Bulk-edge correspondence in the presence of a mobility gap

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May 16-19, 2017, Minneapolis

based on joint work with A. Elgart, J. Schenker; J. Shapiro

Outline

Goal of the talk

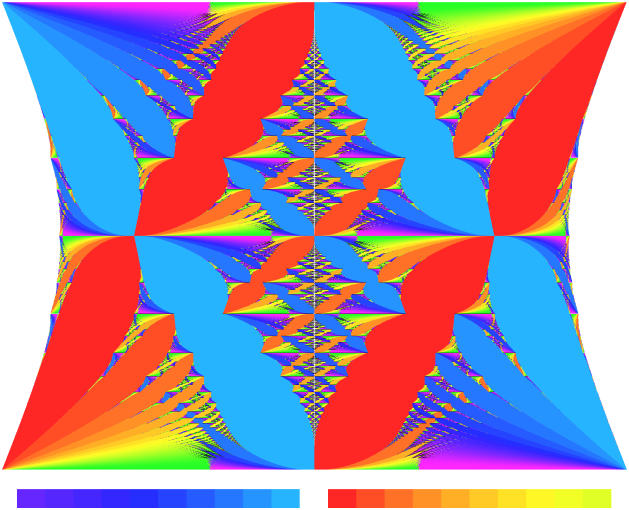
Quantum Hall systems

Chiral systems

Goals of the talk

- Difference between spectral and mobility gap
- Bulk-edge correspondence for quantum Hall Hamiltonians (2 dim)
- Bulk-edge correspondence for chiral Hamiltonians (1 dim)

The quantum Hall phase diagram of graphene



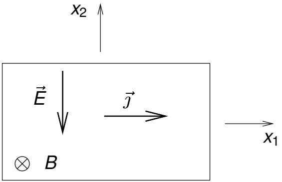
(with Agazzi and Eckmann)

Goal of the talk

Quantum Hall systems

Chiral systems

The experiment (von Klitzing, 1980)



$\underline{B}$: magnetic field
$\vec{E}$: electric field
$\vec{\jmath}$: current density

Electron gas of
- density n
- Fermi level μ

Hall-Ohm law

$$\vec{\jmath} = \underline{\sigma}\vec{E}\ , \qquad \underline{\sigma} = \begin{pmatrix} \sigma_{\mathrm{D}} & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & \sigma_{\mathrm{D}} \end{pmatrix}$$

σ_{H}: Hall conductance
σ_{D}: ohmic (dissipative) conductance, ideally $= 0$

The experiment (von Klitzing, 1980)

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$$\vec{j} = \underline{\sigma}\vec{E}\ , \qquad \underline{\sigma} = \begin{pmatrix} \sigma_D & \sigma_H \\ -\sigma_H & \sigma_D \end{pmatrix}$$

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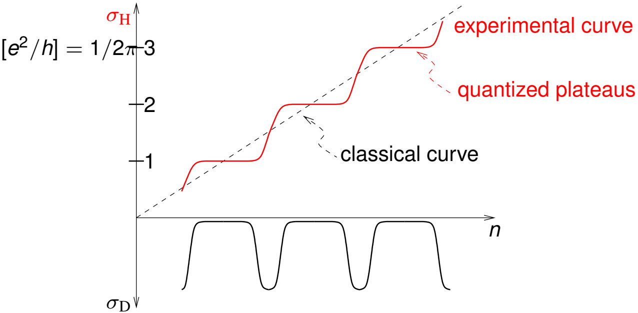
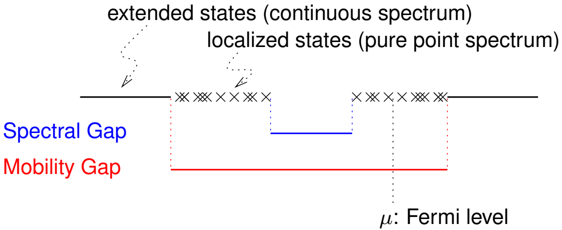


Width of plateaus increases with disorder

Spectral vs. Mobility Gap

The spectrum of a single-particle Hamiltonian



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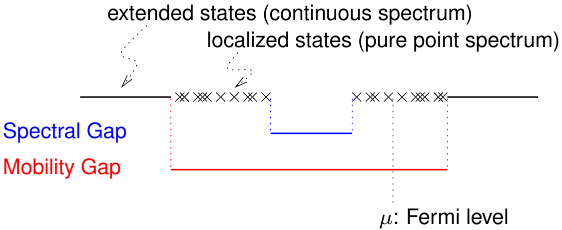

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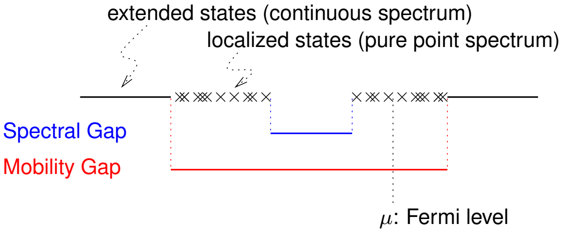

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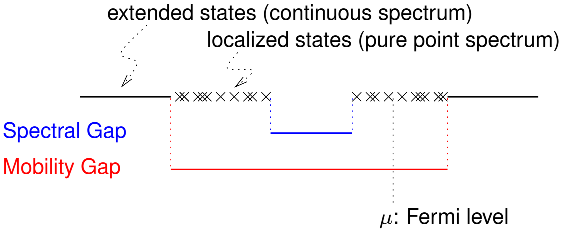

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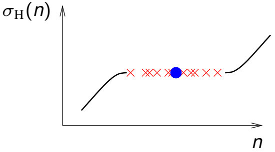

Plateaus arise because of a Mobility Gap only!

Mobility gap, technically speaking

Hamiltonian H_B on $\ell^2(\mathbb{Z}^d)$
$P_\mu = E_{(-\infty,\mu)}(H_B)$ Fermi projection,
(not to be confused with eigenprojection $\mathcal{P}_\lambda = E_{\{\lambda\}}(H_B)$)

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Assumption. Fermi projection has strong off-diagonal decay:

$$\sup_{x'} \mathrm{e}^{-\varepsilon|x'|} \sum_{x} \mathrm{e}^{\nu|x-x'|} |P_\mu(x,x')| < \infty$$

(some $\nu > 0$, all $\varepsilon > 0$)

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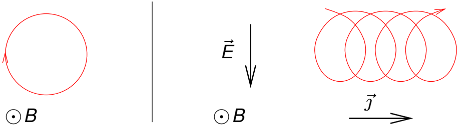
- Proven in (virtually) all cases where localization is known.
- Trivially false for extended states at $E = \mu$.

IQHE as a Bulk effect

Paradigm: Cyclotron orbit drifting under a electric field $\vec{E}$

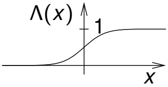


Hamiltonian H_B in the plane. Kubo formula (linear response to $\vec{E}$)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

where

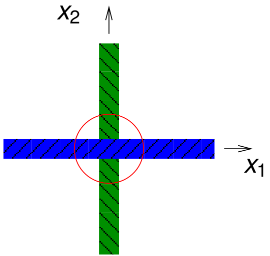
$\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches



IQHE as a Bulk effect (remarks)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr} P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2] \big]$$

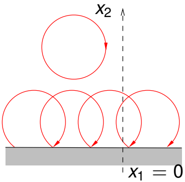
where $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches. Supports of $\vec{\nabla}\Lambda_i$:

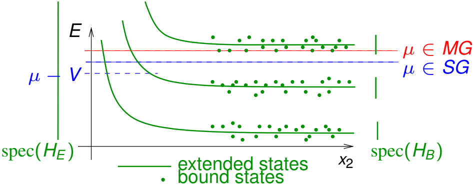


Remarks.

- The trace is well-defined. Roughly: An operator has a well-defined trace if it acts non-trivially on finitely many states only. Here the intersection contains only finitely many sites.
- The operator is localized in x_1, x_2 ($x_1 = x_2 = 0$) but not in energy ($E \leq \mu$)

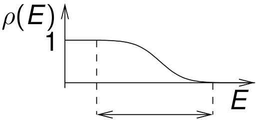
IQHE as an edge effect (spectral gap)





Hamiltonian H_E on the upper half-plane: restriction of H_B through boundary conditions at $x_2 = 0$.

State $\rho(H_E)$: 1-particle density matrix, e.g. $\rho(H_E) = E_{(-\infty,\mu)}(H_E)$, or (actually) smooth



supp $\rho' \subset$ Spectral Gap for H_B (not for H_E)

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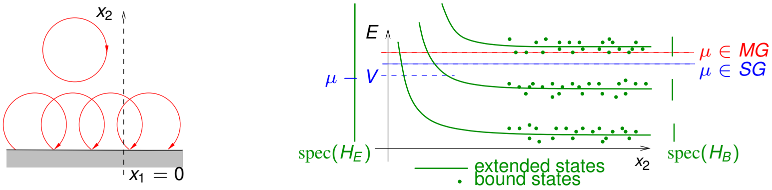

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Current operator across $x_1 = 0$: $\mathrm{i}[H_E, \Lambda_1]$

$$I = \mathrm{i}\ \mathrm{tr}(\rho(H_E + V) - \rho(H_E))[H_E, \Lambda_1]$$

As $V \to 0$: $I/V \to \sigma_\mathrm{E}$

$$\sigma_\mathrm{E} = \mathrm{i}\ \mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

Equality of conductances

Theorem (Schulz-Baldes, Kellendonk, Richter). Ergodic setting. If the Fermi level μ lies in a Spectral Gap of H_B, then

$$\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}.$$

In particular, σ_{E} does not depend on ρ', nor on boundary conditions.

What about the case of a Mobility Gap?

Is

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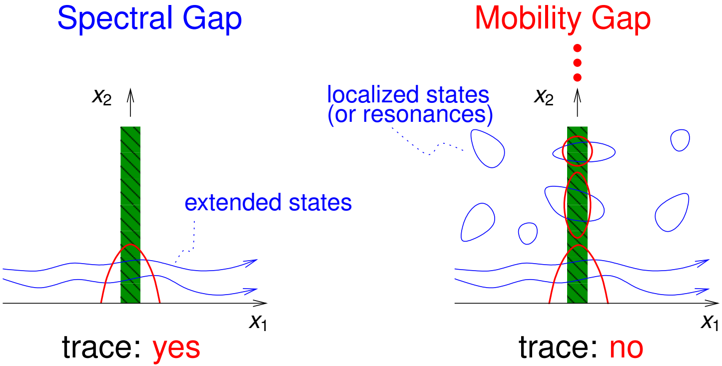

$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

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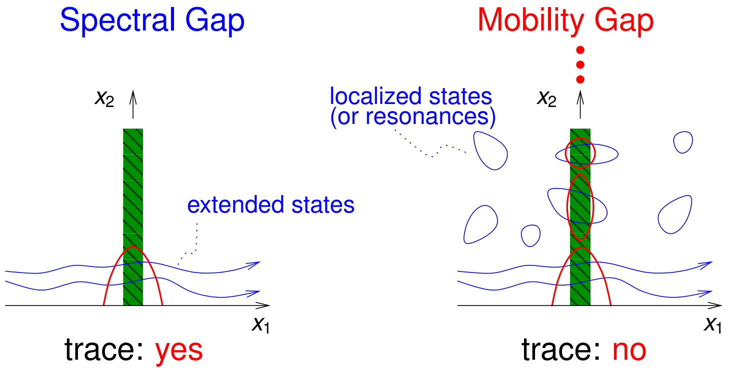


$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

Guiding principle: Localized states should not contribute to the edge current

Equality of conductances

For a suitable definition of σ_{E}:

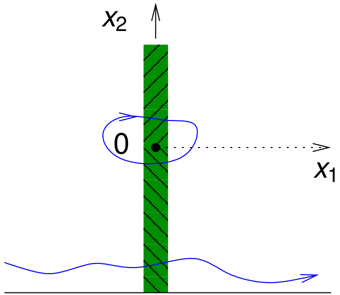
Theorem (Elgart, G., Schenker). If $\operatorname{supp} \rho'$ lies in a Mobility Gap, then

$$\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$$

In particular σ_{E} does not depend on ρ', nor on boundary conditions.

Definition of σ_{E} in case of a Mobility Gap

Replace H_E to H_a ($a > 0$) as follows



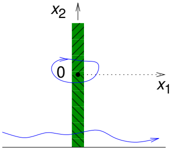
edge: $x_2 = -a$
(eventually: $-a \to -\infty$)

- Current across the portion  of $x_1 = 0$:

$$-\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) \quad \text{(exists!)}$$

- Current across the portion :

Definition of σ_{E} in case of a Mobility Gap



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$$\rho'(H_a) \rightsquigarrow \rho'(H_B) = \sum_{\lambda} \rho'(\lambda)\mathcal{P}_\lambda$$

(sum over eigenvalues λ of H_B: $H_B\mathcal{P}_\lambda = \lambda\mathcal{P}_\lambda$; mob. gap!)

$$\mathrm{tr}(\mathcal{P}_\lambda[H_B, \Lambda_1](1 - \Lambda_2)) =$$
$$\mathrm{tr}(\underbrace{\mathcal{P}_\lambda[H_B, \Lambda_1]\mathcal{P}_\lambda}_{=0 \text{ by virial theorem}}(1 - \Lambda_2)) + \mathrm{tr}(\mathcal{P}_\lambda[H_B, \Lambda_1][\mathcal{P}_\lambda, \Lambda_2])$$

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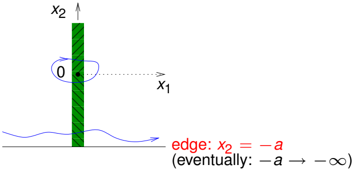

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- Together:

$$\sigma_{\mathrm{E}} = \lim_{a\to\infty} -\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) - \mathrm{i}\sum_\lambda \rho'(\lambda)\,\mathrm{tr}(\mathcal{P}_\lambda[H_B, \Lambda_1][\mathcal{P}_\lambda, \Lambda_2])$$

Bulk-edge correspondence in case of a Mobility Gap

Summary:

$$\sigma_{\mathrm{B}} = \mathrm{i}\ \mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

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$$\sigma_{\mathrm{B}} = \sigma_{\mathrm{E}}$$

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Remarks.

- All operators localized in x_1 near $x_1 = 0$
- Red operators localized in x_2 near $x_2 = 0$
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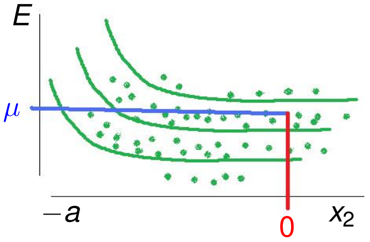
Proof of $\sigma_\mathrm{B} = \sigma_\mathrm{E}$.

- $\mathrm{tr}[A, B] = 0$ yields $\sigma_\mathrm{B} - \sigma_\mathrm{E} = 0$
- Viewed as non-commutative Stokes theorem on (x_1, x_2, E) applied to $x_1 = 0$, $x_2 \lesssim 0$, $E \lesssim \mu$



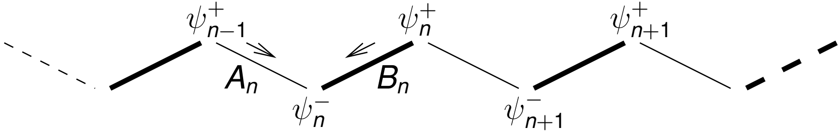
Goal of the talk

Quantum Hall systems

Chiral systems

The model (1 dimensional)

Alternating chain with nearest neighbor hopping
(Su-Schrieffer-Heeger model)



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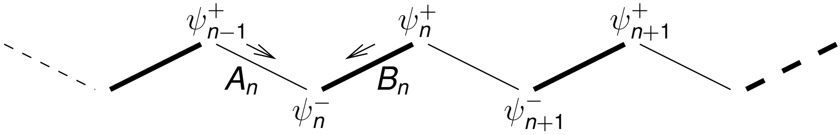

Hilbert space: sites arranged in dimers

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}_{n\in\mathbb{Z}}$$

Hamiltonian

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

with S, S^* acting on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$ as

$$(S\psi^+)_n = A_n\psi^+_{n-1} + B_n\psi^+_n, \qquad (S^*\psi^-)_n = A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n$$

($A_n, B_n \in \mathrm{GL}(N)$ almost surely)

Chiral symmetry

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0$$

hence

$$E_I(H)\Pi = \Pi E_{-I}(H) \qquad (E_I(H) \text{ spectral projection for } I \subset \mathbb{R})$$

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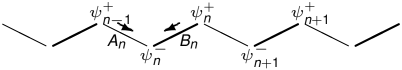


- Eigenvalue equation $H\psi = \lambda\psi$ is $S\psi^+ = \lambda\psi^-$, $S^*\psi^- = \lambda\psi^+$, i.e.

$$A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n$$

is one 2nd order difference equation, but two 1st order for $\lambda = 0$

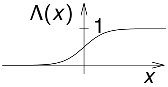
Bulk index

Let

$$\Sigma = \operatorname{sgn} H$$

Definition. The Bulk index is

$$\mathcal{N} = \frac{1}{2}\operatorname{tr}(\Pi\Sigma[\Lambda, \Sigma])$$



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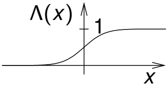

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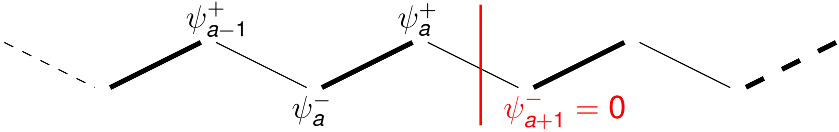
Equivalently

$$-\mathcal{N} = \operatorname{tr}(\Pi P_+[\Lambda, P_-]) + \operatorname{tr}(\Pi P_-[\Lambda, P_+])$$

using $P_+ := E_{(0,+\infty)}$, $P_- := E_{(-\infty,0)}$ and $\Sigma = P_+ - P_-$

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

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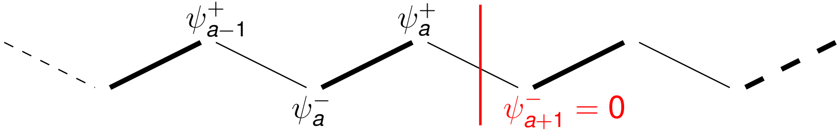

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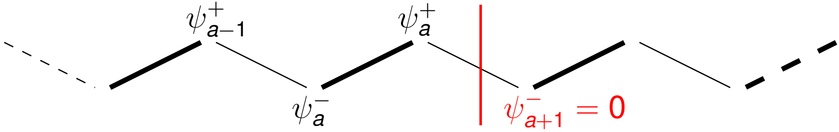

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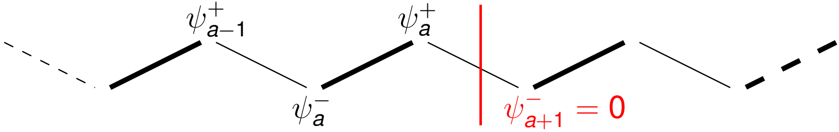
$$\mathcal{N}^\pm_a := \dim\{\psi \mid H_a\psi = 0, \Pi\psi = \pm\psi\}$$

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \le a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

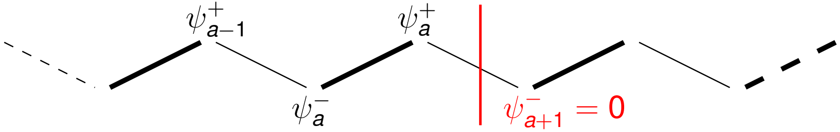
Eigenspace $\operatorname{ran} \mathcal{P}_{0,a}$ invariant under Π.

$$\mathcal{N}^\pm_a := \dim\{\psi \mid H_a\psi = 0, \Pi\psi = \pm\psi\}$$

Definition. The Edge index is

$$\mathcal{N}_a = \mathcal{N}^+_a - \mathcal{N}^-_a = \mathrm{tr}(\Pi \mathcal{P}_{0,a})$$

A vanishing lemma



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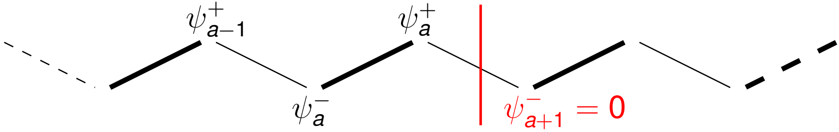

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Eigenvalue equation $H_a\psi = 0$, i.e., two 1st order eqs.

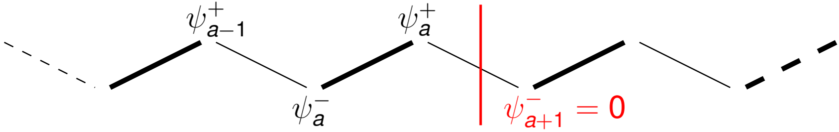
$$A_n\psi^+_{n-1} + B_n\psi^+_n = 0, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = 0$$

Lemma.

$$\mathcal{N}_a^{+} = \dim\{\psi^+ : \mathbb{Z} \to \mathbb{C}^N \mid S\psi^+ = 0, \psi^+_n \text{ is } \ell^2 \text{ at } n \to -\infty\}$$
$$\mathcal{N}_a^{-} = 0$$

In particular $\mathcal{N}_a$ is independent of a.

A vanishing lemma



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In particular $\mathcal{N}_a$ is independent of a. Call it $\mathcal{N}^\sharp$.

Bulk-edge duality

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

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Remark. Consider the dynamical system $A_n \psi^+_{n-1} + B_n \psi^+_n = 0$ with Lyaponov exponents

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Phase boundaries correspond to $\gamma_i = 0$ (cf. Prodan et al.)

Proof

Recall $\mathcal{N}_a = \mathrm{tr}(\Pi \mathcal{P}_{0,a})$

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Lemma. The common value of $\mathcal{N}_a$ is

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Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\operatorname{tr}(\Pi \Lambda) = N\Big(\sum_{n \leq a} \Lambda(n)\Big) \operatorname{tr}_{\mathbb{C}^2} \Pi = 0$$

though $\|\Pi \Lambda\|_1 = \|\Lambda\|_1 \to \infty$, $(a \to +\infty)$

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$$\begin{aligned}\operatorname{tr}(\Pi \Lambda P_{+,a}) &= \operatorname{tr}(P_{+,a} \Pi \Lambda P_{+,a}) = \operatorname{tr}(\Pi P_{-,a} \Lambda P_{+,a}) \\ &= \operatorname{tr}(\Pi P_{-,a} [\Lambda, P_{+,a}])\end{aligned}$$

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Lemma. The common value of $\mathcal{N}_a$ is

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Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\operatorname{tr}(\Pi \Lambda) = 0$$

So,

$$\operatorname{tr}(\Pi\Lambda) = \underbrace{\operatorname{tr}(\Pi\Lambda\mathcal{P}_{0,a})}_{\to \mathcal{N}^\sharp} + \underbrace{\operatorname{tr}(\Pi\Lambda P_{+,a}) + \operatorname{tr}(\Pi\Lambda P_{-,a})}_{\to \operatorname{tr}(\Pi P_-[\Lambda, P_+]) + \operatorname{tr}(\Pi P_+[\Lambda, P_-]) = -\mathcal{N}}$$

q.e.d.

Summary

Elementary methods used to establish bulk-edge correspondence in simple models of topological insulators in presence of a mobility gap