Linear-Quadratic Control of Stochastic Equations in a Hilbert Space with Fractional Brownian Motions

Tyrone E. Duncan

University of Kansas

March 2016

Outline

1. Linear-quadratic control for stochastic equations in a Hilbert space
2. Some noise process generalizations of Brownian motion
3. Fractional Brownian motions (FBMs) and other noise processes as additive noise for controlled linear systems
4. Finite and infinite time horizon control problems
5. Explicit optimal controls with FBMs
6. A linear stochastic system with multiplicative (Volterra type) Gaussian noise and controls from the family of linear state feedback operators

This is joint work with B. Maslowski and B. Pasik-Duncan.

Fractional Brownian Motions

Let $H \in (0,1)$ be fixed. The process $(B(t), t \geq 0)$ is a real-valued standard fractional Brownian motion with the Hurst parameter index $H \in (0,1)$ if it is a Gaussian process with continuous sample paths that satisfies

$$\mathbb{E}\left[B(t)\right] = 0$$
$$\mathbb{E}\left[B(s)B(t)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for all $s, t \in \mathbb{R}_+$.

The formal derivative $\frac{dB}{dt}$ is called fractional Gaussian noise.

Some properties

1. Self-similarity

$$(B^H(\alpha t), t \geq 0) \overset{L}{\sim} (\alpha^H B^H(t), t \geq 0)$$

   for $\alpha > 0$

2. Long range dependence for $H \in (\frac{1}{2}, 1)$

$$r(n) = \mathbb{E}[B^H(1)(B^H(n+1) - B^H(n))]$$

$$\Sigma_{n=0}^{\infty} r(n) = \infty$$

3. A sample path property
   $(B^H(t), t \geq 0)$ is of unbounded variation so the sample paths are not differentiable a.s.

$$\Sigma_i |B^H(t_{i+1}^{(n)}) - B^H(t_i^{(n)})|^p \to \begin{cases} 0 & pH > 1 \\ c(p) & pH = 1 \\ +\infty & pH < 1 \end{cases}$$

   $c(p) = \mathbb{E}|B^H(1)|^p$
   $(t_i^{(n)}, i = 0, 1, \cdots, n; n \in \mathbb{N})$ is a sequence of nested partitions of $[0, 1]$ such that $t_0^{(n)} = 0$ and $t_n^{(n)} = 1$ for all $n \in \mathbb{N}$ and the sequence of partitions becomes dense in $[0, 1]$.
4. For $H \neq \frac{1}{2}$ a FBM is neither Markov nor semimartingale.

Some Applications of FBMs

1. Turbulence
2. Hydrology
3. Economic Data
4. Telecommunications
5. Earthquakes
6. Epilepsy
7. Cognition
8. Biology

$$\begin{aligned} dX(t) &= (AX(t) + Bu(t))dt + dB_H(t) \\ X(0) &= x \end{aligned}$$

where $x \in V$, $X(t) \in V$, V is an infinite dimensional real separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. The process $(B_H(t), t \geq 0)$ is a V-valued fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ and having the incremental covariance $\widetilde{Q}$ where $\widetilde{Q}$ is trace class ($Tr(\widetilde{Q}) < \infty$) so that

$$\mathbb{E} < B_H(t), x >< B_H(s), y >= \frac{1}{2} < \widetilde{Q}x, y > (t^{2H} + s^{2H} - |t-s|^{2H})$$

for $x, y \in V$. The operator $A : Dom(A) \to V$ with $Dom(A) \subset V$ is a linear, densely defined operator on V which is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$.

Admissible Controls

The fractional Brownian motion B_H is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \geq 0)$ is the filtration for B_H.

$$\mathcal{U} = \{u : \mathbb{R}_+ \times \Omega \to U, u \text{ is progressively measurable,} \\ \mathbb{E} \int_0^T |u(t)|_U^2 dt < \infty \text{ for all } T > 0\}$$

Ergodic Quadratic Cost Functional

$$J_T(x,u) := \frac{1}{2}\int_0^T (|LX(s)|^2 + < Ru(s), u(s) >_U) ds$$

where $L \in \mathcal{L}(V)$, $R \in \mathcal{L}(U)$, R is self-adjoint and invertible. The control problem is to minimize the following ergodic cost

$$lim\ sup_{T\to\infty} \frac{1}{T}\mathbb{E}J_T(x,u).$$

Assumptions

(A1) There are $K \in \mathcal{L}(V), M_K > 0,$ and $\omega_K > 0$ such that

$$|e^{(A+KL)t}|_{\mathcal{L}(V)} \leq M_K e^{-\omega_K t}$$

for all $t > 0$ (detectability).
(A2) There are $F \in \mathcal{L}(V, U), M_F > 0,$ and $\omega_F > 0$ such that

$$|e^{(A+BF)t}|_{\mathcal{L}(V)} \leq M_F e^{-\omega_F t}$$

for all $t > 0$ (stabilizability).

Stationary Riccati Equation

The stationary Riccati equation has a weak solution as follows

$$< Px, Ay > + < Ax, Py > + < L^* Lx, y > - < R^{-1} B^* Px, B^* Py >= 0$$

for all $x, y \in Dom(A)$. Moreover the strongly continuous semigroup $(\Phi(t), t \geq 0)$ generated by $A_P = A - BR^{-1}B^*P$ is exponentially stable, that is

$$|\Phi(t)|_{\mathcal{L}(V)} \leq M_P e^{-\widetilde{\omega} t}$$

for some constants $M_P > 0$ and $\widetilde{\omega} > 0$.

Finite Time Horizon Optimal Control

Theorem. An optimal admissible control $\hat{u}$ is

$$\hat{u}(t) = -R^{-1}B^*P(t)X(t) + \psi(t)$$
$$\psi(t) = \mathbb{E}[\phi(t)|\mathcal{F}(t)]$$
$$= \int_0^t s^{-(H-\frac{1}{2})}(I_{t-}(I_{T-}^{(H-\frac{1}{2})}u_{H-\frac{1}{2}}U_P(\cdot,t)P(\cdot)C))(s)dB_H(s)$$

$$\begin{aligned} (I_{b-}^{\alpha}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}dt \\ u_a(s) &= s^a \end{aligned}$$

Let $\Psi(t) = \Phi^*(t)$ be the adjoint semigroup of $(\Phi(t)), t \geq 0)$ that is generated by A_P^*. It is known that the stochastic integral

$$\varphi_T(t) = \int_t^T \Psi(s-t)PdB_H(s)$$

for $t \in [0, T]$ is a well defined centered Gaussian process in $L^p(\Omega \times (0, T), V)$ for each $p \in [1, \infty)$. Define V_T and W as

$$V_T(t) = \mathbb{E}[\varphi_T(t)|\mathcal{F}(t)]$$
$$W(t) = \mathbb{E}[\varphi(t)|\mathcal{F}(t)]$$

where

$$\varphi(t) = \int_t^\infty \Psi(s-t)PdB_H(s).$$

A Hilbert Space for a FBM

Let L^2_H be the Hilbert space whose inner product $< \cdot, \cdot >_H$ is given by

$$< f, g >_H = \rho(H) \int_0^T u^2_{\frac{1}{2}-H}(r)(I^{H-\frac{1}{2}}_{T-} u_{H-\frac{1}{2}} f)(r)(I^{H-\frac{1}{2}}_{T-} u_{H-\frac{1}{2}} g)(r) dr$$

where $\rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}$. The term $I^{H-\frac{1}{2}}_{T-}$ is a fractional integral for $H \in (\frac{1}{2}.1)$ and a fractional derivative for $H \in (0, \frac{1}{2})$. This Hilbert space is naturally associated with a fractional Brownian motion with Hurst parameter H by the covariance factorization.

$$\left(I^{\alpha}_{T-}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \varphi(s)\, ds$$

$$\left(D^{\alpha}_{T-}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{\psi(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\, ds \right)$$

Theorem. Let (A1)-(A2) be satisfied and let $u \in \mathcal{U}$ be a control satisfying

$$lim_{T\to\infty}\frac{1}{T}\mathbb{E} < PX^u(T), X^u(T) >= 0 \qquad (1)$$

where $(X^u(T), T \in [0,\infty))$ is the solution to the system equation with the control $u \in \mathcal{U}$. Then

$$lim\ sup_{T\to\infty}\frac{1}{T}\mathbb{E}J_T(x,u) \geq J_\infty$$

where

$$J_\infty := lim\ sup_{T\to\infty}\frac{-1}{2T}\mathbb{E}\int_0^T |R^{\frac{1}{2}}B^*W(s)|_U^2 ds + \int_0^\infty Tr(\widetilde{Q}P\Phi(t))\phi_H(r)dr$$

for each $x \in V$ where $\phi_H(r) = H(2H-1)|r|^{2H-2}, r \in \mathbb{R}$, $W(t) = \mathbb{E}[\varphi(t)|\mathcal{F}(t)]$. Moreover, the feedback control $\hat{u}(t) = -R^{-1}B^*(PX^{\hat{u}}(s) + W(s))$ is admissible and satisfies (1).

Stochastic Evolution Equations with a Multiplicative Gaussian Noise

$$
\begin{aligned}
dX(t) &= (A(t)X(t) + B(t)K(t)X(t))dt + \sigma(t)X(t)dB(t) \\
X(0) &= x_0
\end{aligned}
$$

where $X(t) \in V$ a real, separable Hilbert space, $(B(t), t \geq 0)$ is a real-valued Volterra-type Gaussian process, $(A(t), t \geq 0)$ is a family of closed, unbounded operators on V such that $Dom(A(t)) = Dom(A(0))$ and $Dom(A^*(t)) = Dom(A^*(0))$ for each $t \in \mathbb{R}_+$ and the family generates a strongly continuous evolution operator, $B \in C_s(\mathbb{R}_+, \mathcal{L}(U, V))$ and $K \in C_s(\mathbb{R}_+, \mathcal{L}(V, U))$, σ is a real-valued continuous function. The control is

$$u(t) = K(t)X(t)$$

where K is to be determined. This can be described as a Markov type control.

Noise Process

The noise B is generated from a Wiener process W as follows

(R2) $\qquad B(t) = \int_0^t K(t,r)dW(r) \qquad t \in \mathbb{R}_+$

There is a continuous version of the process B.

Assume that $K(\cdot, s)$ has bounded variation on (s, T) and

(R3) $\qquad \int_0^T |K|^2((s,T], s)ds < \infty$

This family of noise processes includes the family of FBMs for $H \in (\frac{1}{2}, 1)$.

Let R be the covariance function for the noise B given by

$$R(s,t) = \mathbb{E}[B(s)B(t)] = \int_0^{min(s,t)} K(t,r)K(s,r)dr$$

and let K satisfy the following two conditions

$$\begin{aligned} (K1) \qquad & K(t,s) = 0 \qquad s > t \qquad \text{(causality)} \\ & K(t,\cdot) \in L^2(0,t) \qquad t \in \mathbb{R}_+ \end{aligned}$$

$$(K2) \qquad \int_0^T (K(t,r) - K(s,r))^2 dr \leq C|t-s|^\beta$$

for each $T > 0$ and some constants $C > 0, \beta > 0$ for $s, t \in [0, T]$

The stochastic integral is constructed from V-valued step functions

$$I_T(\phi) = \int_0^T \phi dB = \Sigma \phi_i (B(t_{i+1}) - B(t_i))$$

so

$$\mathbb{E}|I_T^2(\phi)|^2 = \int_0^T |\mathcal{K}_T^K \phi|^2 dt$$

where

$$(\mathcal{K}_T^K \phi)(s) = K(s+, s)\phi(s) + \int_s^T \phi(r) K(dr, s)$$

$$< \phi, \psi >_{\mathcal{H}} = < \mathcal{K}_T^K \phi, \mathcal{K}_T^K \psi >_{L^2([0,T),V)}$$

It is assumed that $\mathcal{K}_T^K$ is injective which is satisfied for the noise examples given. I_T is extended to $\mathcal{H}_r$ which is continuously embedded in $\mathcal{H}$ where

$$||\phi||^2_{\mathcal{H}_r} = \int_0^T \phi^2(s) K^2(s+, s) ds + \int_0^T (\int_0^T |\phi(t)||K|(dt, s))^2 ds$$

(K3) $K(t,s)$ is differentiable in the first variable for $\{0 < s < t < T\}$ and both K and $\frac{\partial K}{\partial t}$ are continuous.

$$|\frac{\partial K}{\partial t}(t,s)| \leq c(t-s)^{\alpha-1}s^{-\alpha}$$

$$\int_s^t K^2(t,u)du \leq c(t-s)^{1-2\alpha}$$

on the set $\{0 < s < t < T\}$ for some constants $c > 0$ and $\alpha \in (0, \frac{1}{2})$.

If K satisfies (K1)-(K3) then $L^{\frac{2}{1+2\alpha}}$ in $\mathcal{H}_r$

Two processes that satisfy the assumptions:

1. Fractional Brownian motion (FBM) with the Hurst parameter $H \in (\frac{1}{2}, 1)$.
(R9) $\quad K(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du \quad$ for $s < t$
The kernel satisfies conditions (K1)-(K3) with $\alpha = H - \frac{1}{2}$.

2. Liouville fractional Brownian motion (LFBM) with $H \in (\frac{1}{2}, 1)$.
$K(t,s) = \widetilde{c_H}(t-s)^{H-\frac{1}{2}} 1_{(0,T]}(s) \qquad s, t \in \mathbb{R}_+$
satisfies (K1)-(K3) with $\alpha = H - \frac{1}{2}$.

Strong and Weak Solutions

The V-valued process $(X(t), t \geq 0)$ is a strong solution to the equation if

$$X(t) = x + \int_0^t (A(s)X(s) + B(s)K(s)X(s))ds + \int_0^t \sigma(s)X(s)dB(s)$$

and a weak solution to the equation exists if for each $z \in \mathcal{D}, z \in Dom(A^*(0))$ the following equality is satisfied

$$< X(t), z > = < x, z > + \int_0^t < X(s), A^*(s)z > ds$$

$$+ < B(s)K(s)X(s), z > ds + \int_0^t \sigma(s) < X(s), z > dB(s)$$

$$\widetilde{A}(t) = A(t) + B(t)K(t) - \alpha(t)I \qquad t \geq 0$$
$$\widetilde{A}_\lambda(t) = A(t) + B_\lambda(t)K(t) - \alpha(t)I \qquad t \geq 0$$

$\widetilde{A}$ and $\widetilde{A}_\lambda$ generate mild and strong evolution operators respectively on V denoted $(U(t,s))$ and $(U_\lambda(t,s))$, $B_\lambda = \lambda(\lambda I - A)^{-1}B$ and

$$U(t,s) = exp[-\int_s^t \alpha(r)dr]U_K(t,s)$$
$$U_\lambda(t,s) = exp[-\int_s^t \alpha(r)dr]U_K^\lambda(t,0)$$
$$X_\lambda(t) = exp[Z(t)]U_\lambda(t,s) \qquad t \geq 0$$
$$X(t) = exp[Z(t)]U(t,0) \qquad t \geq 0$$

where

$$Z(t) = \int_0^t \sigma(r)dB(r)$$

$$\alpha(t) = \sigma^2 \frac{\partial}{\partial t}\left(\int_0^t K(t,r)dr\right)^2 = \sigma^2 \frac{\partial}{\partial t} R(t,t) = \sigma^2 \frac{\partial}{\partial t}(\mathbb{E}B^2(t))$$

For fractional Brownian motion or Liouville fractional Brownian motion $\alpha(t) = c_H t^{2H-1}$ where the constant depends on whether it is FBM or LFBM.

For FBM if σ is not a constant
$\alpha(t) = \sigma(t) \int_0^t \sigma(s)\phi_H(t-s)ds$
where $\phi(t) = H(2H-1)t^{2H-2}$ so that for continuous σ the condition (K3) is satisfied.

Finite Time Horizon Control

The cost functional, J_T, is the following

$$J_T(K) = \mathbb{E}\int_0^T (|L(t)X(t)|^2 + < R(t)K(t)X(t), K(t)X(t) >_U)dt$$
$$+\mathbb{E} < GX(T), X(T) >$$

where $L \in C_s([0,T], \mathcal{L}(V)), G = G^*, G \in \mathcal{L}(V), G > 0, R \in C_s([0,T], \mathcal{L}(U)), R(t) = R^*(t), < R(t)u, u > \geq \lambda_0 |u|_U^2, u \in U, t \in [0,T]$.

The family of admissible controls is $K \in C_s([0,T], \mathcal{L}(V,U))$.

Riccati Equation

The Riccati differential equation associated with the control problem is

$$\frac{dP}{dt} + A^*P + PA - PBR^{-1}B^*P + L^*L - 2\alpha(t)P = 0 \qquad t \in [0, T]$$
$$P(T) = G$$

Lemma. With the assumptions given above, there is a unique weak solution $(P(t), t \in [0, T])$ to the Riccati equation that satisfies $P \in C_s([0, T], \mathcal{L}(V)), P(t) \geq 0, P(t) = P^*(t)\ t \in [0, T]$ such that

$$\frac{d<P(t)x,y>}{dt} + < A(t)x, P(t)y > + < P(t)x, A(t)y >$$
$$- < R^{-1}(t)B^*(t)P(t)x, B^*(t)P(t)y >$$
$$+ < L(t)x, L(t)y > -2\alpha(t) < P(t)x, y >= 0 \qquad P(T) = G$$

for $t \in [0, T], x, y \in D$.

Theorem. Let (A1), (K1)-(K3) be satisfied. The feedback control

$$u(t) = -R^{-1}(t)B^*(t)P(t)X(t)$$
$$K(t) = -R^{-1}(t)B^*(t)P(t)$$

is an optimal control for the control problem.
The optimal cost is

$$J_T(K) =< P(0)x_0, x_0 >$$

Proof. Apply an Ito formula to $(< P(t)X_\lambda(t), X_\lambda(t) >, t \in [0, T])$ and then let $\lambda \to \infty$.

Infinite Time Horizon Discounted Control

The cost functional is

$$J_\infty(K) = \mathbb{E} \int_0^\infty e^{-\beta(t)}(|L(t)X(t)|^2 + < R(t)K(t)X(t), K(t)X(t) >)dt$$

The function $\beta : \mathbb{R}_+ \to \mathbb{R}$ is assumed to be continuously differentiable and has the interpretation of the discount rate. Considering a scalar geometric equation with an FBM, it follows that $f(t) = exp[-\beta(t) + \sigma^2 t^{2H}]$ must be integrable so that the control problem is well posed. The conditions of stabilizability and detectability guarantee the existence and uniqueness in the family of self-adjoint, nonnegative, strongly continuous and uniformly bounded operators of the following Riccati equation

$$\frac{dP}{dt} + PA + A^*P + A - PBR^{-1}B^*P - \gamma(t)P = 0$$

where $\gamma(t) = \frac{d\beta}{dt} - 2\alpha(t)$.

Theorem. The feedback control $u(t) = -R^{-1}(t)B^*(t)P(t)X(t)$ minimizes the cost functional $J_\infty(K)$ in the family of strongly continuous and uniformly bounded operators $K(\cdot)$.

- T.E. Duncan, B. Maslowski and B. Pasik-Duncan, Stochastic equations in Hilbert spaces with multiplicative fractional Gaussian Noise, *Stochastic Process Appl.* 115 (2005), 1357-1383.
- T. E. Duncan, B. Maslowski, and B. Pasik-Duncan, Linear-quadratic control for stochastic equations in a Hilbert space with a fractional Brownian motion, *SIAM J. Control Optim.*, 50 (2012), 507-531.
- T. E. Duncan, B. Maslowski and B. Pasik-Duncan, Some ergodic control problems for linear stochastic equations in a Hilbert space with fractional Brownian motion, *Banach Center Publ.* 105 (2015), 91-102.

Thank You