Virtual Elements for magneto-static problems

Donatella Marini

Dipartimento di Matematica, Università di Pavia, Italy

IMATI- C.N.R., Pavia, Italy

coauthors: L. Beirão da Veiga, F. Brezzi, F. Dassi and A. Russo

Recent Advances and Challenges in Discontinuous Galerkin Methods
and related approaches
in honour of Bernardo Cockburn 60th birthday
Minneapolis, June 29 July 1, 2017

Outline

1. The problem and the variational formulation
2. VEM spaces and degrees of freedom
3. The discrete problem and error estimates
4. Serendipity version
5. Numerical results
6. Hints on 3D

The continuous problem

$\Omega \subset \mathbb{R}^2$ (polygonal) computational domain
given $j \in L^2(\Omega)$(with $\int_\Omega j = 0$), and $\mu = \mu(x) \geq \mu_0 > 0$

$$\begin{cases} \text{find } \mathbf{H} \in H(\mathrm{rot};\Omega) \text{ and } \mathbf{B} \in H(\mathrm{div};\Omega) \text{ such that:} \\ \mathrm{rot}\mathbf{H} = j \text{ and } \mathrm{div}\mathbf{B} = 0, \text{ with } \mathbf{B} = \mu\mathbf{H}, \text{ in } \Omega \\ \text{with the boundary conditions } \mathbf{H}\cdot\mathbf{t} = 0 \text{ on } \partial\Omega \end{cases}$$

$$\mathrm{rot}\mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}, \qquad \mathbf{rot}\, q = (\frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x})^T$$

and

$$H_0(\mathrm{rot};\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^2 \text{ with } \mathrm{rot}\mathbf{v} \in L^2(\Omega), \mathbf{H}\cdot\mathbf{t} = 0 \text{ on } \partial\Omega\}.$$

Variational formulation

Among the various formulations we chose (see Kikuchi 89)

$$\begin{cases} \text{find } \mathbf{H} \in H_0(\mathrm{rot};\Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that:} \\ \int_\Omega \mathrm{rot}\mathbf{H}\,\mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega + \int_\Omega \nabla p \cdot \mu\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega j\,\mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega \quad \forall \mathbf{v} \in H_0(\mathrm{rot};\Omega) \\ \int_\Omega \nabla q \cdot \mu\mathbf{H}\,\mathrm{d}\Omega = 0 \quad \forall q \in H_0^1(\Omega). \end{cases}$$

Unique solution $(\mathbf{H}, p)$ with $p \equiv 0$, $\mathrm{rot}\mathbf{H} = j$, $\mathrm{div}\mu\mathbf{H} = 0$.

Rotated RT-like virtual elements

- $\mathcal{T}_h$ = decomposition of Ω into elements E

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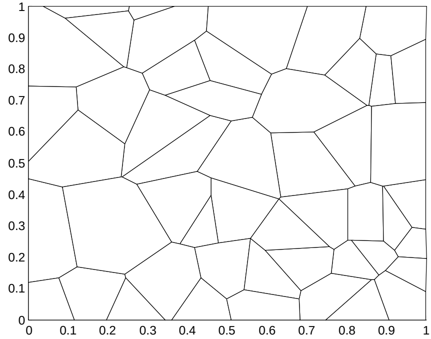

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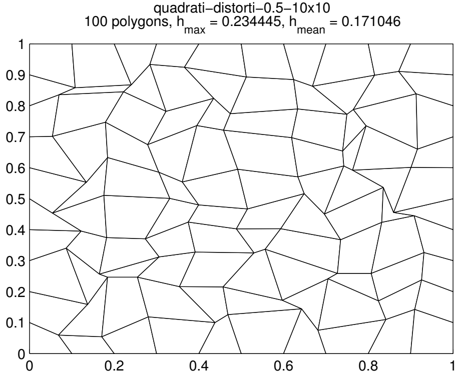

Rotated RT-like virtual elements

- $\mathcal{T}_h$ = decomposition of Ω into elements E

Nodal space:

$$k \geq 1 \ \rightarrow \ V_k^n(E) := \Big\{ q \in C^0(\overline{E}) : \, q_{|e} \in \mathbb{P}_k(e) \forall e \in \partial E, \, \Delta q \in \mathbb{P}_{k-1}(E) \Big\}.$$

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degrees of freedom:

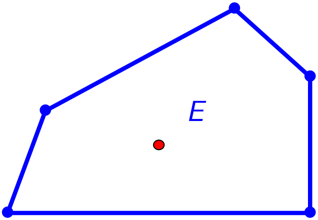
- the nodal values $q(\nu)$ at all vertexes ν of E, (1)
- for each edge e, the moments $\int_e q \, p_{k-2} \, \mathrm{d}s \quad \forall p_{k-2} \in \mathbb{P}_{k-2}(e)$, (2)
- $\int_E (\nabla q \cdot \mathbf{x_E}) \, p_{k-1} \mathrm{d}E \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(E)$, (3)

where $\mathbf{x_E} = \mathbf{x} - \mathbf{b}_E$, with $\mathbf{b}_E$ = barycenter of E. The d.o.f. are unisolvent, and are computationally equivalent to the moments $\int_E q \, p_{k-1} \mathrm{d}E$.

Example of d.o.f. for $k = 1$

$$V_1^n(E) := \left\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_1(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_0(E) \right\}$$

Example: $k = 1$

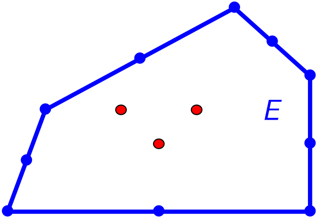


$$\bullet = \int_E \nabla q \cdot \mathbf{x_E} \, \mathrm{d}E$$

Example of d.o.f. for $k = 2$

$$V_2^n(E) := \left\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_2(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_1(E) \right\}$$

Example: $k = 2$



$$\bullet = \int_E \nabla q \cdot \mathbf{x_E} p_1 \mathrm{d}E$$

Rotated RT-like virtual elements

Edge space:

$$V^e_{k-1}(E) := \left\{ \mathbf{v} | \mathrm{div}\mathbf{v} \in \mathbb{P}_{k-1}(E), \mathrm{rot}\mathbf{v} \in \mathbb{P}_{k-1}(E),\ \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_{k-1}(e) \forall e \in \partial E \right\}$$

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degrees of freedom

$$\bullet \text{ on each egde } e,\ \int_e (\mathbf{v} \cdot \mathbf{t}_e) p_{k-1}\, \mathrm{d}s \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(e) \tag{4}$$

$$\bullet \text{ the moments } \int_E \mathbf{v} \cdot \mathbf{x_E}\, p_{k-1} \mathrm{d}E \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(E) \tag{5}$$

$$\bullet \int_E \mathrm{rot}\mathbf{v}\, p^0_{k-1} \mathrm{d}E \quad \forall p^0_{k-1} \in \mathbb{P}^0_{k-1}(E), \tag{6}$$

where

$$\mathbb{P}^0_s := \{ q \in \mathbb{P}_s \text{ with } \int_E q \mathrm{d}E = 0 \}$$

The d.o.f. (4)-(6) are unisolvent.

Example of d.o.f.

$$V_0^e(E) := \Big\{\mathbf{v}|\ \mathrm{div}\mathbf{v} \in \mathbb{P}_0(E),\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_0(E),\ \mathbf{v}_{|e}\cdot\mathbf{t}_e \in \mathbb{P}_0(e)\ \forall e \in \partial E\Big\}.$$

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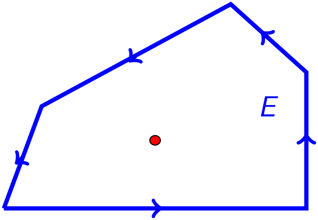

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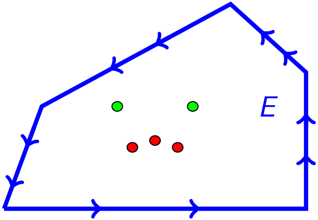

● $= \int_E \mathbf{v} \cdot \mathbf{x_E} p_1 \mathrm{d}E$

● $= \int_E rot\mathbf{v} p_1^0 \mathrm{d}E$

The global spaces

$$V_k^n = \{q \in H_0^1(\Omega) \text{ such that } q_{|E} \in V_k^n(E)\, \forall E \in \mathcal{T}_h\Big\},$$

$$V_{k-1}^e = \{\mathbf{v} \in H_0(\mathrm{rot};\Omega) \text{ such that } \mathbf{v}_{|E} \in V_{k-1}^e(E)\, \forall E \in \mathcal{T}_h\Big\}.$$

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We would like to write

$$\begin{cases} \text{find } \mathbf{H}_h \in V_{k-1}^e \text{ and } p_h \in V_k^n \text{ such that:} \\ \sum_E \int_E \mathrm{rot}\mathbf{H}_h\, \mathrm{rot}\mathbf{v} \mathrm{d}E + \sum_E \int_E \nabla p_h \cdot \mu \mathbf{v} \mathrm{d}E = \sum_E \int_E j\, \mathrm{rot}\mathbf{v} \mathrm{d}E \quad \forall \mathbf{v} \in V_{k-1}^e \\ \sum_E \int_E \nabla q \cdot \mu \mathbf{H}_h \mathrm{d}E = 0 \quad \forall q \in V_k^n. \end{cases}$$

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Problem: for functions $q \in V_k^n$ we do not know $\nabla q_{|E}$.
Similarly, vectors $\mathbf{v} \in V_{k-1}^e$ are not known inside the elements.

How to compute $\displaystyle\int_E \nabla q \cdot \mu \mathbf{v} \mathrm{d}E$?

How to compute $\int_E \nabla q \cdot \mu \mathbf{v} \, dE$?

μ p.w. constant

How to compute $\int_E \nabla q \cdot \mu \mathbf{v} \mathrm{d}E$? μ p.w. constant

Note 1: the L^2-projection $\Pi^0_{k-1} : V^e_{k-1}(E) \to [\mathbb{P}_{k-1}(E)]^2$ is computable from the d.o.f. (4)-(6).

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Indeed, any $\mathbf{p}_{k-1} \in [\mathbb{P}_{k-1}(E)]^2$ can be split in a unique way as

$$\mathbf{p}_{k-1} = \mathbf{rot} q_k + \mathbf{x_E} q_{k-2}, \quad q_k \in \mathbb{P}_k(E), q_{k-2} \in \mathbb{P}_{k-2}(E).$$

Hence:

$$\int_E \Pi^0_{k-1}\mathbf{v} \cdot \mathbf{p}_{k-1} \mathrm{d}E := \int_E \mathbf{v} \cdot \mathbf{p}_{k-1} \mathrm{d}E = \int_E \mathbf{v} \cdot (\mathbf{rot} q_k + \mathbf{x_E} q_{k-2}) \mathrm{d}E$$
$$= \int_E (\mathrm{rot}\mathbf{v}) q_k \mathrm{d}E + \sum_{e \in \partial E} \int_e (\mathbf{v} \cdot \mathbf{t}) q_k \,\mathrm{d}s + \int_E \mathbf{v} \cdot \mathbf{x_E} q_{k-2} \mathrm{d}E$$

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Note 2: $\nabla V^n_k \equiv \{\mathbf{v} \in V^e_{k-1}$ such that $\mathrm{rot}\mathbf{v} = 0\}$.

Scalar product

$$[\mathbf{v},\mathbf{w}]_{e,E} := (\Pi^0_{k-1}\mathbf{v},\Pi^0_{k-1}\mathbf{w})_{0,E} + S_E((I-\Pi^0_{k-1})\mathbf{v},(I-\Pi^0_{k-1})\mathbf{w})$$

where S_E is such that

$$\alpha_*(\mathbf{v},\mathbf{v})_{0,E} \le [\mathbf{v},\mathbf{v}]_{e,E} \le \alpha^*(\mathbf{v},\mathbf{v})_{0,E} \qquad \forall \mathbf{v} \in V^e_{k-1}(E).$$

Example:

$$S_E((I-\Pi^0_{k-1})\mathbf{v},(I-\Pi^0_{k-1})\mathbf{w}) = h_E \int_{\partial E} [(I-\Pi^0_{k-1})\mathbf{v}]\cdot\mathbf{t}[(I-\Pi^0_{k-1})\mathbf{w}]\cdot\mathbf{t}$$

Consistency holds:

$$[\mathbf{v},\mathbf{p}_{k-1}]_{e,E} \equiv (\mathbf{v},\mathbf{p}_{k-1})_{0,E} \quad \forall \mathbf{v} \in V^e_{k-1}(E),\ \forall \mathbf{p}_{k-1} \in [P_{k-1}(E)]^2.$$

The discrete problem

Given $j \in L^2(\Omega)$, with $\int_\Omega j \, \mathrm{d}\Omega = 0$,

$$\begin{cases} \text{find } \mathbf{H}_h \in V^e_{k-1} \text{ and } p_h \in V^n_k \text{ such that:} \\ \displaystyle\int_\Omega \mathrm{rot}\mathbf{H}_h \, \mathrm{rot}\mathbf{v} \, \mathrm{d}\Omega + [\nabla p_h, \mu\mathbf{v}]_e = \int_\Omega j \, \mathrm{rot}\mathbf{v} \, \mathrm{d}\Omega \quad \forall \mathbf{v} \in V^e_{k-1}, \\ [\nabla q, \mu\mathbf{H}_h]_e = 0 \quad \forall q \in V^n_k. \end{cases}$$

Theorem

The problem has a unique solution ($p_h = 0$) and the following estimate holds:

$$\|\mu(\mathbf{H} - \mathbf{H}_h)\|_{0,\Omega} \leq C \left(\|\mu(\mathbf{H} - \mathbf{H}_I)\|_{0,\Omega} + \|\mu(\mathbf{H} - \Pi^0_{k-1}\mathbf{H})\|_{0,\Omega} \right),$$

$$\|\mathrm{rot}(\mathbf{H} - \mathbf{H}_h)\|_{0,\Omega} = \|j - \Pi^0_{k-1} j\|_{0,\Omega},$$

Numerical results

Test case 1: $\Omega = [0,1]^2$

exact solution: $$\mathbf{H}(x,y) := \begin{pmatrix} \pi \sin(\pi x)\cos(\pi y) \\ -\pi \cos(\pi x)\sin(\pi y) \end{pmatrix}$$

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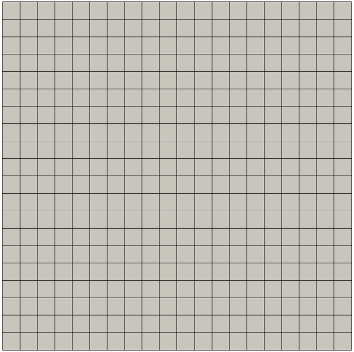

Figure : Example of uniform mesh

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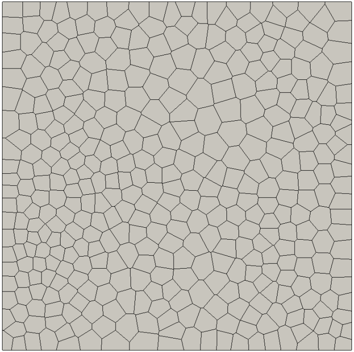

Figure : Example of Voronoi mesh

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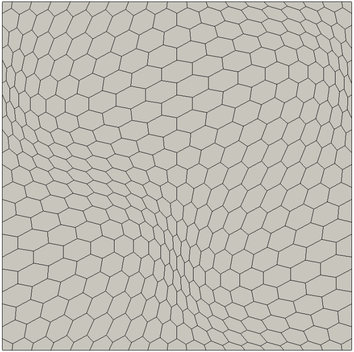
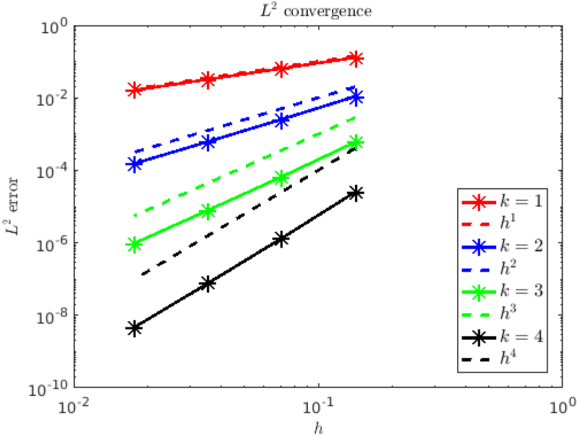


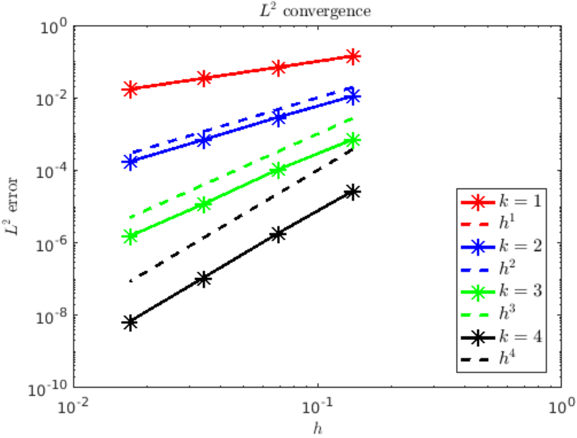
Figure : Example of distorted hexagons

Numerical results: L^2-convergence



convergence in h: uniform mesh

Numerical results: L^2-convergence



convergence in h: Voronoi mesh

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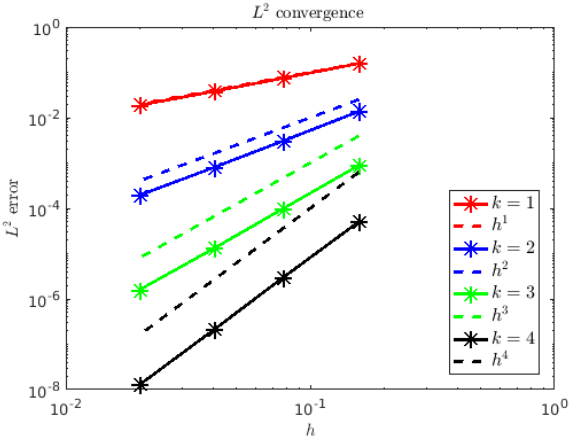

convergence in h: distorted hexagons

Serendipity spaces

The Serendipity procedure is, as in Finite Elements, a way of reducing the degrees of freedom by changing the space. (The most famous example is the 8-node square)

Aim:

• we want to keep the boundary d.o.f. to preserve conformity: H^1 for nodal VEMs, and $\mathbf{H}(\mathrm{rot})$ for edge VEMs

• we try to eliminate as many internal d.o.f. as we can, keeping only those needed for the expected accuracy

NOTE: When dealing with 3D problems, this allows to eliminate d.o.f. on the faces, otherwise impossible (or extremely hard) with static condensation

Serendipity nodal spaces (simplest case)

Define a projection $\Pi_S^n : V_k^n(E) \to \mathbb{P}_k(E)$ by

$$\begin{cases} \displaystyle\int_{\partial E} \partial_t(q - \Pi_S^n q)\partial_t p \,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_k(E), \\ \displaystyle\int_{\partial E} (\mathbf{x_E} \cdot \mathbf{n})(q - \Pi_S^n q) \,\mathrm{d}s = 0. \end{cases}$$

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Π_S^n is well defined if $\mathbb{P}_k(E)$ does not contain bubbles, i.e.,

$k < \eta :=$ number of straight lines necessary to cover the boundary

Serendipity nodal spaces (simplest case)

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Serendipity nodal space:

$$SV^n_k(E) = \Big\{ q \in V^n_k(E) : \int_E \nabla q \cdot \mathbf{x_E} p_{k-1} \mathrm{d}E = \int_E \nabla \Pi^n_S q \cdot \mathbf{x_E} p_{k-1} \mathrm{d}E \quad \forall p_{k-1} \in \mathbb{P}_{k-1} \Big\}$$

$$\mathbb{P}_k(E) \subseteq SV^n_k(E) \subseteq V^n_k(E)$$

Serendipity edge spaces (simplest case)

Define a projection $\Pi^e_S : V^e_{k-1}(E) \to (\mathbb{P}_{k-1}(E))^2$ as follows:

$$\int_{\partial E} [(\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{t}][\nabla p \cdot \mathbf{t}]\,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_{k-1}(E),$$
$$\int_{\partial E} (\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{t}\,\mathrm{d}s = 0,$$
$$\int_E \mathrm{rot}(\mathbf{v} - \Pi^e_S \mathbf{v}) p^0_{k-1} \mathrm{d}E = 0 \quad \forall p^0_{k-1} \in \mathbb{P}^0_{k-1}(E).$$

Serendipity edge space:

$$SV^e_{k-1}(E) = \left\{ \mathbf{v} \in V^e_{k-1}(E) : \int_E (\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{x}_E\, p_{k-1} \mathrm{d}E = 0 \quad \forall p_{k-1} \in \mathbb{P}_{k-1} \right\}$$

$$(\mathbb{P}_{k-1}(E))^2 \subseteq SV^e_{k-1}(E) \subseteq V^e_{k-1}(E)$$

Serendipity nodal spaces (general case)

If $k \geq \eta :=$ number of straight lines necessary to cover the boundary, i.e.,

if $\mathbb{P}_k(E)$ contains bubbles

Define $\Pi^n_S : V^n_k(E) \to \mathbb{P}_k(E)$ by

$$\begin{cases} \displaystyle\int_{\partial E} \partial_t (q - \Pi^n_S q) \partial_t p \, \mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_k(E), \\ \displaystyle\int_{\partial E} (\mathbf{x_E} \cdot \mathbf{n})(q - \Pi^n_S q) \, \mathrm{d}s = 0, \\ \displaystyle\int_E \nabla (q - \Pi^n_S q) \cdot \mathbf{x_E} p_{k-\eta} \mathrm{d}E = 0 \quad \forall p_{k-\eta} \in \mathbb{P}_{k-\eta}(E) \end{cases}$$

Serendipity nodal spaces (general case)

If $k \geq \eta :=$ number of straight lines necessary to cover the boundary, i.e.,

if $\mathbb{P}_k(E)$ contains bubbles

Define $\Pi^n_S : V^n_k(E) \to \mathbb{P}_k(E)$ by

$$\begin{cases} \int_{\partial E} \partial_t(q - \Pi^n_S q)\partial_t p \,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_k(E), \\ \int_{\partial E} (\mathbf{x_E} \cdot \mathbf{n})(q - \Pi^n_S q)\,\mathrm{d}s = 0, \\ \int_E \nabla(q - \Pi^n_S q) \cdot \mathbf{x_E} p_{k-\eta} \mathrm{d}E = 0 \quad \forall p_{k-\eta} \in \mathbb{P}_{k-\eta}(E) \end{cases}$$

Serendipity nodal space:

$$SV^n_k(E) = \Big\{ q \in V^n_k(E) : \int_E \nabla q \cdot \mathbf{x_E} p \mathrm{d}E = \int_E \nabla \Pi^n_S q \cdot \mathbf{x_E} p \mathrm{d}E \quad \forall p \in \mathbb{P}_{k-\eta/k-1} \Big\}$$

$$\mathbb{P}_k(E) \subseteq SV^n_k(E) \subseteq V^n_k(E)$$

Serendipity edge spaces (general case)

If $k \geq \eta :=$ number of straight lines necessary to cover the boundary, i.e.,

if $\mathbb{P}_k(E)$ contains bubbles

Define a projection $\Pi^e_S : V^e_{k-1}(E) \to (\mathbb{P}_{k-1}(E))^2$ as follows:

$$\int_{\partial E} [(\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{t}][\nabla p \cdot \mathbf{t}] \,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_{k-1}(E),$$

$$\int_{\partial E} (\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{t} \,\mathrm{d}s = 0,$$

$$\int_E \mathrm{rot}(\mathbf{v} - \Pi^e_S \mathbf{v}) p^0_{k-1} \mathrm{d}E = 0 \quad \forall p^0_{k-1} \in \mathbb{P}^0_{k-1}(E),$$

$$\int_E (\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{x}_E \, p_{k-\eta} \mathrm{d}E = 0 \quad \forall p_{k-\eta} \in \mathbb{P}_{k-\eta}(E)$$

Serendipity edge spaces (general case)

If $k \geq \eta :=$ number of straight lines necessary to cover the boundary, i.e.,

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$$\int_E (\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{x}_E \, p_{k-\eta} \mathrm{d}E = 0 \quad \forall p_{k-\eta} \in \mathbb{P}_{k-\eta}(E)$$

Serendipity edge space:

$$SV^e_{k-1}(E) = \left\{ \mathbf{v} \in V^e_{k-1}(E) \; : \; \int_E (\mathbf{v} - \Pi^e_S \mathbf{v}) \cdot \mathbf{x}_E \, p \mathrm{d}E = 0 \quad \forall p \in \mathbb{P}_{k-\eta/k-1} \right\}$$

$$(\mathbb{P}_{k-1}(E))^2 \subseteq SV^e_{k-1}(E) \subseteq V^e_{k-1}(E)$$

Coupling of Nodal and Edge Serendipity spaces

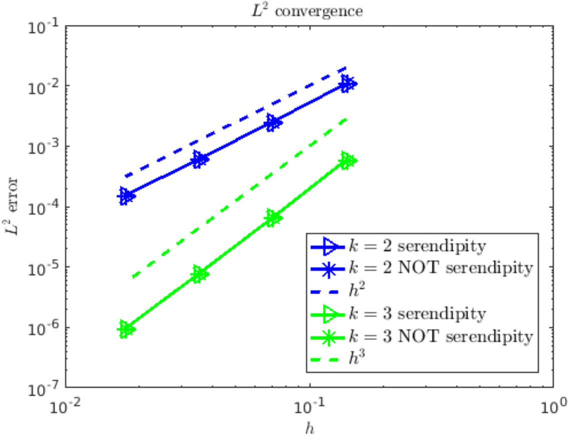
We still have

$$\nabla\, SV_k^n(E) = \Big\{ \mathbf{v} \in SV_{k-1}^e(E) \;:\; \mathrm{rot}\,\mathbf{v} = 0 \Big\}.$$

$$\begin{cases} \text{find } \mathbf{H}_h \in SV_{k-1}^e \text{ and } p_h \in SV_k^n \text{ such that:} \\ \displaystyle\int_\Omega \mathrm{rot}\mathbf{H}_h\,\mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega + [\nabla p_h, \mu\mathbf{v}]_e = \int_\Omega j\,\mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega \quad \forall \mathbf{v} \in V_{k-1}^e, \\ [\nabla q, \mu\mathbf{H}_h]_e = 0 \quad \forall q \in V_k^n. \end{cases}$$

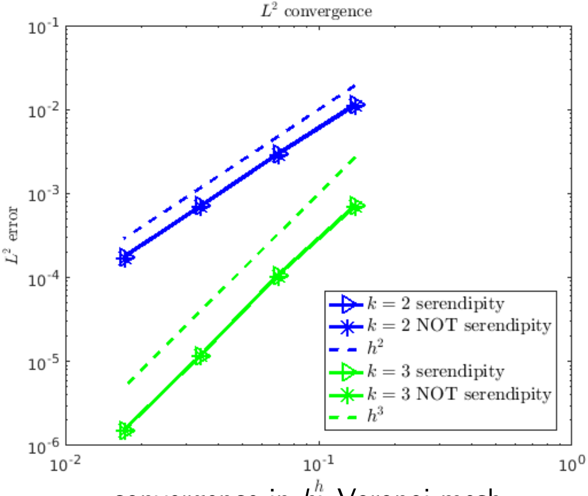
Unique solution and same error estimates as before.

Numerical results: original vs Serendipity- L^2-convergence



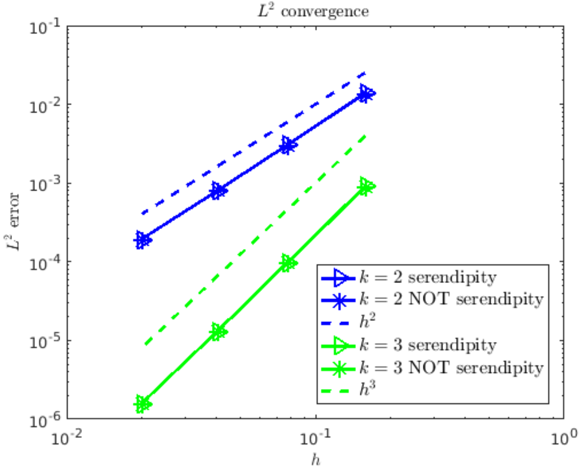
convergence in h: uniform mesh

Numerical results: original vs Serendipity- L^2-convergence



convergence in h: Voronoi mesh

Numerical results: original vs Serendipity- L^2-convergence



convergence in h: distorted hexagons

Numerical results: original vs Serendipity- “pressure”

$$\max |p_h - 0|$$

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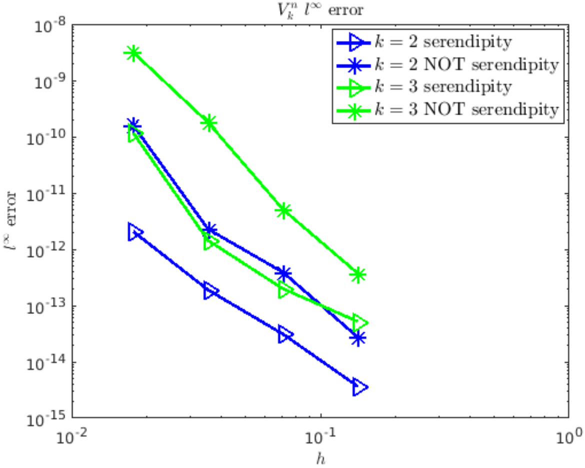

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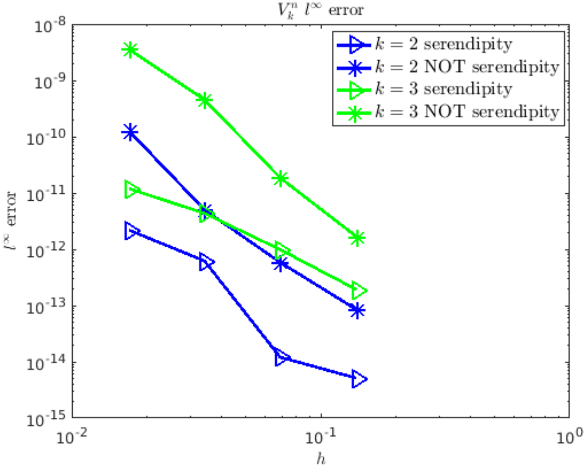

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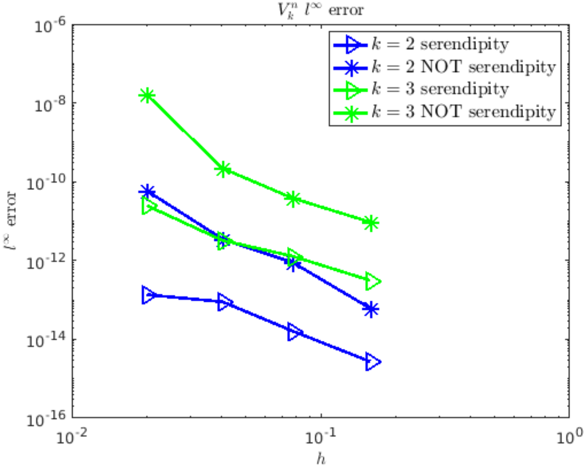

convergence in h: distorted hexagons

Counting the d.o.f.

mesh	degree k	standard	serendipity	gain
quad100		1181	981	16%
quad400	2	4561	3761	17%
quad1600		17921	14721	17%
quad6400		71041	58241	18%
voro100		1505	1305	13%
voro400	2	6005	5205	13%
voro1600		24005	20805	13%
voro6400		96005	83205	13%
hexa94		1415	1227	13%
hexa389	2	5840	5062	13%
hexa1415		21230	18400	13%
hexa5711		85670	74248	13%

Counting the d.o.f.

mesh	degree k	standard	serendipity	gain
quad100		2321	1721	25%
quad400	3	9041	6641	26%
quad1600		35681	26081	26%
quad6400		141761	103361	27%
voro100		2807	2207	21%
voro400	3	11207	8807	21%
voro1600		44807	35207	21%
voro6400		179207	140807	21%
hexa94		2639	2075	21%
hexa389	3	10899	8565	21%
hexa1415		39627	31137	21%
hexa5711		159915	125649	21%

Hint on 3D

$\Omega \subset \mathbb{R}^3$

$$\begin{cases} \text{given } \mathbf{j} \in H(\mathrm{div};\Omega) \quad (\text{with } \mathrm{div}\mathbf{j} = 0 \text{ in } \Omega), \quad \text{and } \mu = \mu(\mathbf{x}) \geq \mu_0 > 0, \\ \text{find } \mathbf{H} \in H(\mathbf{curl};\Omega) \text{ and } \mathbf{B} \in H(\mathrm{div};\Omega) \text{ such that:} \\ \mathbf{curl}\mathbf{H} = \mathbf{j} \text{ and } \mathrm{div}\mathbf{B} = 0, \text{ with } \mathbf{B} = \mu\mathbf{H} \text{ in } \Omega \\ \text{with the boundary conditions } \mathbf{H} \wedge \mathbf{n} = 0 \text{ on } \partial\Omega. \end{cases}$$

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Variational formulation (always Kikuchi 89)

$$\begin{cases} \text{find } \mathbf{H} \in H_0(\mathbf{curl};\Omega) \text{ and } p \in H^1_0(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \mathbf{curl}\mathbf{H} \cdot \mathbf{curl}\mathbf{v} \, \mathrm{d}\Omega + \int_\Omega \nabla p \cdot \mu\mathbf{v} \, \mathrm{d}\Omega = \int_\Omega \mathbf{j} \cdot \mathbf{curl}\mathbf{v} \, \mathrm{d}\Omega \quad \forall \mathbf{v} \in H_0(\mathbf{curl};\Omega) \\ \displaystyle\int_\Omega \nabla q \cdot \mu\mathbf{H} \, \mathrm{d}\Omega = 0 \quad \forall q \in H^1_0(\Omega) \end{cases}$$

Hint on 3D serendipity discretization

On a polyhedron P we define:
the nodal space:

$$SV_k^n(P) := \{q \in C^0(P) : q_{|f} \in SV_k^n(f) \, \forall face \ f, \Delta q \in \mathbb{P}_{k-2}(P)\}$$

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d.o.f. inside P: $\int_P q p_{k-2}\,\forall p_{k-2} \in \mathbb{P}_{k-2}(P)$

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the edge space:

$$SV_{k-1}^e(P) := \{\mathbf{v} : \mathbf{v} \cdot \mathbf{t} \text{ continuous}, \mathbf{v}_{|f} \in SV_{k-1}^e(f), \operatorname{div}\mathbf{v} \in \mathbb{P}_{k-2}(P), \mathbf{curl}(\mathbf{curl}\mathbf{v}) \in (\mathbb{P}_{k-2}(P))^3\}$$

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d.o.f. inside P: $\int_P \mathbf{v} \cdot \mathbf{x}_P p_{k-2}\, \forall p_{k-2} \in \mathbb{P}_{k-2}(P)$

$\int_P \mathbf{curl v} \cdot (\mathbf{x}_P \wedge \mathbf{p}_{k-2})\, \forall \mathbf{p}_{k-2} \in (\mathbb{P}_{k-2}(P))^3$

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the face space:

$$V_{k-1}^f(P) = \{\mathbf{w} : \mathbf{w}\cdot\mathbf{n}_f \in \mathbb{P}_{k-1}(f), \operatorname{div}\mathbf{w} \in \mathbb{P}_{k-1}(P), \mathbf{curl}\mathbf{v} \in (\mathbb{P}_{k-2}(P))^3\}$$

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$$\text{for each } f :\ \int_f \mathbf{w}\cdot\mathbf{n}_f\, p_{k-1} \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(f)$$

$$\text{for } k \geq 2\ \int_P \mathbf{w}\cdot\nabla p_{k-1} \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(P)$$

$$\text{for } k \geq 2\ \int_P \mathbf{w}\cdot(\mathbf{x}_P \wedge \mathbf{p}_{k-2})\, \forall \mathbf{p}_{k-2} \in (\mathbb{P}_{k-2}(P))^3$$

Hint on 3D serendipity discretization

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the nodal space:

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the edge space:

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the face space:

$$V_{k-1}^f(P) = \{\mathbf{w} : \mathbf{w} \cdot \mathbf{n}_f \in \mathbb{P}_{k-1}(f), \operatorname{div}\mathbf{w} \in \mathbb{P}_{k-1}(P), \mathbf{curl}\mathbf{v} \in (\mathbb{P}_{k-2}(P))^3\}$$

Exact sequence:

$$\nabla SV_k^n(P) = \{\mathbf{v} \in SV_{k-1}^e(P) : \mathbf{curl}\mathbf{v} = 0\}$$

$$\mathbf{curl} SV_{k-1}^e(P) = \{\mathbf{w} \in V_{k-1}^f(P) : \operatorname{div}\mathbf{w} = 0\}, \qquad \operatorname{div} V_{k-1}^f(P) = \mathbb{P}_{k-1}(P)$$

Saving in interelement d.o.f.

	dofs k=2		
Mesh	VEM$\mathcal{S}_2$	VEM$_2$	$\mathbb{Q}_2$
8^3	2,673	7,857	4,401
16^3	18,785	57,953	31,841
32^3	140,481	444,609	241,857

	dofs k=3		
Mesh	VEM$\mathcal{S}_3$	VEM$_3$	$\mathbb{Q}_3$
8^3	4,617	14,985	11,529
16^3	32,657	110,993	84,881
32^3	245,025	853,281	650,529

Table : Number of inter-element dofs for cubic uniform mesh: $k = 2$ and $k = 3$

Saving in interelement d.o.f.

	dofs k=4		
Mesh	VEM$\mathcal{S}_4$	VEM$_4$	$\mathbb{Q}_4$
8^3	8,289	23,841	22,113
16^3	59,585	177,089	164,033
32^3	450, 945	1,363,329	1,261,953

	dofs k=5		
Mesh	VEM$\mathcal{S}_5$	VEM$_5$	$\mathbb{Q}_5$
8^3	15,417	34,425	36,153
16^3	112,625	256,241	269,297
32^3	859,617	1,974,753	2,076,129

Table : Number of inter-element dofs for cubic uniform mesh: $k = 4$ and $k = 5$

HAPPY BIRTHDAY BERNARDO

JOIN US: IT'S FUN!!!