Singularity Formation in Derivative Nonlinear Schrödinger Equations

Gideon Simpson
School of Mathematics
Drexel University
November 1, 2016

Joint with
Y. Cher (Toronto), X. Liu (Toronto), C. Sulem (Toronto)
Outline

1. Background & Overview
   - Structural Properties
   - Previous Results
   - Challenges & Results

2. Time Dependent Simulations of Finite Time Singularities
   - Direct Evidence for Singularity Formation
   - Dynamic Rescaling

3. Inferences from Computation of Blowup Profiles
   - Computed Profiles
   - Refined Local Analysis

4. Time Dependent Simulations Revisited – Adaptive Methods
   - Computational Challenge
   - Adaptive Meshing
Derivative Nonlinear Schrödinger Equation (DNLS)

Characteristic Form: $i\psi_t + i|\psi|^2\psi_x + \psi_{xx} = 0$ \hspace{1cm} (1a)

Conservation Form: $i\Psi_t + i(|\Psi|^2\Psi)_x + \Psi_{xx} = 0$ \hspace{1cm} (1b)

(1a) becomes (1b) via the Gauge transformation

$$\psi = \Psi \exp\left\{-\frac{i}{2} \int_{-\infty}^{x} |\Psi|^2 \, dx \right\}.$$ \hspace{1cm} (2)

Generalized DNLS (gDNLS)

Characteristic Form: $i\psi_t + i|\psi|^{2\sigma}\psi_x + \psi_{xx} = 0$ \hspace{1cm} (3a)

Conservation Form: $i\Psi_t + i(|\Psi|^{2\sigma}\Psi)_x + \Psi_{xx} = 0$ \hspace{1cm} (3b)

No known analog of the Gauge transformation for $\sigma \neq 1$. 

Physical Origin of DNLS

- Weakly nonlinear Alfvén waves in plasma physics, in a long wave length approximation, are governed by

\[ i\psi_t + i(|\psi|^2\psi)_x + \psi_{xx} = 0 \]

where \( \psi = \psi_y + i\psi_z \) is the magnetic field in the transverse direction.

- Self-steepening optical pulses can be modeled by

\[ i\psi_t + i|\psi|^2\psi_x + \psi_{xx} = 0 \]

where \( \psi \) represents the dimensionless, complex valued, slowly varying envelope of the electric field (CLLE).

- \( g\text{DNLS} \) has not (yet) appeared in a physical model.
Integrability

- Cubic DNLS equation is completely integrable, Kaup-Newell (1978)
- “Orbitally stable” solitons in modulated $H^1$, Colin-Ohta (2006)
- An $L^2$ “critical” integrable equation
- Recent work on well-posedness via inverse scattering: Liu, Perry, & Sulem (2015), Pelinovksy & Shimabukuro (2016)
gDNLS Scaling

Scaling

Given a solution \( \psi(x, t) \)

\[
\psi_\lambda(x, t) = \lambda^{\frac{1}{2\sigma}} \psi(\lambda x, \lambda^2 t)
\]  

is also a solution

Norm Scaling

\( L^2 \) based Sobolev norms scale as

\[
\| \psi_\lambda \|_{\dot{H}^s} = \lambda^{s + \frac{1}{2} \left( \frac{1}{\sigma} - 1 \right)} \| \psi \|_{\dot{H}^s}
\]  

The scale invariant Sobolev norm, \( \| \psi_\lambda \|_{\dot{H}^{s_c}} = \| \psi \|_{\dot{H}^{s_c}} \), is

\[
s_c = \frac{1}{2} \left( 1 - \frac{1}{\sigma} \right), \quad \text{Critical Sobolev Exponent}
\]
**$L^2$ Critical Equations and Singularities**

- **Focusing NLS**
  
  \[ iu_t + \Delta u + |u|^{2\sigma} u = 0 \]  
  
  has finite time singularities for $\sigma d \geq 2$; $\sigma d = 2$ is $L^2$ critical

- **gkDV**
  
  \[ u_t + u^n u_x + u_{xxx} = 0 \]  
  
  has finite time singularities for $n \geq 4$; $n = 4$ is $L^2$ critical
gDNLS Invariants

Mass, $L^2$, (Critical norm for $\sigma = 1$)

$$Q[\psi] = \int |\psi|^2 dx \quad (9)$$

Momentum

$$P[\psi] = \int \bar{\psi} D_x \psi, \quad D_x \equiv \frac{1}{i} \partial_x \quad (10)$$

Hamiltonian

$$E[\psi] = \int |\psi_x|^2 + \frac{1}{2(\sigma+1)} \bar{\psi}^{\sigma+1} D_x \psi^{\sigma+1} \quad (11a)$$

$$\partial_t \psi = -i \frac{\delta E}{\delta \bar{\psi}} \quad (11b)$$
Well-Posedness Results

Cubic Nonlinearity (Hayashi, Hayashi & Ozawa (1992, 1993), . . . )

By a change of variables, cubic DNLS is equivalent to

\[ iU_t + U_{xx} = iU^2 \bar{V}, \quad iV_t + V_{xx} = -iV^2 \bar{U}, \]

GWP for small data \( \| \psi_0 \|_{L^2} < \sqrt{2\pi} \) in

\[ C(\mathbb{R}; H^m) \cap C^1(\mathbb{R}; H^{m-2}), \quad m \in \mathbb{N} \]
Well-Posedness Results

Cubic Nonlinearity (Hayashi, Hayashi & Ozawa (1992, 1993),…)

By a change of variables, cubic DNLS is equivalent to

\[ iU_t + U_{xx} = iU^2 \bar{V}, \quad iV_t + V_{xx} = -iV^2 \bar{U}, \]

GWP for **small data** \( \| \psi_0 \|_{L^2} < \sqrt{2\pi} \) in

\[ C(\mathbb{R}; H^m) \cap C^1(\mathbb{R}; H^{m-2}), \quad m \in \mathbb{N} \]

Other Nonlinearities

- Tsutsumi & Fukuda (1980)
- Kenig, Ponce, & Vega (1993)
- Hao (2007)
- Others…
Challenges for gDNLS

- Is the cubic problem global in time for large data or is there finite time singularity formation?
- Best result (to date) due to Y. Wu (2015): For \( u_0 \in H^1 \) with \( \| u_0 \|_{L^2}^2 < 4\pi \), the solution is global.
- Are there finite time singularities for sufficiently strong nonlinearities (large enough \( \sigma \) )?
Main Results

- For sufficiently large initial conditions, we have numerical evidence of a finite time singularity when $\sigma > 1$
- Asymptotics and numerics indicate the blowup is of the form

$$
\psi(x, t) \sim \left[ \frac{1}{2a(t_*-t)} \right]^{\frac{1}{4\sigma}} Q \left( \frac{x-x_*}{\sqrt{2a(t_*-t)}} + \frac{b}{a} \right) e^{i(\theta + \frac{1}{2a} \log \frac{t_*}{t_*-t})}
$$

(12)

where $x_*$, $t_*$ and $\theta$ depend on the data, but $Q$, $a$ and $b$ do not.
- The blowup rate is

$$
\| \psi \|_{\dot{H}^s} \sim |t_* - t|^{\frac{\sigma}{4\sigma} - \frac{s}{2}}
$$

(13)
The blowup profile \((Q, a, b)\) solve
\[
Q_{\xi \xi} - Q + ia \left( \frac{1}{2\sigma} Q + \xi Q_{\xi} \right) - ibQ_{\xi} + i|Q|^{2\sigma} Q_{\xi} = 0 \tag{14}
\]

As \(\sigma \rightarrow 1\), the blowup profile
\[
Q(\xi) \sim L(\xi) \exp \left\{ -i \left( \frac{a\xi^2}{4} - \frac{b\xi}{2} + \frac{1}{4} \int_0^\xi |L|^2 \right) \right\} \tag{15}
\]
with \(L\) corresponding to the algebraic soliton
\[
L_{\xi \xi} - L^3 + \frac{3}{16}L^5 = 0, \quad L(\xi) = \sqrt{\frac{8}{1 + 4\xi^2}} \tag{16}
\]

Parameters scale as
\[
a \propto (\sigma - 1)^{\gamma_a}, \quad \gamma_a \approx 3.2 \tag{17}
\]
\[
2 - b \propto (\sigma - 1)^{\gamma_b}, \quad \gamma_b = 2 \tag{18}
\]
Ongoing Work

- Direct observation of singularity in time dependent simulations for $\sigma \geq 1.1$
- Blowup profile computed for $\sigma \geq 1.044$
- **Conjecture:** $\sigma = 1$ is global for arbitrary $H^1$ data
- **Challenge:** Directly observe singularity and make inferences about blowup rate for $\sigma \searrow 1$ – only slightly supercritical problems
1 Background & Overview

2 Time Dependent Simulations of Finite Time Singularities
   - Direct Evidence for Singularity Formation
   - Dynamic Rescaling

3 Inferences from Computation of Blowup Profiles

4 Time Dependent Simulations Revisited – Adaptive Methods
Gaussian Initial Condition, \( u_0(x) = 3 \exp(-2x^2) \)

\( \sigma = 2 \), Quintic Nonlinearity
\[ u_0(x) = 3 \exp(-2x^2) \]–Derivative Animation

\[ \sigma = 2, \text{ Quintic Nonlinearity} \]

- Suggests both shock formation and wave collapse
- Blowup has both dispersive and hyperbolic elements
Metrics for Gaussian Initial Condition

- Reasonable conservation of the invariants
- Norms grow and appear to be going singular
Dynamic Rescaling Formulation

Decomposition

\[ \psi(x, t) = L(t)^{-\frac{1}{2\sigma}} u \left( \frac{x - x_0(t)}{L(t)}, \int_0^t \frac{ds}{L(s)^2} \right) = L(t)^{-\frac{1}{2\sigma}} u(\xi(x, t), \tau(t)) \]  \hspace{1cm} (19)

\[ iu_{\tau} + i|u|^{2\sigma} u_{\xi} + u_{\xi\xi} + i\frac{a}{2\sigma} u - ibu_{\xi} + a\xi U_{\xi} = 0 \]  \hspace{1cm} (20)

\[ a \equiv -L \frac{dL}{dt}, \quad b \equiv L \frac{dx_0}{dt} \]
Dynamic Rescaling Formulation

Decomposition

\[ \psi(x, t) = L(t)^{-\frac{1}{2\sigma}} u \left( \frac{x - x_0(t)}{L(t)}, \int_0^t \frac{ds}{L(s)^2} \right) = L(t)^{-\frac{1}{2\sigma}} u(\xi(x, t), \tau(t)) \]  

(19)

\[ iu_{\tau} + i|u|^{2\sigma} u_{\xi} + u_{\xi\xi} + i\frac{a}{2\sigma} u -ibu_{\xi} + a\xi U_{\xi} = 0 \]  

(20)

\[ a \equiv -L \frac{dL}{dt}, \quad b \equiv L \frac{dx_0}{dt} \]

Goal of dynamic rescaling is to evolve a smooth PDE with \( t_* \) corresponding to \( \tau \to +\infty \).
Dynamic Rescaling Formulation

Decomposition

\[ \psi(x, t) = L(t)^{-\frac{1}{2\sigma}} u \left( \frac{x - x_0(t)}{L(t)}, \int_0^t \frac{ds}{L(s)^2} \right) = L(t)^{-\frac{1}{2\sigma}} u(\xi(x, t), \tau(t)) \]

\[ iu_\tau + i|u|^{2\sigma} u_\xi + u_{\xi\xi} + i \frac{a}{2\sigma} u - ibu_\xi + a\xi U_\xi = 0 \]

\[ a \equiv -L \frac{dL}{dt}, \quad b \equiv L \frac{dx_0}{dt} \]

- Goal of dynamic rescaling is to evolve a smooth PDE with \( t_* \) corresponding to \( \tau \to +\infty \).
- Permits a careful examination of the solution near \( t_* \).
Dynamic Rescaling Formulation

Decomposition

\[ \psi(x, t) = L(t)^{-\frac{1}{2\sigma}} u \left( \frac{x - x_0(t)}{L(t)}, \int_0^t \frac{ds}{L(s)^2} \right) = L(t)^{-\frac{1}{2\sigma}} u(\xi(x, t), \tau(t)) \]  

(19)

\[ iu_\tau + i|u|^{2\sigma} u_\xi + u_{\xi\xi} + i \frac{a}{2\sigma} u - ibu_\xi + a\xi U_\xi = 0 \]  

(20)

\[ a \equiv -L \frac{dL}{dt}, \quad b \equiv L \frac{dx_0}{dt} \]

- Goal of dynamic rescaling is to evolve a smooth PDE with \( t_* \) corresponding to \( \tau \to +\infty \).
- Permits a careful examination of the solution near \( t_* \).
- Closures are needed for \( a \) and \( b \).
Closures for Dynamic Rescaling

**Motivation:** Choose parameters keeping $u$ smooth and centered
Closures for Dynamic Rescaling

Motivation: Choose parameters keeping $u$ smooth and centered
Define $L$ to evolve such that $\|u_\xi\|_{L^2}$ remains constant:

$$L(t) = \|u_\xi(\cdot, 0)\|_{L^2}^q \|\psi_x(\cdot, t)\|_{L^2}^{-q}, \quad q = \frac{2\sigma}{\sigma + 1}$$

Define $x_0$ to follow the maximum amplitude:

$$x_0(t) = \frac{\int x|u_x|^2 dx}{\int |u_x|^2 dx}$$
Motivation: Choose parameters keeping $u$ smooth and centered

Define $L$ to evolve such that $\|u_\xi\|_{L^2}$ remains constant:

$$L(t) = \|u_\xi(\cdot, 0)\|_{L^2}^q \|\psi_x(\cdot, t)\|_{L^2}^{-q}, \quad q = \frac{2\sigma}{\sigma + 1}$$

Define $x_0$ to follow the maximum amplitude:

$$x_0(t) = \frac{\int x|u_x|^2dx}{\int |u_x|^2dx}$$

Many other choices possible
Implications of $a$ and $L$

- $a(t) = -L(t)\dot{L}(t)$

Assume that $a \to A > 0$, a constant, very rapidly.

Then $L(t)^2 \approx -2At + K$.

Take $K = 2At^\star > 0$ since $L(0) > 0$.

Hence, $L(t) = \sqrt{2A(t^\star - t)}$; length scale goes to zero.

By our choice of $L$, $L(t) = \|u_\xi(x,0)\|\|\psi_x(x,t)\| - q, q > 0$, if $L \to 0$, $\|\psi_x(x,t)\| \to \infty$. 
Implications of $a$ and $L$

- $a(t) = -L(t)\dot{L}(t)$
- Assume that $a \to A > 0$, a constant, very rapidly
Implications of $a$ and $L$

- $a(t) = -L(t)\dot{L}(t)$
- Assume that $a \to A > 0$, a constant, very rapidly
- Then

$$L(t)^2 \approx -2At + K$$

Take $K = 2At_* > 0$ since $L(0) > 0$. 
Implications of $a$ and $L$

- $a(t) = -L(t)\dot{L}(t)$
- Assume that $a \to A > 0$, a constant, very rapidly
- Then
  \[
  L(t)^2 \approx -2At + K
  \]
  Take $K = 2At_\ast > 0$ since $L(0) > 0$.
- Hence,
  \[
  L(t) = \sqrt{2A(t_\ast - t)};
  \]
  Length scale goes to zero.
Implications of $a$ and $L$

- $a(t) = -L(t)\dot{L}(t)$

Assume that $a \to A > 0$, a constant, very rapidly

Then

$$L(t)^2 \approx -2At + K$$

Take $K = 2At_\star > 0$ since $L(0) > 0$.

Hence,

$$L(t) = \sqrt{2A(t_\star - t)};$$

Length scale goes to zero.

By our choice of $L$,

$$L(t) = \|u_\xi(\cdot, 0)\|_2^q \|\psi_x(\cdot, t)\|^{-q}, \quad q > 0,$$

if $L \to 0$, $\|\psi_x(\cdot, t)\| \to \infty$. 
Dynamic Rescaling Results

$\sigma = 2$, Quintic Case

Since $a \to A > 0$, we conclude a collapse occurs in finite time. Appears to be generic; other initial conditions lead to similar behavior.

$u(\xi, \tau) \to S(\xi) e^{iC \tau}$, a fixed profile. After rescaling $S \to Q$, $Q\xi\xi - Q + i\alpha(\frac{1}{4}Q + \xi Q\xi) - i\beta Q\xi + i|Q|^4 Q\xi = 0 \quad (21)$
Dynamic Rescaling Results

\( \sigma = 2, \) Quintic Case

- Since \( a \to A > 0, \) we conclude a collapse occurs in finite time.
- Appears to be generic; other initial conditions lead to similar behavior.
Dynamic Rescaling Results

\( \sigma = 2, \text{ Quintic Case} \)

- Since \( a \to A > 0 \), we conclude a collapse occurs in finite time.
- Appears to be generic; other initial conditions lead to similar behavior.
Dynamic Rescaling Results

$\sigma = 2$, Quintic Case

Since $a \to A > 0$, we conclude a collapse occurs in finite time.

Appears to be generic; other initial conditions lead to similar behavior

$u(\xi, \tau) \to S(\xi)e^{iC\tau}$, a fixed profile. After rescaling $S \to Q$,

$$Q_{\xi\xi} - Q + i\alpha \left(\frac{1}{4}Q + \xi Q_\xi\right) - i\beta Q_\xi + i|Q|^4 Q_\xi = 0 \quad (21)$$
Scaling Parameters for other Values of $\sigma \geq 1.1$

- In all cases, we find $a \to A(\sigma) > 0$ and $b \to B(\sigma)$
- As $\sigma \to 1$, the asymptotic is harder to resolve
- Recall the rescaled blowup profile:

$$Q_{\xi\xi} - Q + i\alpha \left(\frac{1}{2\sigma} Q + \xi Q_{\xi}\right) - i\beta Q_{\xi} + i|Q|^{2\sigma} Q_{\xi} = 0$$

with $Q$, $\alpha$ and $\beta$ only depending on $\sigma$. 

1. Background & Overview

2. Time Dependent Simulations of Finite Time Singularities

3. Inferences from Computation of Blowup Profiles
   - Computed Profiles
   - Refined Local Analysis

4. Time Dependent Simulations Revisited – Adaptive Methods
Blowup Profile

Inverting coordinate transformations, as $t \to t_*$

$$\psi(x, t) \sim \left[\frac{1}{2a(t^*-t)}\right]^{\frac{1}{4\sigma}} Q \left(\frac{x-x_*}{\sqrt{2a(t^*-t)}} + \frac{b}{a}\right) e^{i(\theta + \frac{1}{2a} \log \frac{t^*}{t^*-t})}$$

- $Q$, $a$ and $b$ are universal; they do not depend on $\psi_0$; $Q$ is an attractor
- $x_*$, $t_*$ and $\theta$ will depend on the data
Blowup Profile


Inverting coordinate transformations, as $t \to t_*$

$$
\psi(x, t) \sim \left[ \frac{1}{2a(t_* - t)} \right]^{\frac{1}{4\sigma}} Q \left( \frac{x - x_*}{\sqrt{2a(t_* - t)}} + \frac{b}{a} \right) e^{i(\theta + \frac{1}{2a} \log \frac{t_*}{t_* - t})}
$$

- $Q, a$ and $b$ are universal; they do not depend on $\psi_0$; $Q$ is an attractor
- $x_*, t_*$ and $\theta$ will depend on the data
- Understanding $Q$ gives insight into singularity formation
Inverting coordinate transformations, as $t \rightarrow t_*$

$$
\psi(x, t) \sim \left[ \frac{1}{2a(t_*-t)} \right]^{\frac{1}{4\sigma}} Q \left( \frac{x-x_*}{\sqrt{2a(t_*-t)}} + \frac{b}{a} \right) e^{i(\theta + \frac{1}{2a} \log \frac{t_*}{t_*-t})}
$$

- $Q$, $a$ and $b$ are universal; they do not depend on $\psi_0$; $Q$ is an attractor
- $x_*$, $t_*$ and $\theta$ will depend on the data
- Understanding $Q$ gives insight into singularity formation
- Existence/Uniqueness of $(Q, a, b)$ is an open problem – would provide rigorous proof of finite time singularity, **but would not be in energy space**
Numerically Computed Blowup Profiles
Liu, Simpson & Sulem, Physica D (2013), $\sigma \geq 1.08$

- Profiles appear to converge as $\sigma \to 1$
- $\alpha$ appears to go to zero; $\alpha = 0$ at $\sigma = 1$ would be inconclusive
- $\beta$ tends to a finite, nonzero constant
Numerically Computed Blowup Profiles
Liu, Simpson & Sulem, Physica D (2013), $\sigma \geq 1.08$

- Profiles appear to converge as $\sigma \to 1$
- $\alpha$ appears to go to zero; $\alpha = 0$ at $\sigma = 1$ would be inconclusive
- $\beta$ tends to a finite, nonzero constant
**Profile Properties as $\sigma \searrow 1$**

Cher, Simpson, & Sulem, arXiv:1602.02381

- New code and refined asymptotics allow $\sigma \geq 1.044$
- Time independent nonlinear solver – Parallel (large mesh/large domain) finite difference scheme
\( A_{\pm} \) as \( \sigma \downarrow 1 \\
Cher, Simpson, & Sulem, arXiv:1602.02381

- Large \( \xi \) behavior of \( Q \):

\[
Q \approx A_{\pm} |\xi|^{-\frac{1}{2\sigma}} \left( 1 \pm \frac{b}{2a\sigma|\xi|} \right) \exp \left\{ -\frac{i}{a} \left( \log |\xi| \pm \frac{b}{a|\xi|} \right) \right\} \tag{22}
\]

with

\[
A_- \approx \sqrt{4\pi(\sigma - 1)} \tag{23}
\]

\[
A_+ \approx 4\epsilon^{3/4} a^{-1/2} \exp \left\{ -\frac{\pi}{a} + \frac{2}{3} \frac{(2 - b)^{3/2}}{a} \right\} \tag{24}
\]
**A_± as σ ↘ 1, Continued**

Cher, Simpson, & Sulem, arXiv:1602.02381

---

**NOTE:** \(|Q|\) has a much larger prefactor for \(\xi < 0\)
\( a \) and \( b \) Parameters

Cher, Simpson, & Sulem, arXiv:1602.02381

Combining asymptotics and numerics, we predict

\[
\begin{align*}
a & \propto (\sigma - 1)^{\gamma_a}, \quad \gamma_a \approx 3.2 \\
\epsilon &= 2 - b \propto (\sigma - 1)^{\gamma_b}, \quad \gamma_b = 2
\end{align*}
\]

**Note:** Blowup solutions will have zero momentum (and energy) – used to close the system of equations for the constants
1. **Background & Overview**

2. **Time Dependent Simulations of Finite Time Singularities**

3. **Inferences from Computation of Blowup Profiles**

4. **Time Dependent Simulations Revisited – Adaptive Methods**
   - Computational Challenge
   - Adaptive Meshing
Boundary Conditions and Long Time Integration

- To the left, $u$ is “large”
Boundary Conditions and Long Time Integration

- To the left, $u$ is “large”
- Slow, $|\xi|^{-1/(2\sigma)}$ decay, and large $A_- \sim \sqrt{\sigma - 1}$ constant shows up in the rescaled coordinates
- Large domain needed for Robin Boundary Conditions

\[
Q_{\xi\xi} - Q + ia \left( \frac{1}{2\sigma} Q + \xi Q_\xi \right) - ibQ_\xi + i|Q|^{2\sigma} Q_\xi = 0 \quad (25)
\]
Boundary Conditions and Long Time Integration

- To the left, $u$ is "large"
- Slow, $|\xi|^{-1/(2\sigma)}$ decay, and large $A_- \sim \sqrt{\sigma - 1}$ constant shows up in the rescaled coordinates
- Large domain needed for Robin Boundary Conditions
- Original $x$ coordinate allows simple Dirichlet conditions, but resolution needed at singularity

\[ Q_{\xi\xi} - Q + ia \left( \frac{1}{2\sigma} Q + \xi Q_{\xi} \right) - ibQ_{\xi} + i|Q|^{2\sigma} Q_{\xi} = 0 \]  

(25)
Boundary Conditions

Time Independent Problem

Do not need Dirichlet conditions, but domain must be large enough that Robin Boundary conditions can be developed for:

\[ Q_{\xi\xi} - Q + ia \left( \frac{1}{2\sigma} Q + \xi Q_\xi \right) - ibQ_\xi + \left| Q \right|^{2\sigma} Q_\xi = 0 \]  \hspace{1cm} (26)

Time Dependent Problem

- Moderate unscaled domain (periodic/Dirichlet BCs) with a uniform mesh on original problem cannot resolve singularity
- Dynamic rescaling still requires a large domain to accommodate the BCs
- Absorbing boundary conditions?
- Unscaled problem on moderate domain with adaptive mesh
Aside, gKdV

\[ u_t + u^n u_x + u_{xxx} = 0, \]

- \( n \geq 4 \) corresponds to \( L^2 \) critical/supercritical
- Finite time singularities known to occur
- Mixture of hyperbolic/dispersive terms like gDNLS,

\[ \psi_t + |\psi|^{2\sigma} \psi_x - i\psi_{xx} = 0 \]
gKdV Simulations from Klein & Peter (2015)

\( n = 5 \), Supercritical

- Trailing edge has slower decay
- Complicates boundary conditions for dynamic rescaling method

Fig. 18: Solution to the gKdV equation (1) with \( \epsilon = 1 \) for \( n = 5 \) and the perturbed soliton initial data \( 1.01 Q(x + 3)(3) \) for several values of \( t \).
Equal Arc Length Placement

- Pick the location of the mesh,

\[ x_{\text{min}} = x_0 < x_1 < x_2 < \ldots < x_N < x_{N+1} = x_{\text{max}} \quad (27) \]

such that

\[ \int_{x_i}^{x_{i+1}} \sqrt{1 + |\psi_x|^2} \, dx = \text{Constant} \quad (28) \]

- Dynamic rescaling uniformly distributes mesh points near singularity
- Boundary conditions are better handled – no longer computing just near the singularity
**Modified Equation**

- Based on $\|u\|_{\dot{H}^1} \propto (t^* - t)^{-\alpha}$, with $\alpha = \alpha(\sigma)$, take

$$
\lambda(t) = \|u\|_{\dot{H}^1}^{-1/\alpha}, \quad s = \int_0^t \frac{dt'}{\lambda(t')} \quad (29)
$$

- Modified equation

$$
iu_s + i\lambda(s)|u|^{2\sigma}u_x + \lambda(s)u_{xx} = 0 \quad (30)
$$

- Singularity moved to $s \to \infty$, but spatial scale unchanged.
Adaptive Algorithm

- Strang splitting of hyperbolic & dispersive piece:

  - Hyperbolic: \( iu_s + i\lambda(s)|u|^{2\sigma}u_x = 0 \) (31)
  - Dispersive: \( iu_s + \lambda(s)u_{xx} = 0 \) (32)

- Hyperbolic piece solved exactly in Lagrangian coordinates by method of characteristics
- Dispersive piece solved by Crank-Nicolson time stepping with finite element discretization & simple Dirichlet conditions
- Mesh adapted to satisfy equal arc length constraint
Adaptive Algorithm, Continued

Half Step of Dispersive Scheme

\[
(M(x) - \frac{i\Delta s \lambda}{4} K(x))u^{(1)} = (M(x) + \frac{i\Delta s \lambda}{4} K(x))u
\]  

(33)

Full Step of Lagrangian Hyperbolic Scheme

\[
x \mapsto x + \Delta s \lambda |u^{(1)}|^{2\sigma} = x^{(1)}
\]  

(34)

Remesh & Update \( \lambda \) Use equal arc length constraint to obtain

\((x, u^{(2)})\)–update \( K(x), M(x) \) and \( \lambda \)

Half Step of Dispersive Scheme

\[
(M(x) - \frac{i\Delta s \lambda}{4} K(x))u = (M(x) + \frac{i\Delta s \lambda}{4} K(x))u^{(2)}
\]  

(35)
Adaptive Meshing, $\sigma = 2$, Real & Imaginary Parts
Adaptive Meshing, $\sigma = 2$, Mesh & Amplitude
Adaptive Meshing, $\sigma = 2$, Dynamic Rescaling Coordinates
Adaptive Meshing, $\sigma = 2$, Scalars
Adaptive Meshing, $\sigma = 1.05$, Real & Imaginary Parts
Adaptive Meshing, $\sigma = 1.05$, Dynamic Rescaling
Adaptive Meshing, $\sigma = 1.05$, Scalars

\begin{align*}
\|u\|_{L^\infty} & \quad \|u\|_{\dot{H}^1} \quad 1/L
\end{align*}

\begin{align*}
10^{-5} & \quad 10^{-4} & \quad 10^{-3} & \quad 10^{-2}
\end{align*}
Summary

- Supercritical gDNLS appears to have finite time singularities with a universal blowup profile
Summary

- Supercritical gDNLS appears to have finite time singularities with a universal blowup profile
- Ongoing work to study the $\sigma \rightarrow 1$ limit via adaptive methods
Summary

- Supercritical gDNLS appears to have finite time singularities with a universal blowup profile
- Ongoing work to study the $\sigma \to 1$ limit via adaptive methods
- Large data well posedness in the energy space in $H^1$ remains unresolved
Acknowledgements

Collaborators  Y. Cher (Toronto), X. Liu (Toronto) & C. Sulem (Toronto)

               Cher, Simpson, Sulem – arXiv:1602.02381

Funding  NSERC, NSF, DOE, NSF DMS-1409018

http://www.math.drexel.edu/~simpson/