Derivation and Justification of the Nonlinear Schrödinger Equation

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1 Maxwell's equations and the formal derivation of the NLSE; (following [6] and [4])

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \,, \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \,, \\ \nabla \cdot \mathbf{D} &= 0 \,, \\ \nabla \cdot \mathbf{B} &= 0 \,. \end{aligned}$$

(Assuming no free charges or currents.)

$\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{D}$ the displacement field, and $\mathbf{H}$, the magnetizing field. These must be supplemented with the constitutive relations.

$$
\begin{aligned}
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \ , \\
\mathbf{B} &= \mu_0 \left(\mathbf{H} + \mathbf{M} \right) \ .
\end{aligned}
$$

For most materials, $\mathbf{M} << \mathbf{P}$, so we'll take $\mathbf{M} = 0$.

Take the curl of the first of Maxwell's equations:

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} \tag{1}$$

Simplifying assumptions:

1. Take $\mathbf{E}(x, y, z, t) = \mathcal{E}(z, t)\hat{\mathbf{e}}_x$. Then
$$\nabla \times \nabla \times \mathbf{E} = (-\partial_z^2 \mathcal{E}, 0, 0) \ . \tag{2}$$
2. The Displacement field and Polarizability: In general
$$\mathbf{D} = \epsilon \mathbf{E} + \chi^{(1)} \mathbf{E} + \chi^{(2)} \mathbf{E}^2 + \chi^{(3)} \mathbf{E}^3 \ . \tag{3}$$
   - Take $\chi^{(2)} = 0$. (Often results from a symmetry in the material.)
   - Assume that $\chi^{(1)}$ is a scalar. (Related to isotropy of the material.)
3. **However, the material does not react spontaneously!**
$$(\chi^{(1)} \mathcal{E})(z, t) = \int_{-\infty}^{t} \chi^{(1)}(t - s) \mathcal{E}(z, s) ds \ . \tag{4}$$
   Thus, we no longer have a PDE but a *delay*-differential equation.
4. We'll take (see [6])
$$\chi^{(3)} \mathbf{E}^3 = K \mathcal{E}^3(z, t) \ . \tag{5}$$

Summing up, we have a *scalar*, but *non-local* and *nonlinear* equation for the evolution of the electric field

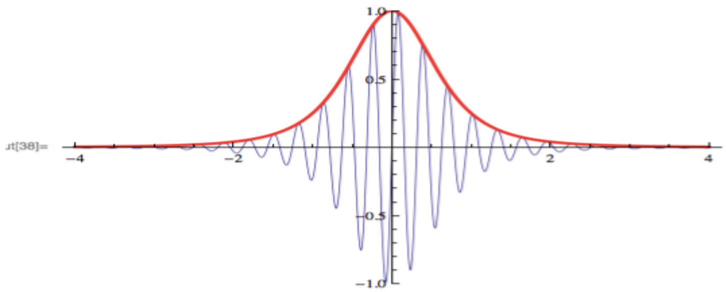
$$\partial_z^2 \mathcal{E} = \partial_t^2 \left(\mu_0 \epsilon + \mu_0 \chi^{(1)} \star \mathcal{E} + (\mu_0 K) \mathcal{E}^3 \right) \ . \tag{6}$$

What sort of solution should we look for?

- Start by shining a laser (monochromatic) down the optical fiber.
- Turn the laser on and off (to give "0"'s or "1"'s). The turning on/off occurs more slowly than the timescale of oscillations in the laser light.

To see the mathematical implications of this wave form, begin by studying the linear equation:

$$\partial_z^2 \mathcal{E} = \partial_t^2 \left(\mu_0 \epsilon + \mu_0 \chi^{(1)} \star \mathcal{E} \right) \ . \tag{7}$$



Take the Fourier transform with respect to t:

$$\partial_z^2 \hat{\mathcal{E}} = -\omega^2 \left(\mu_0 \epsilon_0 + \mu_0 \widehat{\chi^{(1)}} \right) \hat{\mathcal{E}} \equiv -(k(\omega))^2 \hat{\mathcal{E}} \ , \tag{8}$$

where

$$(k(\omega))^2 = \left(\mu_0 \epsilon_0 + \widehat{\chi^{(1)}}(\omega) \right) \omega^2 \ . \tag{9}$$

Remark 1 *The form of $\widehat{\chi^{(1)}}$ is experimentally determined. Fortunately, its exact form in not important for deriving the NLS equation.*

We can write solutions of (8) as

$$\hat{\mathcal{E}}(z,\omega) = \mathcal{E}^{+}(\omega) e^{ik(\omega)z} + \mathcal{E}^{-}(\omega) e^{-ik(\omega)z} \ , \tag{10}$$

where $\mathcal{E}^{\pm}$ are determined by the initial conditions. We expect the initial pulse to have the form:

$$\mathcal{E}(0,t) = A(\epsilon t) e^{i\omega_0 t} + c.c. \tag{11}$$

with an analogous form for the initial "velocity" of the pulse.

Remark 2 Note that the roles of z and t are "interchanged".

Take the Fourier transform of this expression to find the form of $\hat{\mathcal{E}}^{\pm}$:

$$\hat{\mathcal{E}}(0,\omega) = \frac{1}{\epsilon} \left(\hat{A}\left(\frac{\omega - \omega_0}{\epsilon} \right) + \hat{A}\left(\frac{\omega + \omega_0}{\epsilon} \right) \right) \tag{12}$$

Note that the Fourier transform of the initial conditions are localized in a neighborhood of size $\mathcal{O}(\epsilon)$ around $\omega \approx \omega_0$. This is actually the key to deriving the NLS. It is a consequence of the wave-packet form of our equation.

If we assume for simplicity that $\partial_z \hat{\mathcal{E}}(0,t) = 0$, we find that the solution of (8) takes the form

$$\hat{\mathcal{E}}(z,t) = \frac{1}{2\epsilon} \left(\hat{A}\left(\frac{\omega - \omega_0}{\epsilon} \right) + \hat{A}\left(\frac{\omega + \omega_0}{\epsilon} \right) \right) e^{-ik(\omega)z} + \frac{1}{2\epsilon} \left(\hat{A}\left(\frac{\omega - \omega_0}{\epsilon} \right) + \hat{A}\left(\frac{\omega + \omega_0}{\epsilon} \right) \right) e^{ik(\omega)z} \tag{13}$$

Inverting the Fourier transform (and focusing on one term for simplicity) yields:

$$\begin{aligned}\frac{1}{4\pi\epsilon}\int \hat{A}\left(\frac{\omega-\omega_0}{\epsilon}\right)e^{-ik(\omega)z}e^{i\omega t}d\omega &= \frac{1}{4\pi}\int \hat{A}(\nu)e^{-ik(\omega_0+\epsilon\nu)}e^{i(\omega_0+\epsilon\nu)t}d\nu \\ &\approx \frac{1}{4\pi}\int \hat{A}(\nu)e^{-ik(\omega_0)z-i\epsilon k'(\omega_0)\nu z}e^{i(\omega_0+\epsilon\nu)t}d\nu = \frac{e^{i(\omega_0 t-k(\omega_0)z)}}{4\pi}\int \hat{A}(\nu)e^{i\nu(t-k'(\omega_0)z)}d\nu \\ &= \frac{1}{2}A(\epsilon(t-k'(\omega_0)))e^{i(\omega_0 t-k(\omega_0)z)}\ .\end{aligned}$$

Key observation - the solution has two parts:

- The carrier wave $e^{i(\omega_0 t-k(\omega_0)z)}$ which moves with the *phase velocity* $k(\omega_0)/\omega_0$, and
- The envelope which translates with the *group velocity* $c_g = k'(\omega_0)$.

Remark 3 *Recall again that the rolls of space and time are interchanged here - normally one writes the dispersion relation as $\omega = \omega(k)$ so that the group velocity is given as $c_g = \omega'(k_0)$.*

Remark 4 *Although the term we have calculated corresponds to a right moving wave packet, there is also in general a left-moving part of the solution.*

So far we have only considered the lowest order terms in ϵ and ignored the nonlinear terms entirely. Including these effects will lead to more interesting solutions than just translating wave packets. We look for solutions of the form

$$\mathcal{E}(z,t) = \epsilon A(\epsilon(t-k'(\omega_0)z), \epsilon^2 z)e^{i(\omega_0 t-k(\omega_0)z)} + c.c. \tag{14}$$

- We look for solutions of small amplitude so that the nonlinear effects don't overwhelm things.
- In principle, the amplitude function could depend on $\epsilon^2 t$, but it turns out not to be necessary to include this "slow" time scale.
- Note that understanding the behavior of the solution on this scale allows us to "track" the solution much further down the fiber.

We now consider how each part of equation (6) affects solutions of this form:

$$\begin{aligned}\partial_z^2\mathcal{E} &= \epsilon\left(-(k(\omega_0))^2A(\dots)\right)e^{i(\omega_0 t-k(\omega_0)z)} \\ &+ \epsilon^2\left(2ik'(\omega_0)k(\omega_0)\partial_1 A(\dots)\right)e^{i(\omega_0 t-k(\omega_0)z)} \\ &+ \epsilon^3\left((k'(\omega_0)^2\partial_1^2 A(\dots) - 2ik(\omega_0)\partial_2 A(\dots)\right)e^{i(\omega_0 t-k(\omega_0)z)} + c.c.\end{aligned}$$

The effects of the nonlinear term are also simple to compute:

$$\begin{aligned}\mathcal{E}^3 &= 3\epsilon^3|A(\dots)|^2A(\dots)e^{i(\omega_0 t-k(\omega_0)z)} \\ &\quad +\epsilon^3A^3(\dots)e^{3i(\omega_0 t-k(\omega_0)z)} + c.c.\end{aligned}$$

Finally, we examine the dependence of the term $\partial_t^2\left(\mu_0\epsilon + \mu_0\chi^{(1)} \star \mathcal{E}\right)$ on ϵ.

As before, the key to understanding this terms lies in examining the Fourier transform.

If we take the Fourier transform (and again set $(k(\omega))^2 = \left(\mu_0\epsilon_0 + \widehat{\chi^{(1)}}(\omega)\right)\omega^2$), we arrive at the expression

$$-(k(\omega))^2 \hat{A}\left(\frac{\omega-\omega_0}{\epsilon}, \epsilon^2 z\right) e^{-i(\omega-\omega_0)k'(\omega_0)z} e^{-ik(\omega_0)z} + c.c. \tag{15}$$

To systematically the fact that the main contribution to this expression comes from values of ω close to ω_0, write $\nu = (\omega - \omega_0)/\epsilon$, so that

$$(k(\omega))^2 = (k(\omega_0 + \epsilon\nu))^2 = (k(\omega_0))^2 + 2\epsilon k(\omega_0)k'(\omega_0)\nu + \frac{1}{2}C_2(\omega_0)\epsilon^2\nu^2 + \mathcal{O}(\epsilon^3) \ . \tag{16}$$

Remark 5 *Note that we have retained one more order in ϵ than we did in the discussion of the linear problem. This turns out to be sufficient to incorporate the leading order nonlinear effects.*

This gives us the approximation

$$\begin{aligned} &-(k(\omega))^2 \hat{A}\left(\frac{\omega-\omega_0}{\epsilon}\right) e^{-i(\omega-\omega_0)k'(\omega_0)z} e^{-ik(\omega_0)z} + c.c. \\ &\approx -\left((k(\omega_0))^2\hat{A}(\nu,\epsilon^2 z) + 2\epsilon k(\omega_0)k'(\omega_0)\nu\hat{A}(\nu,\epsilon^2 z) + \frac{1}{2}C_2(\omega_0)\epsilon^2\nu^2\hat{A}(\nu,\epsilon^2 z)\right) e^{-i(\omega-\omega_0)k'(\omega_0)z} e^{-ik(\omega_0)z} + c.c. \end{aligned} \tag{17}$$

We now take the inverse Fourier transform of this expression to write things in (z,t) variables again. However the key observation now is that when we take the inverse Fourier transform

$$\mathcal{F}^{-1} : \nu\hat{A}(\nu) \to i(\partial_1 A)(\dots) \tag{18}$$

with a similar expression involving the second derivative for $\nu^2\hat{A}$. Taking the inverse transform of the right hand side of (17) gives:

$$\begin{aligned} &-(\epsilon(k(\omega_0))^2)A(\epsilon(t-k'(\omega_0)),\epsilon^2 z)e^{i(\omega_0 t - k(\omega_0)z)} \\ &\quad +\epsilon^2\left(2ik'(\omega_0)k(\omega_0)(\partial_1 A)(\dots)\right)e^{i(\omega_0 t - k(\omega_0)z)} \\ &\qquad +\frac{\epsilon^3}{2}C_2(\omega_0)\partial_1^2 A(\dots)e^{i(\omega_0 t - k(\omega_0)z)} + c.c. \end{aligned}$$

Remark 6 *Note that one immediate benefit of this procedure is that the nonlocal expression involving convolution with the function $\chi^{(1)}$ has been replaced by a local, differential expression.*

We now insert the expressions we have derived for $\partial_z^2\mathcal{E}$, $\mathcal{E}^3$ and $\mathcal{F}^{-1}\left((k(\omega))^2\hat{\mathcal{E}}\right)$ back into (6) and compare terms with like powers of ϵ:

- The terms of $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ vanish identically - this is a consequence of the *Ansatz* we made for the form of the solution.
- For the terms of $\mathcal{O}(\epsilon^3)$ we have

$$\begin{aligned} \left((k'(\omega_0)^2\partial_1^2 A(\dots) - 2ik(\omega_0)\partial_2 A(\dots)\right)e^{i(\omega_0 t - k(\omega_0)z)} &= \frac{1}{2}C_2(\omega_0)\partial_1^2 A(\dots)e^{i(\omega_0 t - k(\omega_0)z)} \\ &+\mu_0 K|A(\dots)|^2 A(\dots)e^{i(\omega_0 t - k(\omega_0)z)} + \mu_0 K A^3(\dots)e^{3i(\omega_0 t - k(\omega_0)z)} \ , \end{aligned}$$

  with an analogous equation for the complex conjugate.

Note that the terms proportional to our original carrier wave - i.e. the terms proportional to $e^{i(\omega_0 t - k(\omega_0)z)}$ will cancel if

$$i\partial_2 A = c_3(\omega_0)\partial_1^2 A + c_4(\omega_0)|A|^2 A \ . \tag{19}$$

This is of course the nonlinear Schrödinger equation. Thus, we see that if the amplitude function of our wave packet is a solution of the NLS equation, all terms of order $\mathcal{O}(\epsilon^3)$ or lower *that are proportional to the carrier wave* will cancel. In other words, we have a solution of the nonlinear Maxwell's equations, up to terms of $\mathcal{O}(\epsilon^4)$.

Some remarks and caveats:

1. The values of the constants c_3 and c_4 are determined by the dispersion relation - however, this is essentially the only place where the exact form of the dispersion relation enters the argument. This is one of the reasons for the wide applicability of the NLS approximation.
2. Whether or not c_3 and c_4 have the same or different signs determines whether we get the focussing or defocussing NLS equation.
3. The higher order terms in ϵ that we have ignored will enter the problem as higher derivatives of A - i.e. they are singular perturbations. Their effects may not be benign!
4. We have just ignored the $\mathcal{O}(\epsilon^3)$ terms proportional to $e^{3i(\omega_0 t - k(\omega_0)z)}$ - why should they be less important than the terms we eliminated.
5. This whole procedure (to the extent that it can be mathematically justified) relies crucially on the localization of the wave packet function in Fourier space around the frequencies $\omega \approx \pm\omega_0$ - what if the experimental conditions are such that this is not satisfied?

2 Justifying the NLS approximation; (see [7], [8], [9].)

To see how to justify this type of approximation we'll look at a slightly simpler equation. However, the ideas below transfer to (6) without significant difficulty. We'll consider a nonlinear Klein-Gordon equation (NKGE):

$$\partial_t^2 u = \partial_x^2 u - u + u^3 \tag{20}$$

Note that if we take the Fourier transform (with respect to x, this time) we can write this equation as

$$\partial_t^2 \hat{u} = -\omega(k)^2 \hat{u} + \widehat{u^3} \ , \tag{21}$$

which emphasizes the relationship with (6) - though note the interchanged roles of space and time.

Proceeding just as above, we find that if we look for wave packet solutions of the form

$$u^{app}(x,t) \equiv \epsilon A(\epsilon(x - c_g t), \epsilon^2 t) e^{i(k_0 x - \omega(k_0)t)} + c.c. \tag{22}$$

then if A is a solution of the NLSE, all terms in the formal expansion of (20) that are of $\mathcal{O}(\epsilon^3)$ or less and are proportional to the carrier wave will cancel.

The basic idea of showing that this provides an accurate approximation of the NKGE is to write the true solution as this approximation, plus a correction and show that the correction remains small -i.e. we write

$$u(x,t) = u^{app}(x,t) + \epsilon^\beta R(x,t) \ , \tag{23}$$

for some $\beta > 1$, then derive an equation for R and show that the solutions of this equation remain $\mathcal{O}(1)$.

Remark 7 *There's a trap in this innocuous statement - namely for* how long *must we show that R remains $\mathcal{O}(1)$? It's easy to show that R remains bounded for short times, but in order for this result to be interesting we must show that R remains $\mathcal{O}(1)$ for* very long times. *How long?*

- *Note that the point of this approximation is to show that interesting solutions of the NLSE, like solitons, also appear in the nonlinear Maxwell's equations or the NLKG.*
- *If we write the amplitude function $A = A(X, T)$, then*

$$i\partial_T A = c_3 \partial_X^2 A + c_4 |A|^2 A \tag{24}$$

and phenomena like solitons will occur on time scales where $T \sim \mathcal{O}(1)$ - i.e.

$$t \sim \mathcal{O}(\frac{1}{\epsilon^2}) \ . \tag{25}$$

Thus we to control the remainder R for a very long time.

Naively inserting the expansion (23) and trying to control the equation for R fails - mostly, in this case, because of the $\mathcal{O}(\epsilon^3)$ terms that were proportional to $e^{3i(k_0x-\omega(k_0)t)}$ which we just ignored.

To handle this situation, [8] observed, that if one added to u^{app} additional terms of $\mathcal{O}(\epsilon^2)$ or higher, this wouldn't affect the fact that the leading order approximation was given by the NLSE, but might make it easier to control the remainder R. Thus, following [8], we define a new, refined approximation

$$\tilde{u}^{app}(x,t) = \epsilon A(\epsilon(x - c_g t), \epsilon^2 t) e^{i(k_0 x - \omega(k_0)t)} + \epsilon^3 B(\epsilon x, \epsilon t) e^{3i(k_0 x - \omega(k_0)t)} + c.c. \tag{26}$$

and ask if we can choose B to improve our approximation. Here, of course, the form of B is motivated by the terms we failed to eliminate in our previous approximation.

Inserting $\tilde{u}^{app}$ into NLKG one finds that all terms $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^4)$ vanish as before, and at $\mathcal{O}(\epsilon^3)$, the only terms which fail to vanish are

$$-9(\omega(k_0))^2 B = (-9k_0^2 + 1) B e^{3i(k_0 - \omega(k_0)t)} + A^3 e^{3i(k_0 x - \omega(k_0)t)}$$

with an analogous equation for the terms proportional to $e^{-3i(k_0x-\omega(k_0)t)}$.

Thus, if we choose

$$B(\epsilon(x - c_g t), \epsilon^2 t) = \frac{(A(\epsilon(x - c_g t), \epsilon^2 t))^3}{9(k_0^2 - \omega(k_0)^2) - 1} \ ,$$

our improved approximation will satisfy NLKG up to terms $\mathcal{O}(\epsilon^4)$.

Remark 8 *In principle, we can continue to refine this approximation to higher and higher order.*

Remark 9 *It's critical that $9(k_0^2 - \omega(k_0)^2) - 1 \neq 0$. This is always satisfied for this example, but can lead to severe* resonance *problems in equations with quadratic nonlinear terms.*

We can now prove the validity of our NLS approximation. Define

$$\tilde{u}^{app}(x,t) = \epsilon A(\epsilon(x - c_g t), \epsilon^2 t) e^{i(k_0 x - \omega(k_0)t)} + \epsilon^3 \frac{(A(\epsilon(x - c_g t), \epsilon^2 t))^3}{9(k_0^2 - \omega(k_0)^2) - 1} e^{3i(k_0 x - \omega(k_0)t)} + c.c. \tag{27}$$

and set

$$u(x,t) = \tilde{u}^{app}(x,t) + \epsilon^2 R(x,t) \ . \tag{28}$$

Inserting this into the NKGE we find that R satisfies

$$\partial_t^2 R = \partial_x^2 R - R + 3(\tilde{u}^{app})^2 R + 3\epsilon^2(\tilde{u}^{app})R + \epsilon^4 R^3 + \epsilon^{-2} Res \ , \tag{29}$$

where the *residuum*, Res just measures the amount by which $\tilde{u}^{app}$ fails to satisfy the equation, i.e.

$$Res = \partial_x^2 \tilde{a}^{app} - \tilde{a}^{app} + (\tilde{a}^{app})^3 - \partial_t^2 \tilde{a}^{app} \sim \mathcal{O}(\epsilon^4) \ . \tag{30}$$

Recall that our goal is to show that the R remains $\mathcal{O}(1)$ for times $t \sim \epsilon^{-2}$. We will measure the size of R in the H^1 norm since the linear part of the NKGE preserves this norm. (More precisely, we consider the pair, (R, R_t) and measure its size in the $H^1 \times L^2$ norm.) Define

$$E(R) = \frac{1}{2}\int (R_t^2 + R_x^2 + R^2) dx \ . \tag{31}$$

Taking the time derivative we have

$$\partial_t E = \int \left(R_t R_{tt} + R_x R_{xt} + R R_t\right) dx \tag{32}$$

Inserting the equation of motion for R, (29), (and integrating by parts) we find that all the linear terms from the NKGE cancel and we are left with

$$\partial_t E = \int \left(R_t(3(\tilde{u}^{app})^2 R + 3\epsilon^2(\tilde{u}^{app})R + \epsilon^4 R^3 + \epsilon^{-2} Res)\right) dx \ . \tag{33}$$

Although this expression looks slightly messy, the key fact is that every term remaining is at least $\mathcal{O}(\epsilon^2)$ so with a few applications of the Cauchy-Schwartz inequality, we find

$$\partial_t E \leq C_1 \epsilon^2 E + C_2 \epsilon^2 \ . \tag{34}$$

From Gronwall's inequality, we immediately see that

Lemma 1

$$E(R(t)) \leq C_3 e^{C_1 \epsilon^2 t}(E(R(0)) + C_2 \epsilon^2 t) \ . \tag{35}$$

Corollary 1 *If $R(t=0)$ is $\mathcal{O}(1)$ then $R(t)$ remains $\mathcal{O}(1)$ for all $0 \leq t \leq \frac{T}{\epsilon^2}$.*

Our approximation result then follows:

Theorem 2 *If we choose initial conditions for the NKGE of wave packet form, the solution of the NKGE can be approximated by the solution of the NSE for times up to $\mathcal{O}(\epsilon^{-2})$. In particular, all the phenomena of the NSE (e.g. solitons) can also be observed in NKGE, up to higher order corrections.*

3 Extensions, problems, etc.

1. Problems with quadratic nonlinearities are *much* harder. This includes (unfortunately) the water wave problem which is another area in which the NSE approximation has often been used. For some work on this problem see [11], [5], [2], [3]. In particular, these considerations lead to interesting connections with dynamical systems theory.

2. One can refine these approximations to deal with other phenomena of importance in the applications of these ideas to optics. For instance, multiplexing refers to the practice of sending wave-packets with various carrier frequencies down the same optical fiber in order to increase the amount of information that can be sent through each fiber. In order for the information to be uncorrupted at the other end of the fiber, one needs to be sure that the wave packets corresponding to different carrier waves don't "mix". For an investigation of this see [9], [10], [1].

3. Another way to increase the information throughput in a fibre is to make the pulses shorter. However, this can invalidate the essential assumption lying behind the derivation of the NLS approximation which is that the amplitude of the pulse varies slowly in comparison with the carrier wave. In such cases, a different approximation may be preferable [12].

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