Proposition** (Fibich and Shpigelman, 16): The NLS

\[ i\psi_t(t, x, y) + \Delta \psi + |\psi|^2 \psi = 0, \quad -\infty < x, y < \infty \]

does not admit necklace solitary waves. In other words, there are no necklace solutions of

\[ \Delta R(x, y) - \mu R + R^3 = 0, \quad -\infty < x, y < \infty \]
Multi-peak solitary waves

**Theorem** (Ao, Musso, Pacard, Wei, 2012 & 2015) There exist multi-peak, sign-changing solutions of

\[ \Delta R(x, y) - R + R^3 = 0, \quad -\infty < x, y < \infty \]

These solutions do not have the necklace form
Multi-peak solitary waves

**Theorem** (Ao, Musso, Pacard, Wei, 2012 & 2015)

There exist multi-peak, sign-changing solutions of

\[
\Delta R(x, y) - R + R^3 = 0, \quad -\infty < x, y < \infty
\]

These solutions do not have the necklace form.

As with all solitary-wave solutions \( \psi = e^{it} R(x) \) of the critical NLS, they are **strongly-unstable**.
Propagation in PR crystal

- Yang et al. (05): Propagation in a photorefractive (PR) crystal with optically-induced photonic lattice

\[ i\psi_t (t, x, y) + \Delta \psi + \frac{E_0}{1 + I(x, y) + |\psi|^2} \psi = 0, \]

\[ I(x, y) := I_0 \sin^2(x + y) \sin^2(x - y) \]

- Showed theoretically and experimentally that necklace solitary waves exist, and can be stable

- Because of the need to induce a photonic lattice, not applicable for propagation in a Kerr medium
How to stabilize necklace beams in a Kerr medium?

**Idea:** Confine beam to a **hollow-core fiber**

- To leading-order, fiber walls are perfect reflectors
How to stabilize necklace beams in a Kerr medium?

**Idea**: Confine beam to a **hollow-core fiber**
- To leading-order, fiber walls are perfect reflectors
- Propagation governed by the **NLS on a bounded domain** \( D \subseteq \mathbb{R}^2 \)

\[
i \psi_t (t, x, y) + \Delta \psi + |\psi|^2 \psi = 0, \quad (x, y) \in D, \\
\psi = 0, \quad (x, y) \in \partial D
\]

- Typically, \( D \) is the unit disk \( B_1 = \{x^2+y^2 \leq 1\} \)
NLS on a bounded domain

Admits solitary waves $\psi_{\text{solitary}} = e^{i\mu t} R_\mu(x)$, where

$$\Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad (x, y) \in D,$$
$$R_\mu = 0, \quad (x, y) \in \partial D$$

Studied by Fibich and Merle (01), Fukuizumi, Hadj Selem and Kikuchi (12), Noris, Tavares and Verzini (15)

- Mostly for radial solutions on the unit disk $B_1$
Radial solitary waves on the unit disk $B_1$

\[ \Delta R_\mu(r) - \mu R_\mu + R_\mu^3 = 0, \quad 0 \leq r < 1, \]
\[ R_\mu = 0, \quad r = 1 \]

- Unlike free-space, no dilation symmetry

\[ R_\mu^{(1)}(r) \neq \sqrt{\mu} R_\mu^{(1)}(\sqrt{\mu}r) \]

\[ \frac{d}{d \mu} P(R_\mu^{(1)}) > 0, \quad \mu_{\text{lin}} < \mu < \infty \]

\[ \Delta R_\mu - \mu R_\mu = 0, \quad 0 \leq r < 1, \]
\[ R_\mu = 0, \quad r = 1 \]
Linear stability - review

Let \( \psi = e^{i\mu t} [R_\mu + \epsilon [u(x) + iv(x)] e^{\Omega t}] \)

Then the linearized equation for the perturbation reads

\[
\begin{pmatrix}
0 & L_+ \\
L_- & 0
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \Omega
\begin{pmatrix}
u \\
v
\end{pmatrix},
\]

where

\[
L_+ = \Delta - \mu + 3|R_\mu|^2, \quad L_- = \Delta - \mu + |R_\mu|^2
\]

- Linear instability if \( \text{Re}(\Omega) > 0 \)

Lemma (Vakhitov and Kolokolov): If \( R_\mu > 0 \), then \( \psi = e^{i\mu t} R_\mu \) is linearly stable if

\[
\frac{d}{d\mu} P(R_\mu) > 0
\]

and unstable if

\[
\frac{d}{d\mu} P(R_\mu) < 0
\]

VK condition
Lemma If $R_\mu > 0$, then $\psi = e^{i\mu t} R_\mu$ is orbitally stable if and unstable if

$$\frac{d}{d\mu} P(R_\mu) < 0$$

and

$$\frac{d}{d\mu} P(R_\mu) > 0$$

- Weinstein, Grillakis, Shatah, Strauss
Lemma If $R_\mu > 0$, then $\psi = e^{i \mu t} R_\mu$ is orbitally stable if
and unstable if

$$\frac{d}{d \mu} P(R_\mu) > 0$$

Weinstein, Grillakis, Shatah, Strauss

On the unit disc, VK condition is satisfied:
Solitary waves on the unit disk - stability

**THM** (Fibich and Merle, 2001) The ground-state solitary waves

\[ \psi = e^{i \mu t} R^{(1)}(r) \]

on the unit disk are **orbitally stable**:

If \( \| \psi_0 - R^{(1)}_\mu \| < \delta \), then \( \inf_{0 \leq \alpha \leq 2\pi} \| \psi(t) - e^{i \alpha} \psi^{\text{solitary}}(t) \| < \epsilon \)

- **Unlike** in free-space, where they are strongly unstable
- ```Reflecting boundary has a stabilizing effect```
Solitary waves on the unit disk - stability

**THM** (Fibich and Merle, 2001) The ground-state solitary waves 
\[ \psi = e^{i\mu} R_\mu^{(1)}(r) \]
on the unit disk are *orbitally stable*:

\[ \text{If } \|\psi_0 - R_\mu^{(1)}\| < \delta, \text{ then } \inf_{0 \leq \alpha \leq 2\pi} \|\psi(t) - e^{i\alpha} \psi^{\text{solitary}}(t)\| < \varepsilon \]

- **Unlike** in free-space, where they are strongly unstable
- ``Reflecting boundary has a stabilizing effect’’

- But, \( 0 < P(R_\mu^{(1)}) < P_{cr} \)
- **Goal:** Find stable solitary waves with \( P > P_{cr} \)
- Important for applications

- **Idea:** use necklace solitary waves!
Types of necklace solitary waves

\[ \Delta R_\mu - \mu R_\mu + R^3_\mu = 0, \quad (x, y) \in D, \]
\[ R_\mu = 0, \quad (x, y) \in \partial D \]
Types of necklace solitary waves

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad (x, y) \in D, \]
\[ R_\mu = 0, \quad (x, y) \in \partial D \]
Types of necklace solitary waves

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad (x, y) \in D, \]
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Types of necklace solitary waves

\[ \Delta R_{\mu} - \mu R_{\mu} + R_{\mu}^3 = 0, \quad (x, y) \in D, \]
\[ R_{\mu} = 0, \quad (x, y) \in \partial D \]

New solitary waves

circle  rectangle  annulus
Rectangular necklace solitary wave

- Compute a positive/ground-state/single-pearl solitary wave on a square
- Construct rectangular necklace using 3*2 single pearls
Single pearl on a square

$\Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad -1 < x, y < 1$

$R_\mu = 0, \quad x, y = \pm 1$

$R_\mu > 0, \quad -1 < x, y < 1$

- Compute using **non-spectral renormalization method**
Single pearl on a square

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad -1 < x, y < 1, \]
\[ R_\mu = 0, \quad x, y = \pm 1 \]
\[ R_\mu > 0, \quad -1 < x, y < 1, \]

- Compute using **non-spectral renormalization method**
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Single pearl on a square

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\[ R_{\mu} = 0, \quad x, y = \pm 1 \]
\[ R_{\mu} > 0, \quad -1 < x, y < 1, \]

weakly nl regime

\[ R_{\mu} \gg cR_{\mu_{\text{lin}}} \]
Single pearl on a square

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad -1 < x, y < 1, \]
\[ R_\mu = 0, \quad x, y = \pm 1 \]
\[ R_\mu > 0, \quad -1 < x, y < 1, \]

\[ R_{\mu} \square R_{\mu}^{\text{free-space}} = \sqrt{\mu} R^{\text{free-space}}(\sqrt{\mu} x) \]

weakly nl regime

\[ R_{\mu} \square c R_{\mu_{\text{lin}}} \]

strongly nl regime
Single pearl on a square

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad -1 < x, y < 1, \]

\[ R_\mu = 0, \quad x, y = \pm 1 \]

\[ R_\mu > 0, \quad -1 < x, y < 1, \]

\[ \frac{d}{d\mu} P \left( R_\mu^{(1)} \right) > 0 \]
From single pearl to a rectangular necklace
From single pearl to a rectangular necklace
Lemma: $R^{(3\times2)}_{\mu}$ is a necklace solution of

$$\Delta R_{\mu} - \mu R_{\mu} + R^{3}_{\mu} = 0, \quad -3 < x < 3, \quad -2 < y < 2,$$

$$R_{\mu} = 0, \quad x = \pm 3, \quad y = \pm 2$$

with $3 \times 2$ pearls
- Can also construct necklace solitary waves on non-rectangular domains
Circular necklace solitary wave

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad |x| < 1, \]

\[ R_\mu = 0, \quad |x| = 1 \]

- Compute a positive/ground-state/single-pearl on a sector of a circle

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Circular necklace solitary wave

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad |x| < 1, \]
\[ R_\mu = 0, \quad |x| = 1 \]

- Compute a positive/ground-state/single-pearl on a sector of a circle

\[ \frac{d}{d\mu} P(R^{(1)}_\mu(\mu)) > 0 \]

weakly nl regime

strongly nl regime
From single pearl to a circular necklace
From single pearl to a circular necklace

\[ R_\mu = 0 \text{ on interfaces between pearls} \]
Annular necklace solitary wave

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad r_{\text{min}} < |x| < r_{\text{max}} \]

\[ R_\mu = 0, \quad |x| = r_{\text{min}}, r_{\text{max}} \]

- Compute a positive/ground-state/single-pearl on a sector of an annulus
Annular necklace solitary wave

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad r_{\text{min}} < |x| < r_{\text{max}} \]

\[ R_\mu = 0, \quad |x| = r_{\text{min}}, r_{\text{max}} \]

- Compute a positive/ground-state/single-pearl on a sector

\[ \frac{d}{d\mu} P(R_\mu) > 0 \]

Strongly nl regime

weakly nl regime
From single pearl to an annular necklace
From single pearl to an annular necklace

\[ R_\mu = 0 \] on interfaces between pearls
Stability

- Single-pearl solitary waves
- Necklace solitary waves
Single-pearl stability

- Perturbed **single-pearl** solutions $\psi_0 = 1.05 R^{(1)}_\mu$

  - Numerical observation: All single-pearl solitary waves are stable.
  - Indeed, they satisfy the **VK condition** $\frac{d}{d\mu} P \left( R^{(1)}_\mu \right) > 0$
Necklace stability

- Necklace is **stable** under perturbations that preserve its anti-symmetries, since then it inherits the single-pearl stability
  - e.g. $\psi_0 = 1.05 R^{(n)}_{\mu}$

- To excite the instability numerically, use perturbations that break the anti-symmetries, such as
  - $\psi_0 = [1 + noise(x, y)]R^{(n)}_{\mu}$
  - Multiply bottom-left pearl of $R^{(n)}_{\mu}$ by 1.05
Necklace stability

- Since \( P\left(R_{\mu}^{(n)}\right) = nP\left(R_{\mu}^{(1)}\right) \), necklace satisfies the VK condition 
  \[ \frac{d}{d\mu} P\left(R_{\mu}^{(n)}\right) > 0 \]
- This does not imply stability, however, since the VK condition implies stability only for positive solitary waves
- Go back to the original linearized eigenvalue problem:
  Let \( \psi = e^{i\mu t}[R_{\mu}^{(n)} + \epsilon[ u(x) + iv(x)]] e^{\Omega t} \)

  Then
  \[
  \begin{pmatrix}
  0 & L_+ \\
  L_- & 0
  \end{pmatrix}
  \begin{pmatrix}
  u \\
  v
  \end{pmatrix} = \Omega
  \begin{pmatrix}
  u \\
  v
  \end{pmatrix},
  \]

  where
  \[
  L_+ = \Delta - \mu + 3 \left|R_{\mu}^{(n)}\right|^2,
  \quad
  L_- = \Delta - \mu + \left|R_{\mu}^{(n)}\right|^2
  \]
- Instability if \( \text{Re}(\Omega) > 0 \)
Circular necklace with 4 pearls

![Graph showing stability and instability for circular necklace with 4 pearls.](circular_necklace_graph.png)
Circular necklace with 4 pearls

Stable for \( \mu_{\text{lin}} < \mu < \mu_{\text{cr}} \)
Circular necklace with 4 pearls

- Stable for $\mu_{\text{lin}} < \mu < \mu_{\text{cr}}$
- Unstable for $\mu_{\text{cr}} < \mu < \infty$
Circular necklace with 4 pearls

$\square$ Stable for $\mu_{\text{lin}} < \mu < \mu_{\text{cr}}$

$\square$ Unstable for $\mu_{\text{cr}} < \mu < \infty$

$P_{\text{th}}^{\text{necklace}} := P\left( R^{(4)}_{\mu_{\text{cr}}} \right) = ?$
Circular necklace with 4 pearls

- Stable for $\mu_{\text{lin}} < \mu < \mu_{\text{cr}}$
- Unstable for $\mu_{\text{cr}} < \mu < \infty$

\[ P_{\text{th}}^{\text{necklace}} := P \left( R^{(4)} \right) \approx 0.24 P_{\text{cr}} \]

- Necklace becomes unstable when the power of each pearl is $0.08 P_{\text{cr}}$
- Necklace instability not related to collapse!
- Necklace "less stable" than radial solitary waves
Dynamics

- Multiply bottom-left pearl of $R^{(4)}$ by 1.05

$$\mu = \mu_{cr} - 1$$

$$\mu = \mu_{cr} + 1$$
Dynamics

- Multiply bottom-right pearl of $R_{\mu}^{(4)}$ by 1.05

\[ \mu = \mu_{cr} - 1 \]

\[ \mu = \mu_{cr} + 1 \]

Instability but no collapse!
Stable necklace propagation above $P_{cr}$?

- Saw that

\[ P\left( R_{\mu_{cr}}^{(4)} \right) \approx 0.24 P_{cr} \]

- **Idea 1**: take more pearls
Stable necklace propagation above $P_{cr}$?

- Saw that
  \[ P \left( R^{(4)}_{\mu_{\text{cr}}} \right) \approx 0.24P_{cr} \]

- **Idea 1**: take more pearls
  \[ P \left( R^{(6)}_{\mu_{\text{cr}}} \right) \approx 0.23P_{cr}, \quad P \left( R^{(8)}_{\mu_{\text{cr}}} \right) \approx 0.2P_{cr} \]

- Threshold power for necklace instability **decreases** with number of pearls
Stable necklace propagation above $P_{cr}$?

- Saw that

$$P\left(\frac{R^{(4)}}{\mu_{cr}}\right) \approx 0.24P_{cr}$$

- **Idea 1**: take more pearls

$$P\left(\frac{R^{(6)}}{\mu_{cr}}\right) \approx 0.23P_{cr}, \quad P\left(\frac{R^{(8)}}{\mu_{cr}}\right) \approx 0.2P_{cr}$$

- Threshold power for necklace instability *decreases* with number of pearls
Stable necklace propagation above $P_{cr}$?

- Saw that
  \[ P \left( \frac{R^{(4)}}{\mu_{cr}} \right) \approx 0.24 P_{cr} \]

- **Idea 1**: take more pearls
  \[ P \left( \frac{R^{(6)}}{\mu_{cr}} \right) \approx 0.23 P_{cr}, \quad P \left( \frac{R^{(8)}}{\mu_{cr}} \right) \approx 0.2 P_{cr} \]

- Threshold power for necklace instability **decreases** with number of pearls

- **Idea 2**: Use rectangular domain


\[ \text{Diagram showing necklace propagation} \]
Stable necklace propagation above $P_{cr}$?

- Saw that
  \[ P \left( R_{\mu_{cr}}^{(4)} \right) \approx 0.24 P_{cr} \]

- **Idea 1**: take more pearls
  \[ P \left( R_{\mu_{cr}}^{(6)} \right) \approx 0.23 P_{cr}, \quad P \left( R_{\mu_{cr}}^{(8)} \right) \approx 0.2 P_{cr} \]

- Threshold power for necklace instability **decreases** with number of pearls

- **Idea 2**: Use rectangular domain
  \[ P \left( R_{\mu_{cr}}^{(2*2)} \right) \approx 0.55 P_{cr} \]

- **Surprising**, but not good enough
Stable necklace propagation above $P_{cr}$?

- Saw that:
  \[ P \left( R^{(4)}_{\mu_{cr}} \right) \approx 0.24P_{cr} \]

- **Idea 1**: take more pearls
  \[ P \left( R^{(6)}_{\mu_{cr}} \right) \approx 0.23P_{cr}, \quad P \left( R^{(8)}_{\mu_{cr}} \right) \approx 0.2P_{cr} \]

- Threshold power for necklace instability **decreases** with number of pearls

- **Idea 2**: Use rectangular domain
  \[ P \left( R^{(2*2)}_{\mu_{cr}} \right) \approx 0.55P_{cr} \]

- **Surprising**, but not good enough
Instability dynamics

- To raise the threshold power for necklace stability, need to understand the instability dynamics
- Linearized eigenvalue problem:
  \[
  \begin{pmatrix}
  0 & L_+ \\
  L_- & 0
  \end{pmatrix}
  \begin{pmatrix}
  u \\
  v
  \end{pmatrix}
  = \Omega
  \begin{pmatrix}
  u \\
  v
  \end{pmatrix}
  \]
- Instability if \( \text{Re}(\Omega) > 0 \)
  - Yes/No approach
- Fibich, Ilan, Sivan, Weinstein (06-08): Deduce instability dynamics from the unstable eigenfunctions
Instability dynamics
Instability dynamics

(c) 4 pearls

Re(Ω)

μ_∞, μ_∞(2)

μ_cr-22

Π

Re(u)  Im(u)  Re(v)  Im(v)

0  0  0  0

-1  1 -1  1

y

+ - - +

- - + +
Instability dynamics

- Perturbation (pert.)

\[ \text{e.f.} + \text{pert.} = \]
Instability dynamics

e.f. + pert. =

\[ \text{Re}(\Omega) \]

\[ \text{Im}(u) \]

\[ \text{Re}(v) \]

\[ \text{Im}(v) \]

\[ \text{direction of power flow} \]

\[ + - - + \]
Instability dynamics

![Diagram of instability dynamics](image)
Instability dynamics

Graph showing the relationship between $\text{Re}(\Omega)$ and $\mu_{\text{ln}}$ and $\mu_{\text{cr}}$.

Legend:
- +
- -

Direction of power flow:
- + +
- - -
- - +
- + -
Instability dynamics

![Diagram with plots and symbols](image-url)

- **Re(Ω)** vs. **μ**
- **4 pearls**
- **Re(u)** and **Im(u)**
- **Re(v)** and **Im(v)**
- **Direction of power flow**
Instability dynamics

4 pearls

Re(Ω)

μ₁₁

μ₂₂

μ_cr-22

μ_cr

H

Re(u)

Im(u)

Re(v)

Im(v)

direction of power flow

y

-1

0

1

-1

0

1

x

-1

0

1

-1

0

1
Stable necklace propagation above $P_{cr}$?

- Saw that

\[
P \left( R(4) \mid \mu_{cr} \right) \approx 0.24 P_{cr}
\]

- **Idea 1**: take more pearls

\[
P \left( R(6) \mid \mu_{cr} \right) \approx 0.23 P_{cr}, \quad P \left( R(8) \mid \mu_{cr} \right) \approx 0.2 P_{cr}
\]

- Threshold power for necklace instability decreases with number of pearls

- **Idea 2**: Use rectangular domain

\[
P \left( R(2*2) \mid \mu_{cr} \right) \approx 0.55 P_{cr}
\]

- Surprising, but not good enough

- **Idea 3**: Use annular domain, since hole reduces interactions between pearls
Annular necklace with 4 pearls
Annular necklace with 4 pearls

- Stable for $\mu_{\text{lin}} < \mu < \mu_{\text{cr}}$
- $P(R^{(4)}_{\mu_{\text{cr}}}) \approx 0.24 P_{\text{cr}}$
- Hole does not help
Annular necklace with 4 pearls

- Stable for $\mu_{\text{lin}} < \mu < \mu_{\text{cr}}$
- $P\left(R^{(4)}_{\mu_{\text{cr}}}\right) \approx 0.24 P_{\text{cr}}$
- Hole does not help

- Second stability regime!
- $3.1 P_{\text{cr}} < P\left(R^{(4)}_{\mu}\right) < 3.7 P_{\text{cr}}$
- Hole does help!
Dynamics

- $\psi_0 = [1 + 0.02 \text{ noise}(x, y)]R^{(4)}_{\mu}$
Dynamics

- \( \psi_0 = [1 + 0.02 \, noise(x, y)]R_{\mu}^{(4)} \)
- $\psi_0 = [1 + 0.02 \text{noise}(x, y)]R_\mu^{(4)}$

\[\mu = 10\]

\[\mu = 30\]
Dynamics

- $\psi_0 = [1 + 0.02 \text{ noise}(x, y)] R_\mu^{(4)}$

stable solitary wave with $P > 3P_{cr}$

$\mu = 10$

$\mu = 30$
Positive solitary waves on the annulus

\[ \Delta R_\mu (x) - \mu R_\mu + R_\mu^3 = 0, \quad r_{\min} < |x| < r_{\max} \]

- Can look for radial solutions

\[ \Delta R_\mu (r) - \mu R_\mu + R_\mu^3 = 0, \quad r_{\min} < r < r_{\max} \]

- Are these the ground-state (minimal power) solutions?
- Compute ground-state solutions numerically using the non-spectral renormalization method
Ground states on the annulus

\[ \Delta R_\mu(x) - \mu R_\mu + R_\mu^3 = 0, \quad r_{\text{min}} < |x| < r_{\text{max}} \]

\[ R_\mu = 0, \quad |x| = r_{\text{min}}, r_{\text{max}} \]

\[ R_\mu > 0, \quad r_{\text{min}} < |x| < r_{\text{max}} \]

- Radial for \( \mu_{\text{lin}} < \mu < \mu_{\text{cr}} \)
- Non-radial for \( \mu_{\text{cr}} < \mu < \infty \)

- Symmetry breaking
- Related to variational characterization of the ground state

\[ \inf \left\| U \right\|_{P}^2 H(U) \]
R equation on the annulus

\[ \Delta R_\mu(x) - \mu R_\mu + R_\mu^3 = 0, \quad r_{\text{min}} < |x| < r_{\text{max}} \]

\[ R_\mu = 0, \quad |x| = r_{\text{min}}, r_{\text{max}} \]

- First example of
  1. Non-uniqueness of positive solutions of the R equation
  2. A positive solution of the R equation which is not a ground-state

- Previously observed for the inhomogeneous NLS (Kirr et al., 08)
Stability on the annulus

- Radial, $\mu = \mu_{cr} - 0.3$
- Radial, $\mu = \mu_{cr} + 0.3$
- Non-radial, $\mu = \mu_{cr} + 0.3$
Computation of necklace solitary waves

\[ \Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad (x, y) \in D, \]
\[ R_\mu = 0, \quad (x, y) \in \partial D \]

- Nonlinear elliptic problem
- Multidimensional
  - cannot use radial symmetry
- Non-positive solution with necklace structure
- Bounded domain

Plan:
- Compute a single pearl (positive)
- Extend into a necklace using the anti-symmetries
Computation of a single pearl

\[
\Delta R_\mu - \mu R_\mu + R_\mu^3 = 0, \quad (x, y) \in D, \\
R_\mu = 0, \quad (x, y) \in \partial D \\
R_\mu > 0, \quad (x, y) \in D
\]

- Nonlinear elliptic problem
- Multidimensional
- Non-positive solution
- Bounded domain

- When \( D = \mathbb{R}^2 \), can use Petviashvili’s \textit{spectral renormalization method}
- Cannot apply on a bounded domain
- Introduce a \textit{non-spectral version}
Non-spectral renormalization method

- Rewrite equation for $R_\mu$ as
  \[ LR = R^3, \quad L := -\Delta + \mu \]

- Solve using fixed point iterations
  \[ R_{k+1} = L^{-1} R_k^3, \quad k = 0, 1, \ldots \]

- $L$ discretized using finite differences

- Problem: Iterations diverge to zero or to infinity

- To prevent divergence, renormalize at each stage $R_k \rightarrow c_k R_k$
  \[ R_{k+1} = L^{-1} (c_k R_k)^3, \quad k = 0, 1, \ldots \]

- Determine $c_k$ from the requirement that $c_k R_k$ satisfies an integral relation, such as
  \[ \langle R, R \rangle = \langle R, L^{-1} R^3 \rangle \]
Fukuizumi, Hadj Selem, and Kikuchi (12) studied multi-peak solitary waves of 1D NLS on $B_1$:

\[ i\psi_t (t, x) + \psi_{xx} + |\psi|^2 \psi = 0, \quad -1 < x < 1, \]
\[ \psi = 0, \quad x = \pm 1 \]

On $B_1$:

- 2D radial analog are $R^{(n)} (r)$
- 2D non-radial analog are necklace solutions

Theory in 1D and 2D cases is very similar
Summary

- New type of solitary waves of the 2D NLS on bounded domains
- Stable at low powers
- Threshold for necklace instability < critical power for collapse
  - Instability unrelated to collapse
- Second stability region for necklace solitary waves on the annulus
  - Stable solitary waves with $P > P_{cr}$
- Symmetry breaking of the ground state on the annulus
- Non-spectral renormalization method for computing solitary waves on bounded domains

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Positive and necklace solitary waves on bounded domains