Complex Behavior in Coupled Nonlinear Waveguides

Roy Goodman, New Jersey Institute of Technology
Nonlinear Schrödinger/Gross-Pitaevskii Equation

\[ i\psi_t = -\nabla^2 \psi + V(r)\psi \pm |\psi|^2 \psi \]

Two contexts for today:

- **Propagation of light in a nonlinear waveguide**
  - \( \psi(x, z) \) gives the electric field envelope
  - “Evolution” occurs along axis of waveguide (\( t \to z \)) plus one transverse spatial dimension
  - Potential represents waveguide geometry

- **Evolution of a Bose-Einstein condensate (BEC)**
  - Everyone’s favorite nonlinear playground. A “new” state of matter achieved experimentally in the 1990’s.
  - One, two, or three space dimensions
  - Potential represents magnetic or optical trap
Periodic and chaotic tunneling in a 3-well waveguide

Why three wells?

- Other work on two-waveguide arrays shows symmetry-breaking bifurcations and an associated wobbling dynamics.
- Three waveguides provide the simplest system in which Hamiltonian Hopf bifurcations, which lead to complex dynamics, are possible.
- Significant interest in many-waveguide arrays. Useful to proceed: Simple Geometry $\rightarrow$ Complex Geometry, Simple Dynamics $\rightarrow$ Complex Dynamics
What got me thinking: Double well $V(x) = V_0(x + L) + V_0(x - L)$

Stationary

$\psi(x, t) = \Psi(x) e^{-i\Omega t}$

$$\int_{\mathbb{R}} |\Psi(x)|^2 dx = \|\Psi\|^2_2 = \mathcal{N}$$

Time-dependent dynamics

- **Time dependent dynamics in a single or double well**
- **Rigorous result: long-time shadowing of ODE solutions by PDE solutions**

Experiment in Bose-Einstein condensate

Albiez et al. 2005

Marzuola & Weinstein 2010

Pelinovsky & Phan 2012

Goodman, Marzuola, Weinstein 2015

Spontaneous symmetry breaking above critical intensity that is found analytically.

Kerr, Kevrekidis, Shlizerman, Weinstein 2008

see also Fukuizumi & Sacchetti 2011
What got me thinking: Triple well

3-well potential & eigenfunctions

\[ V(x) = V_0(x + L) + V_0(x) + V_0(x - L) \]

Bifurcations of standing waves

(Kapitula/Kevrekidis/Chen SIADS 2006)

Mode unstable for range of \( N \)

Periodic Schrödinger Trimer


\[ \frac{d}{dt} \psi_n + C(\psi_{n-1} - 2\psi_n + \psi_{n+1}) + |\psi_n|^2 \psi_n = 0 \]

subject to \( \psi_{n+3} = \psi_n \)

“Hamiltonian Hopf Bifurcations”

Numerically-generated chaos
Two goals

• Understand what takes place at HH bifurcation as paradigm for nonlinear wave oscillatory instability.

• Flesh out the dynamics of relative periodic orbits in the system. Eventual Goal: Which of these dynamics can we prove exist?
Finite dimensional reduction

Decompose the solution as

$$\psi = c_1(t)\Psi_1(t) + c_2(t)\Psi_2(t) + c_3(t)\Psi_3(t) + \eta(x, t)$$

projection onto eigenmodes

Ignoring contribution of $\eta(x, t)$ gives finite-dimensional Hamiltonian system with (approximate) Hamiltonian

$$\bar{H} = \Omega_1 |c_1|^2 + \Omega_2 |c_2|^2 + \Omega_3 |c_3|^2 - A \left[ \frac{3}{2} \left( |c_1|^2 + |c_3|^2 \right)^2 + 2 |c_2|^4 + 4 |c_2|^2 |c_3 - c_1|^2 + \left( |c_1|^2 + |c_3|^2 \right) (c_1 c_3 + \bar{c}_1 \bar{c}_3) + \frac{3}{2} (c_1^2 \bar{c}_3^2 + \bar{c}_1^2 c_3^2) + ((c_3 - c_1)^2 \bar{c}_2^2 + (\bar{c}_3 - \bar{c}_1)^2 c_2^2) \right]$$

For well-separated potential wells, the spectrum has the form

$$(\Omega_1, \Omega_2, \Omega_3) = (\Omega_2 - \Delta + \epsilon, \Omega_2, \Omega_2 + \Delta + \epsilon)$$

with $\epsilon \ll \Delta \ll 1$
Symmetry reduction

System conserves squared $L^2$ norm $N$

- Reduces # of degrees of freedom from 3 to 2
- Removes fastest timescale

$$\bar{H}_R = (-\Delta + \epsilon) |z_1|^2 + (\Delta + \epsilon) |z_3|^2 - AN \left( z_1^2 + \bar{z}_1^2 + z_3^2 + \bar{z}_3^2 - 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - 4(z_1 \bar{z}_3 + \bar{z}_1 z_3) \right) - A \left[ -\frac{1}{2} |z_1|^4 + 2 |z_1|^2 |z_3|^2 - \frac{1}{2} |z_3|^4 + \frac{3}{2} (z_1^2 \bar{z}_3^2 + \bar{z}_1^2 z_3^2) + \left( |z_1|^2 + |z_3|^2 \right) \left( 5(z_1 \bar{z}_3 + \bar{z}_1 z_3) + 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - z_1^2 - \bar{z}_1^2 - z_3^2 - \bar{z}_3^2 \right) \right].$$

- Relative fixed points in full system $\rightarrow$ fixed points in reduction
- Relative periodic orbits $\rightarrow$ periodic orbits

At $\epsilon = N = 0$, semisimple double frequency $i\Omega = \pm i\Delta$.

When $\epsilon > 0$, non-simple double eigenvalues at $N_{HH1} \approx \frac{\epsilon}{2A}$ and $N_{HH2} \approx \frac{\Delta - 2\epsilon}{2A}$, with instability in between.
Menagerie of standing waves

Three branches continue from linear system

Six branches arise in saddle-node bifurcations

Four stabilizations/destabilizations in HH bifurcations
Lyapunov Center Theorem: (Roughly) For each pair of imaginary eigenvalues of a fixed point, excepting resonance, there exists a one-parameter family of periodic orbits that limits to that fixed point.
Bifurcations in Hamiltonian systems change the topology of Lyapunov branches of periodic orbits

Standard Example: Hamiltonian Pitchfork \( \ddot{x} = \delta x + x^3 \)
ODE & PDE simulations

Trivial solution stable

Real($\lambda$)

Real($z_1$)

Poincaré Section

$|\psi(t)|$
ODE & PDE simulations

Chaotic heteroclinic bursting

Real(\(z_1\))
Poincaré Section

|\(\psi(t)\)|
Reduced Hamiltonian has 41 daunting terms!

\[
\tilde{H}_R = (-\Delta + \epsilon) |z_1|^2 + (\Delta + \epsilon) |z_3|^2 - \\
AN \left( z_1^2 + \bar{z}_1^2 + z_3^2 + \bar{z}_3^2 - 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - 4(z_1 \bar{z}_3 + \bar{z}_1 z_3) \right) - \\
A \left[ -\frac{1}{2} |z_1|^4 + 2 |z_1|^2 |z_3|^2 - \frac{1}{2} |z_3|^4 + \frac{3}{2} (z_1^2 \bar{z}_3^2 + z_3^2 \bar{z}_1^2) + \\
\left( |z_1|^2 + |z_3|^2 \right) \left( 5(z_1 \bar{z}_3 + \bar{z}_1 z_3) + 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - z_1^2 - \bar{z}_1^2 - z_3^2 - \bar{z}_3^2 \right) \right]
\]

Goal: understand periodic orbits of \( \tilde{H}_R \) using Hamiltonian Normal Forms

Given a system with Hamiltonian \( H = H_0(z) + \epsilon \tilde{H}(z, \epsilon) \) find a near-identity canonical transformation \( z = F(y, \epsilon) \) such that the transformed Hamiltonian

\[
K(y, \epsilon) = H(F(y, \epsilon), \epsilon) = H_0(y) + \epsilon \tilde{K}(y, \epsilon)
\]

is “simpler” than \( H(z, \epsilon) \).
What does “simpler” mean?

• Try to remove terms from $H$ to construct $K$
• Eliminating terms at a given order in $\epsilon, y$ introduces new terms of higher order
• A term can be removed if it lies in the range of the adjoint operator of $\text{ad}_{H_0} = \{\cdot, H_0\}$.
• Invoke Fredholm alternative. Resonant terms in adjoint null space. Project Hamiltonian onto this subspace.
• For example in our problem

$$\text{ad}_{H_0} \zeta = \{\zeta, H_0\}$$

and that monomials $\zeta \bar{\zeta}$ are eigenvectors. For our leading-order Hamiltonian $H_0$, which is quadratic,

$$\text{ad}_{H_0} \zeta \bar{\zeta} = i(\zeta_1 + \zeta_3 + \zeta_2^2) \zeta \bar{\zeta}.$$

The vector space $P_m$ has an inner product $\langle F(\zeta, \bar{\zeta}), G(\zeta, \bar{\zeta}) \rangle = F(\partial_{\zeta} \zeta, \partial_{\bar{\zeta}} \bar{\zeta}) G(\zeta, \bar{\zeta})$ under which the monomials (18) form an orthogonal basis. This allows us to choose the complement of $R$.

So, the monomial $\zeta \bar{\zeta}^2$ is in the nullspace, and thus resonant, if and only if $\zeta_1$ and $\zeta_3$ are positive integer solutions to the underdetermined linear system of equations:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 3 \\ 1 & 3 \\ \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_3 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ \end{bmatrix}.$$

The general solution to this system is $\zeta_1 = 1 + 3m$ and $\zeta_3 = 1 + 3m - 1 = 2m$.

This has positive integer-valued solutions only for even values of $m$. Thus, the normal form will contain no cubic terms, and the resonant monomials can be enumerated by specifying $\zeta_1$ and $\zeta_3$ to be integers drawn from the set $\{0, \ldots, m/2\}$.

For quadratic and quartic monomials this yields:

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_3$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\bar{\zeta}_1 \zeta_3$</td>
<td>$</td>
<td>\zeta_3</td>
</tr>
<tr>
<td>1</td>
<td>$</td>
<td>\zeta_1</td>
<td>^2$</td>
</tr>
<tr>
<td>2</td>
<td>$</td>
<td>\zeta_1</td>
<td>^4$</td>
</tr>
</tbody>
</table>

Table 1: Resonant monomials of degree two and four

(h) Degree Two

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_3$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\bar{\zeta}_1^2 \bar{\zeta}_3^2$</td>
<td>$</td>
<td>\zeta_3</td>
<td>^2 \bar{\zeta}_1 \zeta_3$</td>
</tr>
<tr>
<td>1</td>
<td>$</td>
<td>\zeta_1</td>
<td>^2 \bar{\zeta}_1 \zeta_3$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$</td>
<td>\zeta_1</td>
<td>^2 \zeta_1 \zeta_3$</td>
<td></td>
</tr>
</tbody>
</table>

(b) Degree Four

From these tables, we see that the resonant quadratic term (12) contains all four monomials listed in Table 1a, while the resonant quartic terms (13) contain seven of the nine monomials listed in Table 1b, but does not contain...
Three normal form calculations

- Semisimple -1:1 resonance for $\epsilon \ll 1$, $N = O(\epsilon)$
  Gives HH1 at $N_{\text{crit}} = \frac{\epsilon}{2A} + O(\epsilon^2)$

- Nonsemisimple -1:1 resonance at $N_{\text{crit}}$ using a further simplification of above normal form

- Nonsemisimple -1:1 resonance computed numerically at numerical location of HH2
Normal form near semisimple double eigenvalue (Chow/Kim 1988)

\[ H = -\Delta |z_1|^2 + \Delta |z_3|^2 \]

Normal Form

\[ H_{\text{norm}} = -\Delta |z_1|^2 + \Delta |z_3|^2 + \epsilon \left( |z_1|^2 + |z_3|^2 \right) + 2AN(z_1z_3 + \bar{z}_1\bar{z}_3) \]

\[ + A \left[ \frac{1}{2} |z_1|^4 - 2|z_1|^2|z_3|^2 + \frac{1}{2} |z_3|^4 - 2 \left( |z_1|^2 + |z_3|^2 \right) (z_1z_3 + \bar{z}_1\bar{z}_3) \right] \]

In Canonical Polar Coordinates

\[ H = \Delta (-J_1 + J_3) + \epsilon (J_1 + J_3) + 4AN \sqrt{J_1J_3} \cos (\theta_1 + \theta_3) \]

\[ + A \left( \frac{1}{2} J_1^2 - 2J_1J_3 + \frac{1}{2} J_3^2 - 4\sqrt{J_1J_3}(J_1 + J_3) \cos (\theta_1 + \theta_3) \right) \]

Independent of \( (\theta_1 - \theta_3) \) implying the existence of a conserved quantity and the integrability of the Normal Form.

Advantage: Easier to find solution structure in Normal Form.
The system can be further reduced. Periodic orbits \( \begin{pmatrix} J_1 \\ J_3 \end{pmatrix} e^{i\Omega t} \) solve:

\[
\sqrt{J_1 J_3} (2\epsilon - A (J_1 + J_3)) + 2A (N (J_1 + J_3) - J_1^2 - 6J_1 J_3 - J_3^2) \cos \Theta = 0
\]

\[
\sqrt{J_1 J_3} (N - J_1 - J_3) \sin \Theta = 0
\]

With \( \Theta = (\theta_1 + \theta_3) \)

\( J_1 \) and \( J_3 \) act as barycentric coordinates on the triangle of admissible solutions showing relative strength of the three modes.
Sequence of bifurcations in Normal Form

2 Lyapunov families of fixed points + unphysical branch

Unphysical branches cross into physical region

Lyapunov branches "pinch off"

Question: At second bifurcation point HH2, must have Lyapunov families of fixed point. Where do they come from?
Normal form for non-semisimple -1:1 resonances at HH1 and HH2 (Meyer-Schmidt 1974)

In symplectic polar coordinates \((r, \theta, p_r, p_\theta)\), this is:

\[
H = H_0(r, p_r, p_\theta) + \mu^2 \delta H_2(r, p_\theta) + H_4(r, p_\theta)
\]

\[
= \Omega p_\theta + \frac{\sigma}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + \mu^2 \delta \left( a p_\theta + \frac{b}{2} r^2 \right) + \frac{c}{2} p_\theta^2 + \frac{d}{2} p_\theta r^2 + \frac{e}{8} r^4
\]

\[
\delta = \pm 1, \quad \mu \ll 1
\]

Poincaré-Lindstedt argument: periodic orbits with “amplitude” \(\mu r\) and frequency \(\Omega + \mu \omega_1\) when there is a solution to

\[
2\omega_1^2 - \sigma e r^2 = 2\delta \sigma \beta
\]

Two cases:

Hyperbolic \(\sigma e > 0\)

Elliptic \(\sigma e < 0\)
The bifurcation at HH1

Computations using previous normal form

Increasing $N$ ➜
Numerically Computed Periodic orbits (not normal form)
Some computed PDE solutions on this branch.
The bifurcation at HH2

Numerically Computed Periodic orbits

ODE Computation

PDE Computation

New family of periodic orbits arises in “elliptic” HH bifurcation
Increasing $N$

Solutions must satisfy $|z_1|^2 + |z_3|^2 < N$. 

PDE

$N = 0.82$

ODE

$N = 0.8135$
What's going on?
Getting close to other fixed points

\[ N = 0.8125 \]

What about the other Lyapunov branches of periodic orbits?
I thought saddle-node bifurcations were boring.
Normal form for $0^2 i \omega$ bifurcation

\[
H = \left(\frac{q_1^2}{2} + \frac{p_1^2}{2}\right) + \alpha \left(\frac{p_2^2}{2} + \delta q_2 - \frac{q_2^3}{3}\right) + \beta I q_2 + H_{\text{higher}}(q_2, I) + R_\infty(q_1, q_2, p_1, p_2)
\]

Fast & Oscillatory
Saddle node bifurcation at $\delta = 0$

Three families of periodic orbits:
- Fast
- Slow
- Mixed

Small beyond all orders remainder

where $I = \left(\frac{q_1^2}{2} + \frac{p_1^2}{2}\right)$
Perturbation expansion shows two regimes

\[ N < N_{\text{crit}} \]

\[ N > N_{\text{crit}} \]

Saddle-node 1

Saddle-node 2

Mixed periodic orbits bifurcate when \( \delta = \frac{1}{\alpha^4 n^4} \)
Saddle-node

(a) $N=0.2496$
(b) $N=0.2496$
(c) $N=0.258$
(d) $N=0.299$
(e) $N=0.3004$
(f) $N=0.31$

Gelfreich-Lerman 2003
Saddle-node 2

(a) $N=0.664$

(b) $N=0.67$

(c) $N=0.673$
A menagerie of periodic orbits computed by continuation
Parting Words

• This problem has an ODE part and a PDE part

• Increasing from two wells to three makes the ODE part of the problem hard

• In addition to standing waves, there is a whole lot of additional structure in solutions that oscillate among the three waveguides

• Normal forms give us a partial picture of the reduced dynamics

• Even saddle-node bifurcations are interesting.

• Big question: What can be proven about shadowing these orbits in NLS/GP?

For re/preprints http://web.njit.edu/~goodman
Thanks!