Fundamental Characteristics of Urban Transportation Services

Anton J. Kleywegt

School of Industrial and Systems Engineering
Georgia Institute of Technology

Smart Urban Transportation Forum
Institute for Mathematics and its Applications
University of Minnesota

15 May 2017
Fundamental Characteristics

This talk illustrates the following 3 simple, but fundamental, characteristics of urban transportation services:

1. **Economies of scale**: The greater the density of supply and demand for a transportation service, the more efficient can the transportation service be.
   
   The statement above assumes that the impact of the transportation service on congestion is small.

2. **The good and the bad**: Waiting times of ride-hailing customers can be in a good regime or a bad regime.

3. **Imbalance of flows**: Imbalance of flows is bad for transportation services.

Some research on the impact of urban transportation services on congestion and vice versa has been done, but more questions remain.
The following model is based on the ideas in R. Arnott, “Taxi Travel Should Be Subsidized”, Journal of Urban Economics, 40, 316–333, 1996 (but, as will become clear, the motivation is different)

1. Think of it as a steady-state, continuous approximation model
2. A mean-value analysis will be conducted
3. Consider a large (to ignore boundary effects) homogeneous (clearly a simplifying assumption) urban region
4. The region is served by vehicles distributed with a density of $\mu$ per unit area
5. Customers request rides at a rate of $\lambda$ per unit time per unit area (of origin and of destination)
6. The mean distance from customer origin to customer destination is $d$
7. The mean vehicle speed is $v$
8. A fraction $\gamma$ of the customers end up taking rides, where $\gamma$ may depend on price, waiting time (more about this later)
Model (continued)

1. Total vehicle time is partitioned into 3 parts:
   1. Fraction of time vehicles spend moving customers
   2. Fraction of time vehicles spend moving empty to pick up customers
   3. Fraction of time vehicles are idle waiting for an assignment to a customer
Model (continued)

1. Fraction of time vehicles spend moving customers
   - Mean amount of time per unit time per unit area that customers spend being moved in vehicles = $\frac{\gamma \lambda d}{\nu}$
   - $\gamma \lambda d$ = mean amount of time per unit time per unit area that vehicles spend moving customers

2. Fraction of time vehicles spend moving customers = $\frac{\gamma \lambda d}{\mu \nu}$

3. Note that this makes sense only if $\frac{\gamma \lambda d}{\mu \nu} < 1$

4. Thus a necessary (but not sufficient) condition for all these customers to be served is that $\gamma \lambda d < \mu \nu$
Model (continued)

1. Fraction of time vehicles spend moving empty to pick up customers and fraction of time vehicles are idle waiting for an assignment to a customer:

   1. Suppose fraction of time vehicle is idle waiting for an assignment to a customer = \( y \)
   2. Suppose mean distance vehicle moves empty to pick up a customer = \( z \)
   3. Mean number of empty.loaded trips per vehicle per unit time
      \[ \frac{\gamma \lambda}{\mu} \]
   4. Fraction of time moves empty to pick up a customer = \( \frac{\gamma \lambda z}{\mu \nu} \)
   5. Fraction of time vehicle is idle waiting for an assignment to a customer \( y = 1 - \frac{\gamma \lambda (d + z)}{\mu \nu} \)
Model (continued)

1. Fraction of time vehicle is idle waiting for an assignment to a customer: \( y = 1 - \frac{\gamma \lambda (d + z)}{\mu v} \)

1. The mean density of idle taxis = \( y \mu \)

2. Note that there is another relationship between \( y \) and \( z \) that we need to exploit to determine \( y \) and \( z \): The greater the fraction \( y \) of time vehicles are idle, the greater the mean density of idle taxis \( y \mu \), and the smaller the mean distance \( z \) vehicles move empty to pick up customers.

3. To facilitate further calculation, we consider the setting in which idle vehicles are distributed according to a 2-dimensional (spatial) Poisson process with rate \( y \mu \) per unit area.

1. This is heuristic, because the numbers of idle vehicles in disjoint (but close) regions are not independent.
Model

Model (continued)

\[ P[\text{Distance from customer origin to closest idle vehicle} > x] \]
\[ = P[0 \text{ idle vehicles in area } kx^2] \]
\[ = e^{-y\mu kx^2} \]

where \( k \) is a constant that depends on the chosen distance metric, for example, for \( L_1 \)-distance, \( k = 2 \), for \( L_2 \)-distance, \( k = \pi \approx 3.14 \), for \( L_\infty \)-distance, \( k = 4 \).
Model (continued)

Assuming that customers are assigned immediately upon request to the closest idle vehicle, the mean distance from a customer origin to the closest idle vehicle

\[ z = \int_{0}^{\infty} P[\text{Distance from customer to closest idle vehicle} > x] \, dx \]

\[ = \int_{0}^{\infty} e^{-y\mu k x^2} \, dx \]

\[ = \sqrt{\frac{\pi}{k\mu y}} \]
Model (continued)

1. Fraction of time vehicle is idle waiting for an assignment to a customer
   \[ y = 1 - \frac{\gamma \lambda (d + z)}{\mu \nu} \]

2. The mean distance from a customer origin to the closest idle vehicle

\[
z = \sqrt{\frac{\pi}{k \mu y}} = \sqrt{\frac{\pi}{k \mu \left[ 1 - \frac{\gamma \lambda (d + z)}{\mu \nu} \right]}} = \sqrt{\frac{\pi \nu}{k \left[ \mu \nu - \gamma \lambda (d + z) \right]}} =: F(z)
\]
Interpretation

\[ F(z) := \sqrt{\frac{\pi \nu}{k [\mu \nu - \gamma \lambda (d + z)]}} \]

can be interpreted as follows:

1. Suppose that due to dispatching/assignment of idle vehicles to requesting customers, the vehicles have to drive distance \( z \) to pick up the customer assigned to them (this \( z \) may not be the distance from each customer to the closest idle vehicle).

2. Then during the next hour, the vehicles will be idle a fraction

\[ y = 1 - \frac{\gamma \lambda (d + z)}{\mu \nu} \]

of the time.

3. As a result, during that hour, the mean distance from a customer origin to the closest idle vehicle will be \( F(z) \).

Questions:

1. For which values of \( z \) does it hold that \( z = F(z) \)?

2. If \( z^{k+1} = F(z^k) \), how does the resulting sequence \( \{z^k\} \) behave?
$F'(z) := \frac{\gamma \lambda}{2 \left[\mu \nu - \gamma \lambda (d + z)\right]^{3/2}} \left(\frac{\pi \nu}{k}\right)^{1/2}$

is positive and increasing in $z$ as long as $\mu \nu > \gamma \lambda (d + z)$.

Thus $F$ is increasing and convex on $[0, (\mu \nu - \gamma \lambda d)/(\gamma \lambda))$. 
Equation $z = F(z)$ may have no solutions (number of vehicles cannot handle demand, even with optimal assignment)

Figure 1: Mean distance from customer to closest idle vehicle
Mean Distance Vehicles Travel Empty

Equation $z = F(z)$ may have 2 distinct solutions

**Figure 2**: Mean distance from customer to closest idle vehicle
Interpretation: The Good and the Bad

\[ F(z) := \sqrt{\frac{\pi \nu}{k [\mu \nu - \gamma \lambda (d + z)]}} \]

1. For reasonable values of supply and demand, equation \( z = F(z) \) has 2 solutions \( z_1^* < z_2^* \).
2. The smaller solution \( z_1^* \) involves less empty travel (good for vehicles), and less waiting time (good for customers), than the larger solution \( z_2^* \).
3. Question: How much of a threat is the nightmare solution \( z_2^* \)?
4. If \( z^{k+1} = F(z^k) \), does the resulting sequence \( \{z^k\} \) converge to \( z_1^* \) or \( z_2^* \), or does it do something else (cycle, or chaos)?
Interpretation

\[ F(z) := \sqrt{\frac{\pi \nu}{k [\mu \nu - \gamma \lambda (d + z)]}} \]

1. Note that \( |F'(z_1^*)| = F'(z_1^*) < 1 \) and \( |F'(z_2^*)| = F'(z_2^*) > 1 \)

2. If the initial empty travel distance \( z^0 > z_2^* \), then after a few steps \( \mu \nu < \gamma \lambda (d + z^k) \) and there will be no more idle vehicles

3. If the initial empty travel distance \( z^0 < z_2^* \), then \( z^k \to z_1^* \), that is, empty travel distances converge to the optimal empty travel distance \( z_1^* \)

4. Thus, fortunately for transportation service providers, the optimal empty travel distance \( z_1^* \) is an attracting fixed point, and the nightmare solution \( z_2^* \) is an unstable fixed point

5. Arnott: “Thus, it appears that the upper equilibrium can be excluded on stability grounds.”
Shortcomings of the Model

1. **Urban regions are not homogeneous: supply (vehicles) and demand (customer origins and destinations) are not uniformly distributed**
   - Maybe a better approximation: A homogeneous region with a boundary

2. **Over time intervals of an hour or two, urban regions have large imbalances of origin-destination flows (morning commutes and evening commutes)**

3. **Origin-destination demands are random, and the numbers of idle vehicles in disjoint (but close) subregions are not independent (pockets of shortages of idle vehicles, or pockets of surpluses of idle vehicles)**

4. **Not every customer can be assigned to the idle vehicle closest to the customer**

Ongoing research to develop more realistic models and vehicle-customer assignment policies
Interpretation

An alternative view:

1. Recall that the fraction of time a vehicle is idle waiting for an assignment to a customer $= y$

2. Thus the fraction of time a vehicle is available to move, either empty or with a customer, is $1 - y$

3. The mean amount of time to move a customer from origin to destination $= \frac{d}{v}$

4. Given $y$, the mean distance from a customer origin to the closest idle vehicle $= \sqrt{\frac{\pi}{k\mu y}}$

5. Given $y$, the mean amount of time to move empty to pick up a customer $= \sqrt{\frac{\pi}{k\mu yv^2}}$

6. Given $y$, the maximum rate at which a vehicle can transport customers (capacity) is $Q(y) := \frac{1 - y}{\frac{d}{v} + \sqrt{\frac{\pi}{k\mu yv^2}}}$
Figure 3: Rate at which a vehicle can transport customers
An alternative view:

1. Note that, as before, the equation supply = demand,

\[ Q(y) := \frac{1 - y}{d + \frac{\pi}{\sqrt{\frac{k\mu yv^2}{\nu}}} = \frac{\lambda}{\mu} \]

can have 0, 1, or 2 solutions.

2. In a reasonable system, there are 2 solutions: One solution has lower idle time, lower density of idle vehicles, larger distances driving empty, and longer customer waiting times, and the other solution has higher idle time, higher density of idle vehicles, shorter distances driving empty, and shorter customer waiting times.

3. The maximum capacity corresponds to a positive fraction of idle time for vehicles.
Economies of Scale

\[ F(z) := \sqrt{\frac{\pi \nu}{k \left[ \mu \nu - \gamma \lambda (d + z) \right]}} \]

1. The region is served by vehicles distributed with a density of \( r \mu \) per unit area, and customers request rides at a rate of \( r \lambda \) per unit time per unit area (of origin and of destination), where \( r \) is a scaling factor.

2. 

\[ F_r(z) := \sqrt{\frac{\pi \nu}{k \left[ r \mu \nu - \gamma r \lambda (d + z) \right]}} \]

\[ = \frac{1}{\sqrt{kr}} \sqrt{\frac{\pi \nu}{\mu \nu - \gamma \lambda (d + z)}} \]

3. Thus, as scale \( r \) increases, the function \( F \) giving the mean distance from a customer origin to the closest idle vehicle shifts down by a factor of \( \frac{1}{\sqrt{kr}} \), as a result the good fixed point \( z_1^* \) gets better, and the bad fixed point \( z_2^* \) gets worse.