A brief introduction to VEMs

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Recent Advances and Challenges
in DG Methods and Related Approaches

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Outline

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F.de Veub.

Fraeijs de Veubeke approach (1965)

"Darcy problem": $\boldsymbol{u} = -\nabla p$, $div \boldsymbol{u} = f$ + B.C. $p = 0$ at the boundary.

Following B. M. Fraeijs de Veubeke (**1965**) we observe that: given an approximation λ_h of p at the inter-element boundaries, and **another** approximation p_h of p inside each element we can deduce an approximation $\mathbf{u}_h$ of $\boldsymbol{u}$ by requiring, in each element E

$$\int_E \mathbf{u}_h \cdot \mathbf{v} = \int_E p_h \,\mathrm{div}\mathbf{v} - \int_{\partial E} \lambda_h \,(\mathbf{v} \cdot \mathbf{n}) \text{ for all } \mathbf{v}.$$

This can be used (among other things!) to define an *approximate gradient*

$$\mathbf{G_h} : \; (\lambda_h, p_h) \rightarrow \mathbf{u}_h = \mathbf{G_h}(\lambda_h, p_h)$$

to be used in various ways...

Form-3-fields

Formulations with three fields: $\lambda_h, p_h, \mathbf{u}_h$

Consider again the Darcy problem: $\boldsymbol{u} = -\nabla p$, $div\boldsymbol{u} = f$, $p = 0$ at $\partial\Omega$.
The *previous formula* reads, in each element E:

$$\int_E \mathbf{u}_h \cdot \mathbf{v} = \int_E p_h \,\mathrm{div}\mathbf{v} - \int_{\partial E} \lambda_h \,(\mathbf{v}\cdot\mathbf{n}) \text{ for all } \mathbf{v}.$$

You can then add a discretized *conservation equation*:

$$\int_E \mathrm{div}\mathbf{u}_h \, q = \int_E f \, q \text{ for all } q.$$

Then you must **close the system**, with *as many equations as there are λ_h's*, requiring "continuity" (for $\mathbf{u}_h \cdot \mathbf{n}$ or for p_h, or for a combination of the two) at the interelement boundaries. **Bernardo** showed *a number* of possible ways of exploiting this freedom in order to have methods with different suitable features, in different DG contexts.

Conn DG

Connection with DG methods (in various sauces)

Moreover, recalling the *previous formulae*

$$\int_E \mathbf{u}_h \cdot \mathbf{v} = \int_E p_h \,\mathrm{div}\mathbf{v} - \int_{\partial E} \lambda_h \,(\mathbf{v} \cdot \mathbf{n}) \text{ for all } \mathbf{v}.$$

$$\int_E \mathrm{div}\mathbf{u}_h \, q = \int_E f \, q \text{ for all } q.$$

you can define new interelement unknowns $\sigma_h \simeq \mathbf{u}_h \cdot \mathbf{n}$ (four variables), and write

$$\int_E \mathbf{u}_h \cdot \mathbf{v} = \int_E p_h \,\mathrm{div}\mathbf{v} - \int_{\partial E} \lambda_h \,\mathbf{v} \cdot \mathbf{n} \,\forall \mathbf{v} \qquad \int_{\mathbf{E}} \mathbf{u_h} \cdot \nabla \mathbf{q} = - \int_{\mathbf{E}} \mathbf{f} \, \mathbf{q} + \int_{\partial \mathbf{E}} \mathbf{q} \, \sigma_{\mathbf{h}} \,\forall \mathbf{q}.$$

Then you must **close the system**, with *as many equations as there are λ_h's and σ_h's* , typically in the form

$$\lambda_h = F_1((\mathbf{u}_h \cdot \mathbf{n})^{\pm}, \, p_h{}^{\pm}, \, (\nabla p_h)^{\pm}) \quad \text{and} \quad \sigma_h = F_2((\mathbf{u}_h \cdot \mathbf{n})^{\pm}, \, p_h{}^{\pm}, \, (\nabla p_h)^{\pm})$$

Here too **Bernardo** showed *a number* of possible ways of exploiting this freedom in order to have new methods with valuable properties. MFD

The main ancestor: MFD Simplest case

Basic observation: given a polygon E, assume that the information you have about a function v is given by

- The nodal values of v at the vertices
- The fact that v is linear on each edge.

Then, without actually knowing v, you can COMPUTE

$$\int_E \nabla v \nabla p_1 \equiv \int_{\partial E} v \frac{\partial p_1}{\partial n} \qquad \text{for every } p_1 \in \mathrm{P}_1$$

using only the above information. In particular:
the L^2 projection of ∇v onto the gradients of polynomials of degree ≤ 1 depends only, and can be computed from, the values of v at the vertexes.

Known originally as "Support Operator Method" (Hyman Shashkov '97)

$\nabla - \nabla$ inn pr

The main ancestor: MFD Simplest case

In the lowest order MFD method, each test (or trial) unknown is just a set of vertex values. As we have seen, however:

$\forall$ polygon E and $\forall$ pair of "set of nodal values" (say: U and V), if

- u is *any* smooth-enough function with nodal values U
- V are the nodal values of a polynomial $p_1 \in \mathbb{P}_1(E)$

then the integral

$$\int_E \nabla u \cdot \nabla p_1$$

is computable using U and V (and depends only on them). This allows to define a sort of *grad-grad* inner product between U and V that is correct whenever either U or V is made of the nodal values of a polynomial in $\mathbb{P}_1$.

Then you stabilize (brutally) to take care of the other cases.

MFD High Ord.

The main ancestor: MFD Another case

Basic observation: given a polygon E, assume that the information you have about a function v is given by

- The nodal values of v at the vertexes and midpoints of the edges.
- The fact that v is quadratic on each edge.
- The mean value of v over E

Then, without actually knowing v, you can COMPUTE

$$\int_E \nabla v \nabla p_2 \equiv -\int_E v\, \Delta p_2 + \int_{\partial E} v \frac{\partial p_2}{\partial n} \qquad \text{for every } p_2 \in \mathrm{P}_2$$

using only the above information. In particular:
the L^2 projection of ∇v onto the gradients of polynomials of degree ≤ 2 depends only, and can be computed from, the values of v at the *nodes* and the mean value of v over the element.

$\nabla - \nabla$ inn pr

The main ancestor: MFD Another case

Now every unknown U or V is a set of the degrees of freedom:

- The nodal values of v at the vertexes and midpoints of the edges.
- The mean value of v over E

From them you can compute

$$\int_E \nabla u \nabla v \equiv - \int_E u\,\Delta v + \int_{\partial E} u \frac{\partial v}{\partial n}$$

for every function u that assumes the dofs U, and for every V made from the values of a $p_2 \in \mathbb{P}_2$. Then you "stabilize" to extend the *grad-grad* bilinear form to the other pairs of unknowns.

MFD gen

The main ancestor: MFD More general case

Basic observation: given a polygon E, assume that the information you have about a function v is given by

- The nodal values of v at vertexes and $k-1$ Gauss-Lobatto of edges.
- The fact that v is in $\mathbb{P}_k$ on each edge.
- The moments of v of order $\leq k-2$ over E

Then, using the above information on v you can COMPUTE

$$\int_E \nabla v \nabla p_k \equiv -\int_E v\,\Delta p_k + \int_{\partial E} v \frac{\partial p_k}{\partial n} \qquad \text{for every } p_k \in \mathrm{P}_k$$

Here too, you can extend the *grad-grad* integral to the other sets of degrees of freedom with a suitable stabilization.

desc 2 field meth

The descendants of MFD: two-fields methods

A way of of converting the above approach in a Finite-Element style is clearly to consider (for instance for $k = 2$) **two functions**:

- One function v_b in $C^0(\partial E)$ that is, say, a $\mathbb{P}_2(e)$ on each edge e,
- Another function v_i in $\mathbb{P}_0(E)$ (i.e. constant).

Then you can COMPUTE

$$\int_E \widetilde{\nabla}\{v_b, v_i\}\nabla p_2 := -\int_E v_i\,\Delta p_2 + \int_{\partial E} v_b \frac{\partial p_2}{\partial n} \qquad \text{for every } p_2 \in \mathrm{P}_2$$

giving a sort of *approximate gradient* $\widetilde{\nabla}$ of the pair $\{v_b, v_i\}$. Obviously, if you start from a $v \in \mathbb{P}_2(E)$ and define $v_b = v_{|\partial E}$ and v_i=mean value of v on E you get that $\widetilde{\nabla}\{v_b, v_i\} = \nabla v$.
Here too you can then have an approximate *grad-grad* scalar product that you will suitably *stabilize*.

VEM pl va

The main feature of VEM - Plain vanilla version

Another way of of converting the above approach in a Finite-Element style is clearly to consider **single (uncomputable!) functions** v such that

- $v \in C^0(\partial E)$ and $v_{|e} \in \mathbb{P}_k(e)$ on each edge e,
- $\Delta v \in P_{k-2} \Leftarrow$ NOTE! It sets the right dimension!

using the degrees of freedom

- Values of v at vertexes and at $k-1$ Gauss-Lobatto points of edges
- **the moments** $\int_E v\, p_{k-2}$ **for** $p_{k-2} \in \mathbb{P}_{k-2} \Leftarrow$ NOTE! dofs to be used!

Then you can **use the degrees of freedom** to COMPUTE

$$\int_E \nabla v \nabla p_k := -\int_E v\, \Delta p_k + \int_{\partial E} v \frac{\partial p_k}{\partial n} \qquad \text{for every } p_k \in \mathrm{P}_k$$

giving the $(L^2(E))^2$ orthogonal projection of ∇v onto $\nabla(\mathbb{P}_k)$.

Hence you get a *grad-grad* scalar product that you'll *stabilize*, etc. C $\simeq$ dofs

Computationally equivalent degrees of freedom

We go back to the previous VEM space

- $v \in C^0(\partial E)$ and $v_{|e} \in \mathbb{P}_k(e)$ on each edge e,
- $\Delta v \in P_{k-2}$

using the degrees of freedom

- (1) Values of v at vertexes and at $k-1$ Gauss-Lobatto points of edges
- (2) **the moments** $\int_E v\, p_{k-2}$ **for** $p_{k-2} \in \mathbb{P}_{k-2}$ (1st choice)

We point out that using (1) and (2) you can compute, for instance

- (3) $\int_E \nabla v \cdot (\mathbf{x} - \mathbf{x}_b) p_{k-2} \mathrm{d}E$ (where $\mathbf{x}_b := \mathbf{x} - \mathbf{b}$ with $\mathbf{b}$=baricenter)

and from (1) and (3) we can compute (2). We say that these two sets of dof's are *computationally equivalent*. On the contrary, knowing (1) and

- (4) $\Delta v \in \mathbb{P}_{k-2}$

you **cannot** compute (2) nor (3), unless you solve the PDE. We say that "((1),(2)) are *equivalent* but **not** *computationally equivalent* to ((1),(4))"

2d Face elements ($H(\mathrm{div})$-conforming)

For k, k_d, k_r integers, with $k \geq 0$, $k_d \geq 0$, $k_r \geq -1$ set:

$$\mathbf{V}^f_{k,k_d,k_r}(E) := \{\mathbf{v} | \, \mathbf{v} \cdot \mathbf{n}_e \in \mathbb{P}_k(e) \, \forall e, \; \mathrm{div}\mathbf{v} \in \mathbb{P}_{k_d}, \; \mathrm{rot}\mathbf{v} \in \mathbb{P}_{k_r}\},$$

with the following degrees of freedom:

$$\int_e \mathbf{v} \cdot \mathbf{n}_e \, q_k \mathrm{d}e \quad \text{for all } q_k \in \mathbb{P}_k(e), \; \text{for all edge } e,$$

$$\text{for } k_d \geq 1\text{:} \; \int_E \mathbf{v} \cdot \mathbf{grad} q_{k_d} \mathrm{d}E \quad \text{for all } q_{k_d} \in \mathbb{P}_{k_d}(E),$$

$$\text{for } k_r \geq 0\text{:} \; \int_E \mathbf{v} \cdot \mathbf{x}^\perp \, q_{k_r} \mathrm{d}E \quad \text{for all } q_{k_r} \in \mathbb{P}_{k_r}(E).$$

The dof's allow to compute the L^2-orthogonal projection Π^0_s on the polynomials of degree s for $s \leq k_r + 1$.

2D edge

2d edge elements ($H(\mathrm{rot})$-conforming)

For k, k_d, k_r integers, with $k \geq 0$, $k_d \geq -1$, and $k_r \geq 0$:

$$\mathbf{V}^e_{k,k_d,k_r}(E) := \{\mathbf{v} | \mathbf{v} \cdot \mathbf{t_e} \in \mathbb{P}_k(e) \forall e,\ \mathrm{div}\mathbf{v} \in \mathbb{P}_{k_d},\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_{k_r}\},$$

with the degrees of freedom:

$$\int_e \mathbf{v} \cdot \mathbf{t}\, q_k \mathrm{d}e \quad \text{for all } q_k \in \mathbb{P}_k(e), \text{ for all edge } e,$$

$$\text{for } k_r \geq 1\text{:} \quad \int_E \mathbf{v} \cdot \mathbf{rot} q_{k_r} \mathrm{d}E \quad \text{for all } q_{k_r} \in \mathbb{P}_{k_r}(E),$$

$$\text{for } k_d \geq 0\text{:} \quad \int_E \mathbf{v} \cdot \mathbf{x}\, q_{k_d} \mathrm{d}E \quad \text{for all } q_{k_d} \in \mathbb{P}_{k_d}(E).$$

The dof's allow to compute the L^2-orthogonal projection Π^0_s on the polynomials of degree s for $s \leq k_d + 1$.

2 Prop

The two properties

For the *typical* continuous problem

$$\text{Find } p \in Q \text{ s.t. } a(p,q) = (f,q) \quad \forall q \in Q$$

we consider discretizations with the **two properties**: $\forall h$, and $\forall E$ in $\mathcal{T}_h$:

H1- $\forall p_k \in \mathbb{P}_k$, $\forall q_h \in Q_h$

$$a_h^E(p_k, q_h) = a^E(p_k, q_h) \qquad (\mathbf{k - Consistency})$$

H2- $\exists$ two positive constants α_* and α^*, independent of h and of E, s. t.:

$$\forall q_h \in Q_h \qquad \alpha_* \, a^E(q_h, q_h) \leq a_h^E(q_h, q_h) \leq \alpha^* \, a^E(q_h, q_h) \quad (\mathbf{Stability})$$

Teor

Convergence

Under these assumptions, we have the *typical* result

Theorem

The discrete problem: Find $p_h \in Q_h$ *such that*

$$a_h(p_h, q_h) = (f_h, q_h), \qquad \forall\, q_h \in Q_h$$

has a unique solution p_h*. Moreover, for every approximation* p_I *of* p *in* Q_h *and for every approximation* p_π *of* p *that is piecewise in* $\mathbb{P}_k$*, we have*

$$\|p - p_h\|_Q \le C\Big(\|p - p_I\|_Q + \|p - p_\pi\|_{h,Q} + \|f - f_h\|_{Q'_h}\Big)$$

where C *is a constant independent of* h*.*

gen feat

The general features of VEM

The general features of VEM,today,could be summarized as follows:

- The Virtual Element Spaces (used to discretize H^1, $H(\mathrm{div})$, $H(\mathbf{curl})$, $(H^1)^d$, $\underline{\underline{\mathbf{H}}}(\mathbf{div})_{symm}$, $H(\textit{You-name-it})$, etc.) are made of **functions** (scalars, or vectors, of tensors) that are, in each element, **solutions of** (systems of) **Partial Differential Equations**.
- The *name of the game* is that these **functions** are **not computed**, not even in an approximate way (apart from one-d elementary cases).
- In practice, **one uses the degrees of freedom** to compute some **quantities of interest** (e.g. projections on polynomial spaces)
- The above **quantities of interest** are then **used to construct the discretized problem** (possibly with the addition of some suitable stabilizations).
- **There is NO reference element** (with all relative pros and cons).

evol path

VEMs - The paths of evolution

The V.E. Spaces today on the market could be (very) roughly classified in **two groups** (with all obvious intersections and intermediate cases)

- Some spaces are built **per se**. For instance, given a decomposition of the computational domain Ω into elements E, I want to find finite dimensional subspaces of $H^1(\Omega)$ whose degrees of freedom allow me to compute a certain amount of relevant quantities with sufficient accuracy. E.g: on each E I want a computable scalar product that coincides with the H^1 scalar product whenever one of the two entries is a polynomial of a certain degree. I don't care, at the moment, about the use that will be done of these spaces.
- Some other spaces (and often pairs or even complexes of spaces) are constructed **having a specific problem in mind**. For instance, I can construct velocity-pressure pairs to be used for the Stokes problem, or displacement-stresses pairs to be used in the Hellinger-Reissner formulations of elasticity problems, or a discrete De Rham complex to be used for Electromagnetic problems, and so on.

Ex

Examples

The examples that we have seen so far (functions that are polynomials on each edge and have a polynomial Laplacian) clearly belong to the first group (Spaces constructed *per se*).
However if we use them to approximate the solution of a Darcy flow, then they would immediately look as being built *ad hoc* for this problem.

On the other hand, if you consider (as in Beirão-Vacca, 2016) on each polygon E the space of vectors $\mathbf{v}$ such that

- Each component of $\mathbf{v}$ is continuous on ∂E and edge-wise in $\mathbb{P}_k$
- There exists an $s \in H^1(E)$ such that $\Delta\mathbf{v} - \nabla s \in (\mathbb{P}_{k-2})^2$
- $\mathrm{div}\mathbf{v} \in \mathbb{P}_{k-1}$

there is little doubt that you have the Stokes problem in mind.

N.B. the second condition could be written as $\mathrm{rot}(\Delta\mathbf{v}) \in \mathbb{P}_{k-3}$.

Ex El

Examples - More complex

Following Artioli-de Miranda-Lovadina-Patruno we start by setting, for each segment e the set $\mathbf{P}_{0,1}(e)$ of vector valued functions, defined on e, such that:

- the tangential component is constant
- the normal component is linear.

Then we consider the space of tensor valued functions $\underline{\underline{\tau}} \in \underline{\underline{\mathbf{H}}}(\mathbf{div})$ s. t.

- $\underline{\underline{\tau}}$ is the symmetric gradient of some $\mathbf{w} \in (H^1)^2$ (that is, $\underline{\underline{\tau}} = \underline{\underline{\varepsilon}}(\mathbf{w})$)
- $\forall e \in \partial E \quad \underline{\underline{\tau}} \cdot \mathbf{n}_e \in \mathbf{P}_{0,1}(e)$
- $\mathbf{div}\underline{\underline{\tau}} \in \mathbf{RM}$ (= **R**igid **M**ovements).

that is clearly aimed at the discretization of the Hellinger-Reissner formulation of linear elasticity problems, here for simplicity with constitutive tensor $\mathbb{C} = \mathbb{I}dentity$ (otherwise, take $\underline{\underline{\tau}} = \mathbb{C}\underline{\underline{\varepsilon}}(\mathbf{w})$)

N.B. Here, the **first** condition could also be written $\mathbf{curl}\,\mathbf{curl}(\mathbb{D}\underline{\underline{\tau}}) = 0$.

Serend

Serendipity spaces (general view)

The Serendipity procedure is, as in Finite Elements, a way of reducing the (internal) degrees of freedom by changing the space. (The most famous example beng the 8-node square)

Aim:

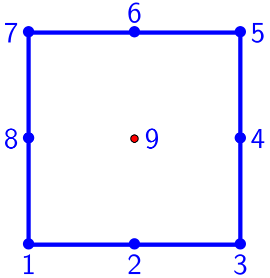
• we want to keep the boundary d.o.f. to preserve conformity: H^1 for nodal VEMs, and $\mathbf{H}(\mathrm{rot})$ for edge VEMs

• we try to eliminate as many internal d.o.f. as we can, keeping only those needed for the expected accuracy

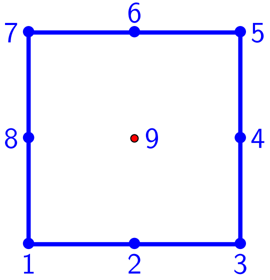
NOTE: When dealing with 3D problems, this will be used to eliminate d.o.f.s internal to the faces: something otherwise impossible (or extremely hard) to do with static condensation

SC vs SERE

Elimination of internal d.o.f.s

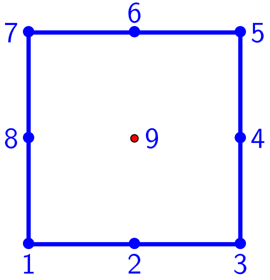


Elimination of internal d.o.f.s



Static condensation is just a way of solving the linear system leaving the approximation space *unchanged.*

Elimination of internal d.o.f.s



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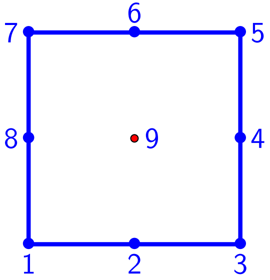
Serendipity changes the approximation space (here $\mathbb{Q}_2 \longrightarrow \mathbb{Q}_2 \setminus x^2y^2$)

Elimination of internal d.o.f.s



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Serendipity changes the approximation space (here $\mathbb{Q}_2 \longrightarrow \mathbb{Q}_2 \setminus x^2y^2$)

In the above case: let $\varphi : \mathbb{R}^8 \to \mathbb{R}$ be a function such that $\varphi\Big(p(1), p(2), \cdots, p(8)\Big) = p(9)\ \forall p \in \mathbb{P}_2$, and then take

$$\mathcal{S} := \{q \in \mathbb{Q}_2 \text{ s.t. } q(9) = \varphi(q(1), q(2), \cdots, q(8))\}$$

$$\Longrightarrow \quad \mathbb{P}_2 \subset \mathcal{S} \subset \mathbb{Q}_2$$

Which dofs must be kept and which may be eliminated

For *nodal* VEMS, and $r \geq 1$, we start from the bigger space

$$\widetilde{V}_r^n(E) := \Big\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_r(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_r(E) \Big\}.$$

with the degrees of freedom:

- the nodal values $q(\nu)$ at all vertexes ν of E,
- for each edge e, the moments $\int_e q \, p_{r-2} \, \mathrm{d}s \quad \forall p_{r-2} \in \mathbb{P}_{r-2}(e)$,
- $\int_E q \, p_r \mathrm{d}E \quad \forall p_r \in \mathbb{P}_r(E)$,

NOTE: out of the above dofs you can compute *almost everything*. But they would be *awfully expensive*!

Split dofs

Which dofs must be kept and which may be eliminated

Recall: For *nodal* VEMS, and $r \geq 1$, we started from the bigger space

$$\widetilde{V}_r^n(E) := \Big\{ q \in C^0(\overline{E}) : \ q_{|e} \in \mathbb{P}_r(e) \, \forall e \in \partial E, \ \Delta q \in \mathbb{P}_r(E) \Big\}.$$

We now split all the above degrees of freedom in two (disjoint) parts

$\mathbb{S} = \{$dofs that we want to keep$\}$ $\quad \mathbb{T} = \{$dofs that we want to throw away$\}$

and we assume that $\mathbb{S}$ contains **all** the boundary degrees of freedom **plus**, possibly, some internal moments, so that the following property holds:

$$(\mathscr{S}) \qquad \forall p_r \in \mathbb{P}_r(E) : \quad \{\delta(p_r) = 0 \text{ for all } \delta \in \mathbb{S}\} \Rightarrow \{p_r \equiv 0\}.$$

Ex-S-Tria

Examples: on triangles

Remember that we want

$$(\mathscr{S}) \qquad \forall p_r \in \mathbb{P}_r(E): \quad \{\delta(p_r) = 0 \text{ for all } \delta \in \mathbb{S}\} \Rightarrow \{p_r \equiv 0\}.$$

This will depend both on r and on the geometry of E.
Can you use **only** the boundary degrees of freedom? On triangles....

Examples: on triangles

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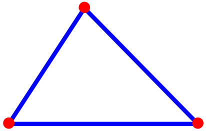

$r = 1$ *yes*

Examples: on triangles

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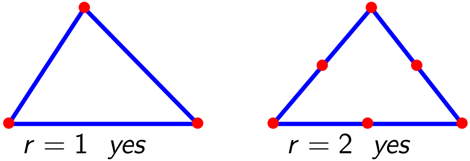

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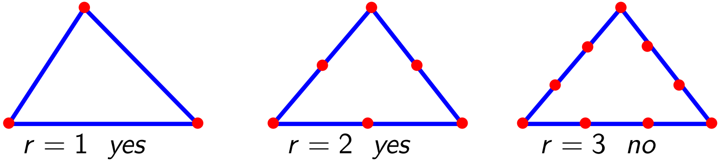

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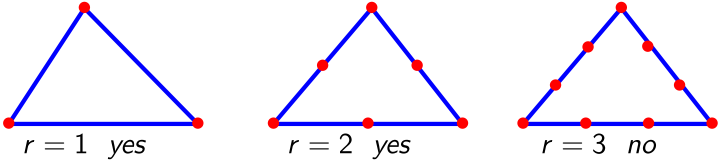

For $r < 3$, property $\mathscr{S}$ holds on triangles just using the boundary d.o.f.
If $r \geq 3$ on triangles we will need, in $\mathbb{S}$, some of the internal d.o.f.
For instance, for $r = 3$ we need just 1 internal d.o.f. (and not 3!!), to "kill" the bubble of $\mathbb{P}_3$.

Ex-S-Qua

Examples: on quadrilaterals

Remember that we want

$$(\mathscr{S}) \qquad \forall p_r \in \mathbb{P}_r(E): \quad \{\delta(p_r) = 0 \text{ for all } \delta \in \mathbb{S}\} \Rightarrow \{p_r \equiv 0\}.$$

This will depend both on r and on the geometry of E.
Can you use **only** the boundary degrees of freedom? On quadrilaterals....

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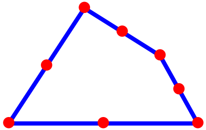

$r = 2$ *yes*

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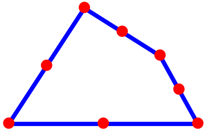

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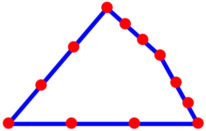

$r = 3$ *yes*

Examples: on quadrilaterals

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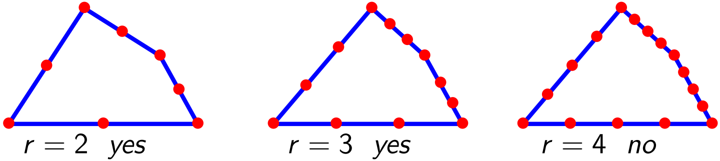

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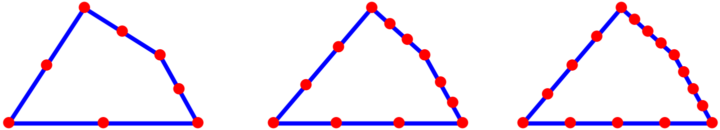

$r = 2$ *yes* $\qquad$ $r = 3$ *yes* $\qquad$ $r = 4$ *no*

For $r < 4$, on quads, property $\mathscr{S}$ holds just using the boundary d.o.f.
If $r \geq 4$ on quads we will need $\mathbb{S}$ some of the internal d.o.f.
For instance, for $r = 4$ we need in $\mathbb{S}$ just 1 internal d.o.f., (and not 6!!), to "kill" the bubble of $\mathbb{P}_4$.

Ex-S-Gen

Examples: General Case

When do we need internal degrees of freedom? And how many of them?

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When do we need internal degrees of freedom? And how many of them? We need to kill the bubbles of $\mathbb{P}_r$: $B_r(E) = \mathbb{P}_r(E) \cap H^1_0(E)$. Our additional internal d.o.f. could then be chosen as

$$\int_E q\, b_r \,\mathrm{d}x, \quad \forall b_r \in B_r(E).$$

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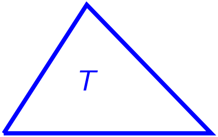

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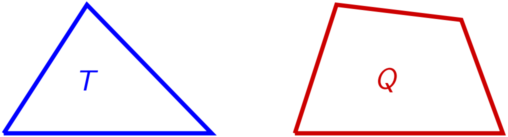

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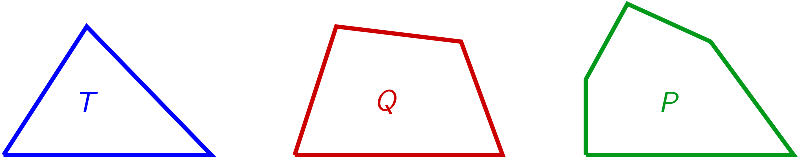

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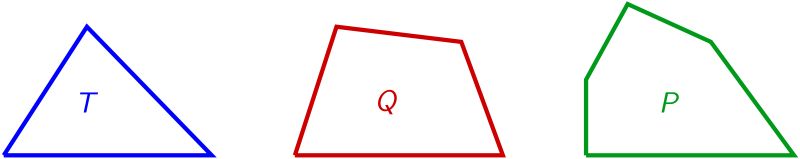

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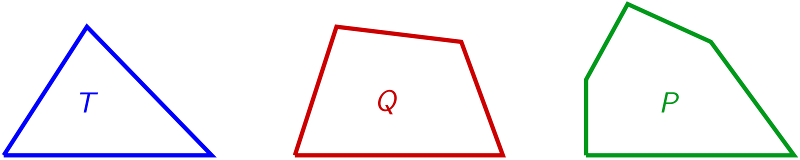

$B_r(T) = \beta_3 p_{r-3}$ $\qquad$ $B_r(Q) = \beta_4 p_{r-4}$

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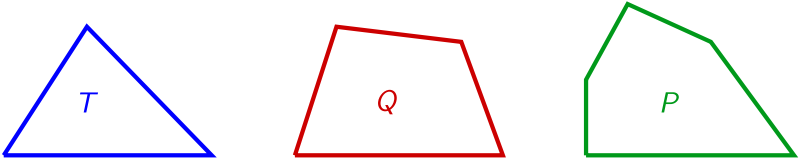
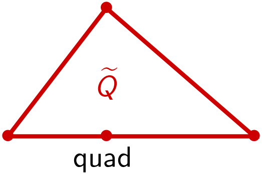
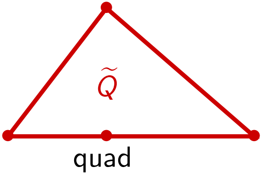


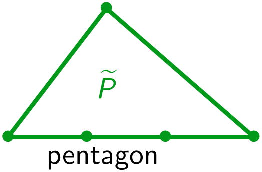
$B_r(T) = \beta_3 p_{r-3}$ $\quad$ $B_r(Q) = \beta_4 p_{r-4}$ $\quad$ $B_r(P) = \beta_5 p_{r-5}$ guai

Examples - Troubles



Examples - Troubles





Examples - Troubles

$$B_r(\widetilde{Q}) = \lambda_1\lambda_2\lambda_3 p_{r-3}$$
$$B_r(\widetilde{P}) = \lambda_1\lambda_2\lambda_3 p_{r-3}$$

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$$B_r(\widetilde{Q}) = \lambda_1\lambda_2\lambda_3 p_{r-3}$$

$$B_r(\widetilde{P}) = \lambda_1\lambda_2\lambda_3 p_{r-3}$$

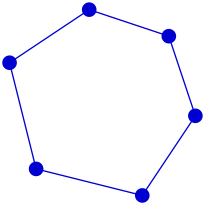
What counts is the number η of straight lines necessary to cover the boundary of E. In both the above cases $\eta = 3$.

other etas

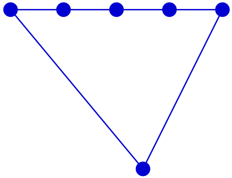
Other examples

η = minimum number of straight lines necessary to cover the boundary

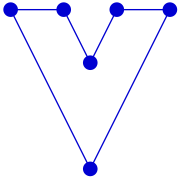
N = number of edges



N=6 η=6



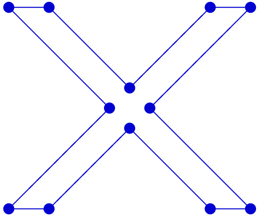
N=6 η=3

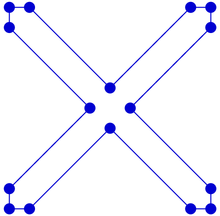


N=6 η=5

more dram

Other examples - Generality presents the BILL





$dim(B_r(E)) = dim(\mathbb{P}_{r-\eta})$. We need as internal dofs, for instance:

either $\int_E q\, p_{r-\eta}\, \mathrm{d}x \;\forall p_{r-\eta} \in \mathbb{P}_{r-\eta}$ or $\int_E q\, b_r\, \mathrm{d}x \;\forall b_r \in B_r$

lazy-stingy

The lazy choice and the stingy choice

Setting $\pi_r := \dim(\mathbb{P}_r)$ we must add, to the boundary dofs:

- on a **triangle** ($\eta = 3$), π_{r-3} internal dofs;
- on a **quad** ($\eta = 4$), π_{r-4} internal dofs;
- on an **η-gon**, $\pi_{r-\eta}$ internal dofs.

In general, even on very distorted polygons, **you must keep in $\mathbb{S}$ as many internal dofs as there are $\mathbb{P}_r$-bubbles**

In practice, in a code, you may either check every element to compute its η (stingy choice) or treat every element as if it were a triangle (lazy choice). Obviously, intermediate choices can be taken as well.

The *best strategy* depends on the circumstances.

from $\mathbb{S}$ to *proj*

A projector $\Pi_r^{n,S}$ - General polygon

Assume that the dof's that we decided to keep in $\mathbb{S}$ are

- The boundary ones
- For $r \geq \eta$ the moments against the *bubbles* of $\mathbb{P}_r$,

Then we can define our **Serendipity projector** $\Pi_r^{n,S}$ as

$$\begin{cases} \int_{\partial E} (q - \Pi_r^{n,S} q) p \, \mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_r(E), \\ \int_E (q - \Pi_r^{n,S} q) \, b_r \mathrm{d}E = 0 \quad \forall b_r \in B_r. \end{cases}$$

It is obvious that, with the above construction:

$$\Pi_r^{n,S} \text{ is a projection } \widetilde{V}_r^n(E) \longrightarrow \mathbb{P}_r$$

Alt dof

The projector $\Pi_r^{n,S}$ - Alternative dof's

However, still keeping as "dofs to be kept"

- The boundary ones
- For $r \geq \eta$ the moments against the *bubbles* of $\mathbb{P}_r$,

we could have started from the *computationally equivalent* dofs

$$\begin{cases} \int_{\partial E} \partial_t (q - \Pi_r^{n,S} q) \partial_t p \, \mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_r(E), \\ \int_{\partial E} (\mathbf{x}_E \cdot \mathbf{n})(q - \Pi_r^{n,S} q) \, \mathrm{d}s = 0, \\ \int_E (\nabla (q - \Pi_r^{n,S} q) \cdot \mathbf{x}_E b_r \mathrm{d}E = 0 \quad \forall b_r \in B_r. \end{cases}$$

It is clear that the new projector still satisfies

$$\Pi_r^{n,S} \text{ is a projection } \widetilde{V}_r^n(E) \longrightarrow \mathbb{P}_r$$

but it is also clear that it is **not the same as before.**

The projector $\Pi_r^{n,S}$ (Convex polygon)

In the case of a **convex** (not necessarily "strictly convex") **polygon** E, with again η = *minimum number of straight lines necessary to cover* ∂E we could instead define

$$\zeta := r - \eta$$

and define $\Pi_r^{n,S}$ as

$$\begin{cases} \int_{\partial E} (q - \Pi_r^{n,S} q) p \, \mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_r(E), \\ \int_E (q - \Pi_r^{n,S} q) \, p_\zeta \mathrm{d}E = 0 \quad \forall p_\zeta \in \mathbb{P}_\zeta. \end{cases}$$

Note that the dimension of $\mathbb{P}_\zeta$ is equal to the dimension of the space $B_r(E)$ of bubbles in E of degree r (that we used before).

Alt X conv

Alternative projector $\Pi_r^{n,S}$ for a convex polygon

Also in the case of a convex polygon we could have use, as an alternative,

$$\begin{cases} \int_{\partial E} \partial_t (q - \Pi_S^n q) \partial_t p \, \mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_k(E), \\ \int_{\partial E} (\mathbf{x}_E \cdot \mathbf{n})(q - \Pi_S^n q) \, \mathrm{d}s = 0, \\ \int_E (\nabla (q - \Pi_S^n q) \cdot \mathbf{x}_E p_\zeta \mathrm{d}E = 0 \quad \forall p_\zeta \in \mathbb{P}_\zeta. \end{cases}$$

Again, we have another projector that does not coincide with the previous ones.

SERE-VEM spaces

Serendipity Nodal VEM-spaces

The operator $\Pi_r^{n,S}$ has the following properties:

- $\Pi_r^{n,S}$ is computable using only the d.o.f. $\delta \in \mathbb{S}$
- $\Pi_r^{n,S} q = q \quad \forall q \in \mathbb{P}_r$.

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At this point we can set:

$$SV_r^n(E) := \{q \in \widetilde{V}_r^n(E) : \text{s.t.}\, \delta(q) = \delta(\Pi_r^{n,S} q)\, \forall \delta \in \mathbb{T}\}$$

From the dofs in $\mathbb{S}$ we can compute $\Pi_r^{n,S}$, and then from $\Pi_r^{n,S}$ we can compute all the other dofs in $\mathbb{T}$.

$$\mathbb{P}_r(E) \subseteq SV_r^n(E) \subseteq \widetilde{V}_r^n(E)$$

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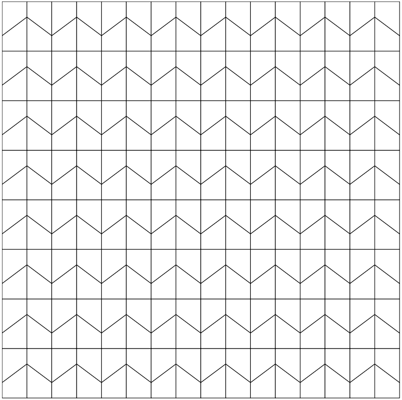
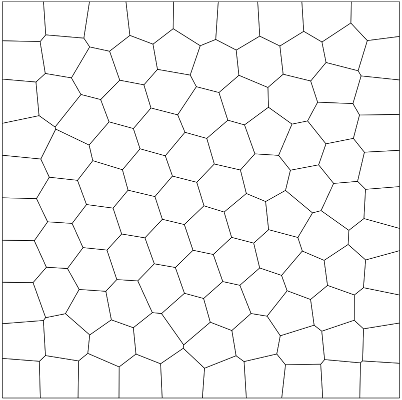
Simple! Isn't it?

Two families of meshes



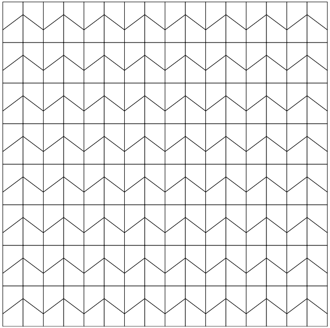
Trapezoidal mesh



Voronoi mesh

Test Trap

Test for the trapezoidal meshes



Trapezoidal mesh

$-\Delta p = f$ in Ω, $p = g$ on Γ

exact solution:

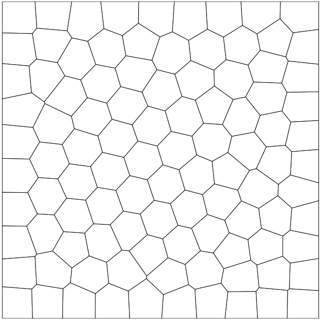
$x^3 + 5y^2 - 10y^3 + y^4 + x^5 + x^4 y$

f and g chosen accordingly

Arnold – Boffi – Falk(2002)

Test Pr LI

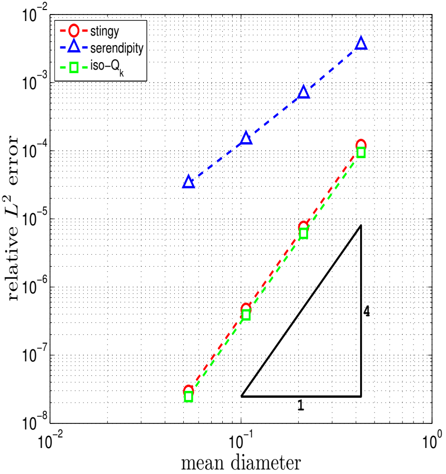
Test for the Voronoi meshes



Voronoi mesh

$$\begin{cases} \mathrm{div}(-\kappa\nabla p + \beta p) + \gamma\, p = f \quad \text{in } \Omega \\ p = g \quad \text{on } \Gamma \end{cases}$$

Trap $k = 3$

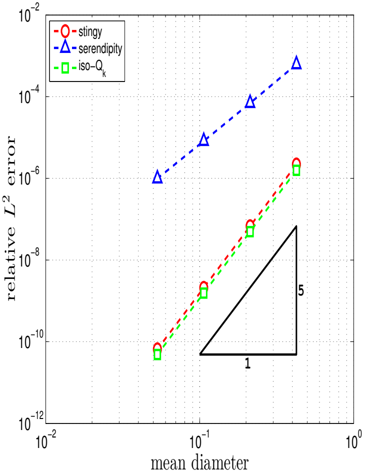


	degrees of freedom		
# el.	stingy	$\mathcal{S}_k$	$\mathbb{Q}_k$
16	105	105	169
64	369	369	625
256	1377	1377	2401
1024	5313	5313	9409

Trapezoidal mesh $k = 3$

Trap $k = 4$

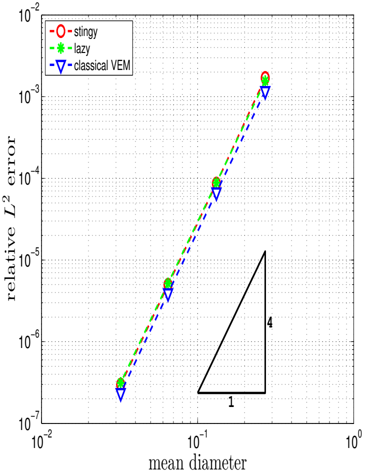
$k = 4$: $\mathbb{Q}_k$-FEM, $\mathcal{S}$-FEM, and $\mathcal{S}$-VEM on quads



	degrees of freedom		
# el.	stingy	$\mathcal{S}_k$	$\mathbb{Q}_k$
16	161	161	289
64	577	577	1089
256	2177	2177	4225
1024	8449	8449	16641

Trapezoidal mesh $k = 4$

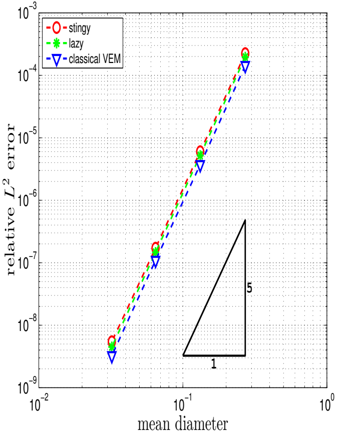
$k = 3$: Classical VEM and $\mathcal{S}$-VEM (stingy, lazy) on Lloyd)



	degrees of freedom		
# el.	stingy	lazy	VEM
25	204	229	279
100	804	904	1104
400	3204	3604	4404
1600	12804	14404	17604

Voronoi-Lloyd mesh $k = 3$

$k = 4$: Classical VEM and S-VEM (stingy, lazy on Voronoi)



	degrees of freedom		
# el.	stingy	lazy	VEM
25	284	355	430
100	1112	1405	1705
400	4408	5605	6805
1600	17614	22405	27205

Voronoi-Lloyd mesh $k = 4$

Wishes 60→120

BEST WISHES
FOR THE NEXT 60s,
BERNARDO!!!!!