A Convex Primal Formulation for Convex Hull Pricing

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Outline

1. Motivation
2. Convex Hull Pricing
3. A Primal Formulation for CHP
4. Results
5. Conclusions
6. Reference
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1. Motivation
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Markets with unit commitment

- Day-ahead and some real-time markets in the US are based on a mixed-integer unit commitment and economic dispatch (UCED) problem solved by the Independent System Operator (ISO).
- The ISO: welfare-maximizing decision maker who sends prices and target quantity instructions to each generator.
- Individual generators can be idealized as price-taking and profit-maximizing.

Ideally, energy prices:
- in the *short term*: decentralize the ISO’s decision by aligning the decision of profit-maximizing market participants with the ISO’s welfare-maximizing decision,
- in the *long term*: incentivize investments in new generation (based on anticipation of future energy prices) at the right place and time.
Non-convexity and uplift payments

If the market model is convex (e.g., the economic dispatch problem):
- prices based on marginal costs align individual profit-maximizing decisions with the ISO’s decision;
- prices support a welfare-maximizing decision.

Because the UCED problem is non-convex,
- sales of energy at LMPs may not cover start-up and no-load costs (e.g., block-loaded units);
- in general, there does not exist a set of uniform prices that support a welfare-maximizing decision.

Uplift payments are:
- side-payments that give incentives for the units to follow the ISO’s decision,
- non-anonymous.
Convex Hull Pricing

- Intuition: Raising the energy price slightly above the marginal cost may reduce uplift payments.

- Convex hull pricing (CHP) [Ring, 1995, Gribik et al., 2007] produces uniform prices that minimize certain uplift payments.

- The Lagrangian dual problem of the UCED problem has been used to determine CHP:
  - Convex hull prices are the dual maximizers.
  - A convex but non-smooth problem.
  - Existing methods do not guarantee polynomial-time solution.
Efficient computation of CHP is challenging

- Obtaining the exact dual maximizers of the Lagrangian dual problem is computationally expensive [Hogan, 2014, Wang et al., 2016].

- Midcontinent ISO (MISO) implemented an approximation that is a single-period integer relaxation of the UCED problem [Wang et al., 2016].

- We propose a polynomially-solvable **primal formulation** of the Lagrangian dual problem, in which we explicitly characterize:
  - **convex hulls** of individual units’ feasible sets (based, in part, on previous literature), and
  - **convex envelopes** of their cost functions.
Outline

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The UCED problem

Consider a $T$-period problem with $|G|$ units

$$\min_{p_g, x_g, u_g, g \in G} \sum_{g \in G} C_g(p_g, x_g, u_g)$$

(1)

$$\sum_{g \in G} p_g = d$$

(2)

$$\forall g \in G.$$

(3)

A basic model, extensions considered later, where:

- $p_g \in \mathbb{R}_+^T$ dispatch levels, $x_g \in \{0, 1\}^T$ on/off, $u_g \in \{0, 1\}^T$ startup.
- The offer cost function $C_g$ is (piecewise) linear or quadratic in $p_g$.
- **System-wide** constraints: only demand constraints.
- **Private** constraints that define $X_g$: generation limits, minimum up/down time, but not (yet) ramping constraints.
Convex Hull Pricing

Profit maximization of market participants

Individual unit’s problem, given specified price vector $\pi$:

$$\max_{p_g, x_g, u_g} \quad \pi^T p_g - C_g(p_g, x_g, u_g)$$

s.t. $$(p_g, x_g, u_g) \in \mathcal{X}_g.$$ (4)

- Assume that unit $g$ is a price-taker in the economics sense that it cannot affect prices.
- Denote its value function by $w_g(\pi)$, which gives the maximum profit under price $\pi$. 
Uplift payment for lost opportunity costs

Profit made by following the ISO’s decision

- Let the ISO’s solution to the UCED problem be $(p^*, x^*, u^*)$.
- Profit made by following the ISO: $\pi^T p_g - C_g(p^*_g, x^*_g, u^*_g)$.

Uplift Payment

- Because of the non-convexity of the UCED problem, the maximum profit $w_g(\pi)$ may be strictly larger than profit made by following the ISO.
- There is a lost opportunity cost (LOC) of following the ISO.
- We define the amount of uplift payment for LOC to be

$$U_g(\pi, p^*, x^*, u^*) = w_g(\pi) - (\pi^T p_g - C_g(p^*_g, x^*_g, u^*_g)),$$

which is non-negative by definition.
CHPs as dual maximizers

- The gap between the UCED problem, $\sum_{g \in G} C_g(p_g^*, x_g^*, u_g^*)$, and its dual:

  The Lagrangian dual function

  $$q(\pi) = \sum_{g \in G} \left( \min_{(p_g, x_g, u_g) \in X_g} C_g(p_g, x_g, u_g) - \pi^\top p_g \right) - \pi^\top d, \quad (7)$$

  is exactly the total lost opportunity cost $\sum_{g \in G} U_g(\pi, p_g^*, x_g^*, u_g^*)$.

- The convex hull prices are the dual maximizers that solves

  The Lagrangian dual problem

  $$\max_{\pi} q(\pi). \quad (8)$$

- Convex hull prices minimize this duality gap. This is a non-smooth problem that is difficult to solve.
A single-period example from [MISO, 2010]

### Table: Units in Example 1

<table>
<thead>
<tr>
<th>Unit</th>
<th>No-load $</th>
<th>Energy $/MWh</th>
<th>$p_g$ MW</th>
<th>$\bar{p}_g$ MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>60</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>600</td>
<td>56</td>
<td>0</td>
<td>100</td>
</tr>
</tbody>
</table>

### Table: Optimal Commitment and Dispatch for Example 1

<table>
<thead>
<tr>
<th>$t$</th>
<th>$d_t$ MW</th>
<th>$x_{1,t}$ MW</th>
<th>$p_{1,t}$ MW</th>
<th>$x_{2,t}$ MW</th>
<th>$p_{2,t}$ MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>170</td>
<td>1</td>
<td>70</td>
<td>1</td>
<td>100</td>
</tr>
</tbody>
</table>

- Both units must be committed since demand is above 100 MW, and optimal dispatch is to operate unit 2 at full output since it has the lower marginal cost.
Example from [MISO, 2010], cont’d

Table: Comparison of Different Pricing Schemes for Example 1

<table>
<thead>
<tr>
<th>Pricing Scheme</th>
<th>$\pi$/MWh</th>
<th>$U_1$</th>
<th>$U_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMP</td>
<td>60</td>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td>CHP</td>
<td>62</td>
<td>60</td>
<td>0</td>
</tr>
</tbody>
</table>

- Under LMP, price is equal to unit 1’s energy offer, and unit 2 would have a negative profit, necessitating uplift to unit 2.
- Under CHP, price is equal to unit 2's average cost, and unit 1 has an incentive to generate more than the ISO’s dispatch, necessitating uplift to unit 1.
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The Lagrangian dual problem in primal space

Structures

- Compact but nonconvex $\mathcal{X}_g$.  
- Quadratic or piecewise linear cost functions $C_g(p_g, x_g, u_g)$.  
- Linear system-wide constraints.  
- Separable across $g$ absent of the system-wide constraints.

Notations

- $\text{conv}(\cdot)$ denotes the **convex hull** of a set.  
- $C^{**}_{g,\mathcal{X}_g}(\cdot)$ is the **convex envelope** of $C_g(\cdot)$ taken over $\mathcal{X}_g$:  
  - the largest convex function on $\text{conv}(\mathcal{X}_g)$ that is an under-estimator of $C_g$ on $\mathcal{X}_g$,  
  - the epigraph of $C^{**}_{g,\mathcal{X}_g}(\cdot)$ is the convex hull of the epigraph of $C_g(\cdot)$ taken over $\mathcal{X}_g$. 

The Lagrangian dual problem

**Theorem 1.**

The optimal objective function value of the Lagrangian dual problem (8) is equal to that of the following problem:

\[
\min_{p_g, x_g, u_g, g \in G} \sum_{g \in G} C_{g,x_g}^*(p_g, x_g, u_g) = d
\]

s.t.

\[
\sum_{g \in G} p_g = d
\]

\[
(p_g, x_g, u_g) \in \text{conv}(x_g), \quad \forall g \in G.
\]

An optimal dual vector associated with (10) is an optimal solution to problem (8).

This is an application of a theorem in [Falk, 1969].
Getting CHPs using the primal formulation

- Convex hull prices are the optimal dual variables associated with the system-wide constraints. To use this primal formulation, we need:

- Explicit characterization of $\text{conv}(\mathcal{X}_g)$:
  - Exponential number of valid inequalities needed for a general mixed-integer linear feasible set [Cornuéjols, 2007].
  - Polyhedral studies of $\mathcal{X}_g$ exploit special structure to find compact characterization [Rajan and Takriti, 2005, Damc-Kurt et al., 2015, Pan and Guan, 2015].
  - First consider convex hull of binary commitment decisions and then include dispatch decisions.

- Explicit characterization of $C_{g,\mathcal{X}_g}(p_g, x_g, u_g)$:
  - In general difficult because of the non-convexity of $\mathcal{X}_g$.
  - Usually the cost functions are piecewise linear or quadratic.
  - If $C_g(p_g, x_g, u_g)$ is affine then its convex envelope is itself.
  - We exhibit convex envelopes for piecewise linear and for quadratic cost functions.
Convex hulls for binary decisions

First consider the set of feasible commitment and start-up decisions.

State-transition constraints

\[ u_{gt} \geq x_{gt} - x_{g,t-1}, \quad \forall t \in [1, T]. \quad (12) \]

Min up/down time constraints

\[ \sum_{i=t-L_g+1}^{t} u_{gi} \leq x_{gt}, \quad \forall t \in [L_g, T], \quad (13) \]
\[ \sum_{i=t-l_g+1}^{t} u_{gi} \leq 1 - x_{gt}, \quad \forall t \in [l_g, T], \quad (14) \]

where \( L_g \) and \( l_g \) are respectively the minimum up and minimum down times of unit \( g \).
Convex hulls for binary decisions

The set of feasible commitment and start-up decisions is

$$\mathcal{D}_g = \{ x_g \in \{0, 1\}^T, u_g \in \{0, 1\}^T \mid (12)–(14) \}.$$  \hfill (15)

Trivial inequality

$$u_{gt} \geq 0, \quad \forall t \in [2, T].$$  \hfill (16)

[Rajan and Takriti, 2005] show that the convex hull of the feasible binary decisions alone is

$$\text{conv}(\mathcal{D}_g) = \{ x_g \in \mathbb{R}^T, u_g \in \mathbb{R}^T \mid (12)–(14), (16) \}.$$  \hfill (17)

The number of these facet-defining inequalities is polynomial in $T$. 

Convex hull for commitment and dispatch decisions

The dispatch variables are constrained by:

\[ x_{gt} \underline{p}_g \leq p_{gt} \leq x_{gt} \overline{p}_g, \quad \forall t \in [1, T], \]  

(18)

**Theorem 2.**

If minimum up/down time and generation limits are considered for each unit, then the polyhedron

\[ C_g = \{ p_g \in \mathbb{R}^T, x_g \in \mathbb{R}^T, u_g \in \mathbb{R}^T \mid (12)-(14), (16), (18) \} \]

describes the convex hull of \( \mathcal{X}_g \).\(^a\)

\(^a\)For \( T = 2 \) and \( T = 3 \), a more general \( \mathcal{X}_g \) in which ramping constraints are included is considered in [Damc-Kurt et al., 2015] and [Pan and Guan, 2015]. The result here holds for an arbitrary \( T \).
Convex envelopes for piecewise linear cost functions

Suppose the interval \([p_g, \bar{p}_g]\) is partitioned into \(|\mathcal{K}|\) intervals:

\[\begin{align*}
[p_g, \bar{p}_g] &= \bigcup_{k \in \mathcal{K}} I_k. \quad (19)
\end{align*}\]

Suppose, when the unit is on, the single-period cost function for \(p_{gt} \in I_k\) is

\[\tilde{C}_{gt}(p_{gt}, 1, 0) = b_{gk}p_{gt} + c_{gk}. \quad (20)\]

Let the start-up cost of this unit be \(h_g\).

**Theorem 3.**

The convex envelope of the piecewise linear cost function taken over \(X_g\) is

\[
\tilde{C}^{**}_{g,X_g}(p_g, x_g, u_g) =
\sum_{t=1}^{T} \left\{ b_{gk}p_{gt} + c_{gk}x_{gt} + h_gu_{gt}, \text{ if } p_{gt} \in I_k \right\}.
\]
Convex envelopes for quadratic cost functions

Suppose the cost functions are given in the form of

$$C_g(p_g, x_g, u_g) = \sum_{t=1}^{T} C_{gt}(p_{gt}, x_{gt}, u_{gt}), \quad (21)$$

where

$$C_{gt}(p_{gt}, x_{gt}, u_{gt}) = a_gp_{gt}^2 + b_gp_{gt} + c_gx_{gt} + h_gu_{gt} \quad (22)$$

is a single-period cost function that includes start-up cost $h_g$ and no-load cost $c_g$.

**Observation**

The convex envelope of a nonlinear convex function taken over a nonconvex domain may not be itself.
A single-period example of convex envelope of quadratic cost function

The convex envelope of the bi-variate single-period cost function 
\[ 0.2p_{gt}^2 + p_{gt} + 4x_{gt} \] taken over \( \{ x_{gt} \in \{0, 1\}, p_{gt} \in \mathbb{R} \mid x_{gt} \leq p_{gt} \leq 5x_{gt} \} \).

The graph of the convex envelope of this bi-variate function at \( (p_{gt}, x_{gt}) \) with \( x_{gt} \in (0, 1) \) is determined by the line connecting \((0, 0, 0)\) and \( \left( \frac{p_{gt}}{x_{gt}}, 1, C_{gt}(\frac{p_{gt}}{x_{gt}}, 1, 0) \right) \).
Generalization to multi-period case for quadratic cost function

Theorem 4.

The convex envelope of the quadratic cost function (21) taken over $X_g$ is

$$C_{g, X_g}^*(p_g, x_g, u_g) = \sum_{t=1}^{T} \begin{cases} a_g \frac{p_g^2}{x_g} + b_g p_g + c_g x_g + h_g u_g, & x_g > 0, \\ 0, & x_g = 0. \end{cases}$$

- Min up/down time constraints complicate $X_g$. However, they do not change the convex envelope.
- Convex envelope is continuously differentiable on $\text{conv}(X_g)$, but not so on the whole Euclidean space.
- Non-linearity comes from the \textbf{quadratic-over-linear} terms.
Casting the problem as a second-order cone program

Moving the nonlinear terms from the objective to the constraints

- Define an additional decision variable \( s_{gt} \in \mathbb{R}_+ \) for each quadratic-over-linear term.
- New constraint:
  \[
  s_{gt} x_{gt} \geq a_g p_{gt}^2.
  \] (23)

Reformulation

- For \( x_{gt} \geq 0 \) and \( s_{gt} \geq 0 \), constraint (23) is equivalent to
  \[
  \| (2 \sqrt{a_g p_{gt}}, x_{gt} - s_{gt}) \|_2 \leq x_{gt} + s_{gt},
  \] (24)
  which is a second-order cone constraint [Lobo et al., 1998].
- Our primal formulation becomes an SOCP, since all other constraints are linear. Off-the-shelf interior-point solvers available.
Extensions

Transmission constraints
- Theorem 1 still holds.
- Any linear system-wide constraints can be included.

Ramping constraints
- Exponential number of valid inequalities needed to describe the convex hulls [Damc-Kurt et al., 2015].
- Convex hulls for $T = 2$ and $T = 3$ have been explicitly described [Pan and Guan, 2015].
- We use valid inequalities for $T = 2$ and $T = 3$ to strengthen our formulation, resulting in a tractable approximation of the convex hulls.
- Convex envelopes will not change because ramping constraints define a convex feasible set. The convex envelope of a convex function taken over a convex domain is itself.
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Example from [MISO, 2010] with three periods

Table: Units in Example 1

<table>
<thead>
<tr>
<th>Unit</th>
<th>No-load $</th>
<th>Energy $/MWh</th>
<th>$p_g$ MW</th>
<th>$\bar{p}_g$ MW</th>
<th>Ramp Rate MW/hr</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>60</td>
<td>0</td>
<td>100</td>
<td>120</td>
</tr>
<tr>
<td>2</td>
<td>600</td>
<td>56</td>
<td>0</td>
<td>100</td>
<td>60</td>
</tr>
</tbody>
</table>

Table: Optimal Commitment and Dispatch for Example 1

<table>
<thead>
<tr>
<th>$t$</th>
<th>$d_t$ MW</th>
<th>$x_{1,t}$</th>
<th>$p_{1,t}$ MW</th>
<th>$x_{2,t}$</th>
<th>$p_{2,t}$ MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>1</td>
<td>70</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>1</td>
<td>40</td>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>170</td>
<td>1</td>
<td>70</td>
<td>1</td>
<td>100</td>
</tr>
</tbody>
</table>
Example from [MISO, 2010], cont’d

We use the primal formulation to obtain CHPs.

- aCHP1: only ramping constraints themselves.
- aCHP2: valid inequalities that describe \( \text{conv}(X_g) \) for \( T = 2 \).
- CHP: valid inequalities that describe \( \text{conv}(X_g) \) for \( T = 3 \).
- Uplift under CHP is smaller than under LMP.

Table: Comparison of Different Pricing Schemes for Example 1

<table>
<thead>
<tr>
<th>Pricing Scheme</th>
<th>( \pi_1 ) /MWh</th>
<th>( \pi_2 ) /MWh</th>
<th>( \pi_3 ) /MWh</th>
<th>( U_1 ) $</th>
<th>( U_2 ) $</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMP</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>0</td>
<td>560</td>
</tr>
<tr>
<td>aCHP1</td>
<td>60</td>
<td>60</td>
<td>64</td>
<td>120</td>
<td>160</td>
</tr>
<tr>
<td>aCHP2</td>
<td>60</td>
<td>60</td>
<td>65.6</td>
<td>168</td>
<td>0</td>
</tr>
<tr>
<td>CHP</td>
<td>60</td>
<td>60</td>
<td>65.6</td>
<td>168</td>
<td>0</td>
</tr>
</tbody>
</table>
ISO New England Example

Consider a 96-period 76-unit 8-bus example that is based on structural attributes and data from ISO New England [Krishnamurthy et al., 2016]:

- Start-up costs, no-load costs, minimum up/down time constraints, and ramping constraints,
- 12 transmission lines,
- quadratic cost functions,
- 2400 dual variables in the Lagrangian dual problem.

We compare uplift payments resulting from:

- LMPs,
- CHP resulting from our primal formulation,
- CHP obtained from MISO’s single-period approximation. (Start-up and no-load costs considered only for fast-start units. 18/76 units fall into this category.)

\(^1\)Optimization problems modeled in CVX and solved by GUROBI on a quad-core laptop. Valid inequalities that characterize \(\text{conv}(\mathcal{X}_g)\) are used when solving UCED.
Results

- **Case 1 (without transmission constraints)**
  - UCED solved in 64.81s. Optimal obj. $41,702,112$
  - SOCP solved in 5.54 s. Optimal obj. $41,697,614$

- **Case 2 (with transmission constraints)**
  - UCED solved in 67.89s. Optimal obj. $41,876,398$
  - SOCP solved in 11.18s. Optimal obj. $41,847,556$

**Table:** Total Uplift Payment ($) Under Different Pricing Schemes

<table>
<thead>
<tr>
<th></th>
<th>LMP</th>
<th>Primal Formulation for CHP</th>
<th>MISO Single-Period Approximation of CHP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>149,105</td>
<td>22,291</td>
<td>89,758</td>
</tr>
<tr>
<td>Case 2</td>
<td>385,753</td>
<td>23,869</td>
<td>302,430</td>
</tr>
</tbody>
</table>

- Uplift under CHP is much smaller than under LMP and MISO approximation.
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Conclusions

- A polynomially-solvable primal formulation for convex hull pricing.
- Exact when ramping constraints are not considered.
- Tractable approximation for the case with ramping constraints.
- Convex hulls and convex envelopes useful for solving UCED.
- Also useful for approximating results of UCED, which can be incorporated into generation and transmission expansion planning formulations.
This work provides a fast implementation of convex hull pricing.

Significant issues remain:

- incentives for generators to reveal truthful start-up and no-load information are not well understood in both LMP with traditional uplift and convex hull pricing with uplift;
- under convex hull prices, profit maximizing generators have incentives to deviate from the ISO-determined dispatch instructions; and
- interaction of day-ahead and real-time markets is not well understood.


Minimum up/down polytopes of the unit commitment problem with start-up costs.
Technical report, IBM Research Division.
