Optimization of distributed feedback controllers for a nonlinear parabolic integro-differential equation

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Joint work with P. Nestler and E. Schöll

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Outline

1. Pyragas type feedback
2. Forward and design problem
3. Oscillating solutions
4. Optimal feedback for periodic targets
5. Numerical examples
Pyrugas feedback control

Control system (Schlögl, Nagumo, ...)

\[
\begin{align*}
\partial_t y - \Delta y + R(y) &= u \quad \text{in } Q := \Omega \times (0, T) \\
y(\cdot, 0) &= \Phi(\cdot), \quad \text{in } \Omega \\
\partial_n y &= 0 \quad \text{in } \Sigma := \partial \Omega \times (0, T)
\end{align*}
\]

with cubic reaction term \( R(y) = \rho (y - y_1)(y - y_2)(y - y_3), \rho > 0, \ y_1 \leq y_2 \leq y_3. \)
Pyragas feedback control

Control system (Schlögl, Nagumo, ...)

\[ \partial_t y - \Delta y + R(y) = u \quad \text{in } Q := \Omega \times (0, T) \]
\[ y(\cdot, 0) = \Phi(\cdot), \quad \text{in } \Omega \]
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with cubic reaction term \( R(y) = \rho (y - y_1)(y - y_2)(y - y_3), \rho > 0, y_1 \leq y_2 \leq y_3. \)

Pyragas feedback:

Let \( \tau > 0 \) and \( \Phi : \Omega \times [-\tau, 0] \rightarrow \mathbb{R} \) be given,

\[ \partial_t y(t) - \Delta y(t) + R(y(t)) = \kappa (y(t - \tau) - y(t)), \quad t > 0, \]
\[ y(\cdot, s) = \Phi(\cdot, s), \quad s \in [-\tau, 0] . \]

If \( y \) has period \( \tau \), then the feedback vanishes. This type of feedback is very popular in Theoretical Physics. → Lasers
Nonlocal Pyragas feedback

Let now $\Phi : \Omega \times [-T, 0] \rightarrow \mathbb{R}$,

$$
\partial_t y(t) - \Delta y(t) + R(y(t)) = \kappa \left( \int_0^T g(\tau) y(t - \tau) d\tau - y(t) \right), \quad t > 0,
$$

$$
y(\cdot, s) = \Phi(\cdot, s), \quad s \in [-\tau, 0],
$$

with given feedback gain $\kappa$ and a kernel $g \in L^\infty(0, T)$ that satisfies

$$
0 \leq g(t) \leq \beta \quad \text{a.e. in } [0, T]
$$

$$
\int_0^T g(\tau) d\tau = 1.
$$
Some references

K. Pyragas,
Continuous control of chaos by self-controlling feedback.

J. Löber, R. Coles, J. Siebert, H. Engel, E. Schöll,
Control of chemical wave propagation in Engineering of Chemical Complexity II.

E. Schöll, H.G. Schuster,
Handbook of Chaos Control.

Many members of the optimal control community considered problems with time-delay, e.g.

Banks, Betts, Bonnans, Burns, Colonius, ... Kappel, Kunisch, Lasiecka, Lenhart, Maurer, ... Tucsnak ...
Forward problem: \( g \mapsto y \)

Depending on the chosen feedback kernel \( g \), different solutions \( y \) are generated.

\[ \Omega = (0, 200), T = 400, \Phi(x, t) := \frac{1}{2} \left( 1 - \tanh \left( \frac{x - vt}{2\sqrt{2}} \right) \right) \]

“Weak gamma delay kernel” \( g(t) = e^{-t}, \)

\[ y_1 = 0, y_3 = 1, y_2 = 0, \]

\[ \kappa = -1.65, \]

\[ y_2 = 0.5, \kappa = -1.4. \]
Forward problem: $g \mapsto y$

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"Weak gamma delay kernel" \( g(t) = e^{-t} \), \( y_1 = 0 \), \( y_3 = 1 \)

\[ y_2 = 0.25, \ \kappa = -1.65 \quad \text{and} \quad y_2 = 0.5, \ \kappa = -1.4 \]
Well-posedness of the problem

**Theorem (Control-to-state mapping)**

For each \( g \in L^\infty(0, T) \), \( \Phi \in C(\Omega \times [−T, 0]) \), the feedback equation has a unique weak solution \( y_g \in C(\bar{Q}) \). The mapping \( G : g \mapsto y_g \) is of class \( C^\infty \).

Idea of the proof:

\[
(\partial_t y - \Delta y + R(y) + \kappa y)(t) = \kappa \int_0^T g(\tau)y(t - \tau) \, d\tau
\]

\[
= \kappa \int_0^t g(\tau)y(t - \tau) \, d\tau + \kappa \int_t^T g(\tau)\Phi(t - \tau) \, d\tau
\]

\[
= \kappa \int_0^t g(t - s)y(x, s) \, ds + \Phi_g(t).
\]

Now we proceed as in Casas, Ryll, T., CMAM 2014. \( T \to \infty \) ?
Optimal feedback design problem (P1)

*Design problem:* Find a kernel $g$ such that the solution $y_g$ associated with $g$ is as close as possible to a desired function $\hat{y}$. 

\[
\min_{g \in C} F(g) := \frac{1}{2} \| y_g - \hat{y} \|_2^2 + \nu \| g \|_2^2 \quad \text{where} \quad y_g \text{ solves} \\
\begin{cases}
\partial_t y(t) - \Delta y(t) + R(y(t)) = \kappa \int_0^T g(\tau) y(t-\tau) d\tau - \kappa y(t) \\
& \text{in } Q, \\
& y(s) = \Phi(s), \quad s \in [-T,0], \\
& \partial_n y(t) = 0 \quad \text{in } \Sigma,
\end{cases}
\]

and $C = \{ g \in L^\infty(0,T) : 0 \leq g(\tau) \leq \beta, \int_0^T g(\tau) d\tau = 1 \}$.

Theorem: There exists at least one optimal “control” $\bar{g}$ for this problem.
Optimal feedback design problem (P1)

**Design problem:** Find a kernel $g$ such that the solution $y_g$ associated with $g$ is as close as possible to a desired function $\hat{y}$.

\[
\min_{g \in C} F(g) := \frac{1}{2} \|y_g - \hat{y}\|_{L^2(Q)}^2 + \frac{\nu}{2} \|g\|_{L^2(0,T)}^2 \quad (P1)
\]

where $y_g$ solves

\[
\begin{cases}
\partial_t y(t) - \Delta y(t) + R(y(t)) &= \kappa \int_0^T g(\tau) y(t - \tau) d\tau - \kappa y(t) \quad \text{in } Q, \\
y(s) &= \Phi(s), \quad s \in [-T, 0], \quad \text{in } \Omega \\
\partial_n y(t) &= 0 \quad \text{in } \Sigma,
\end{cases}
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and $C = \{ g \in L^\infty(0, T) : 0 \leq g(\tau) \leq \beta, \int_0^T g(\tau) \, d\tau = 1 \}$. 

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\partial_n y(t) &=& 0 \quad \text{in } \Sigma,
\end{array} \right.$$\)

and $C = \{ g \in L^\infty(0,T) : 0 \leq g(\tau) \leq \beta, \quad \int_0^T g(\tau) d\tau = 1 \}$.

**Theorem**

There exists at least one optimal "control" $\bar{g}$ for this problem.
Oscillating states

For various feedback kernels $g$, states $y_g$ are obtained that are close to
time-periodic patterns.

$y_1 = y_2 = 0, \ y_3 = 1, \ \kappa = -2$

$\ y_1 = 0, \ y_2 = 0.25, \ y_3 = 1, \ \kappa = -2.43$

$g(t) = t e^{-t}$

g(\tau) = \begin{cases} 
  \frac{1}{t_2 - t_1}, & t_1 \leq \tau \leq t_2, \\
  0, & \text{else.}
\end{cases}

t_1 = 0, \ t_2 = 3$
A simple ODE with delay

\[ x'(t) = \kappa x(t - 1), \quad t > 0 \]
\[ x(s) = \Phi(s), \quad -1 \leq s \leq 0. \]

T. Erneux,

Applied delay differential equations,
Springer, 2009

\[ \kappa = -\pi/2 \]

\[ \kappa = -1.8, \Phi(0) = 1, \Phi(s) = 0, s < 0 \]

\[ \kappa = -1.1 \]
A linear ODE with delay

We follow

Bellman, R.
A survey of the mathematical theory of time-lag, retarded control, and hereditary processes,
With the assistance of John M. Danskin, Jr.
The Rand Corporation, 1954

Hale, J. K., Verduyn Lunel, S.M.
Introduction to functional differential equations,
Springer, 1993

Consider the ODE with delay

\[ x'(t) + ax(t) = \kappa x(t - \tau), \quad t > 0, \]
\[ x(s) = \Phi(s), \quad s \in [-\tau, 0], \]

with given \( a, \kappa \in \mathbb{R} \) and \( \Phi \in C^1[-\tau, 0] \).
The fundamental solution

For the variation of constants formula, we need the

**Fundamental solution** \( X \)

\[
X'(t) + aX(t) = \kappa X(t - \tau), \quad t > 0, \\
X(0) = 1 \\
X(s) = 0, \quad s < 0.
\]
The fundamental solution

For the variation of constants formula, we need the

**Fundamental solution $X$**

\[
X'(t) + a X(t) = \kappa X(t - \tau), \quad t > 0, \\
X(0) = 1 \\
X(s) = 0, \quad s < 0.
\]

**Theorem (Variation of constants formula)**

*If $\Phi \in C^1[-\tau, 0]$, then the linear delay equation above attains a unique differentiable solution $x$. It is given by*

\[
x(t) = X(t)\Phi(0) + \int_{-\tau}^{0} X(t - s - \tau)\Phi(s) \, ds
\]

cf. Bellman or Hale/Verduyn Lunel.*
The characteristic equation for $\lambda \in \mathbb{C}$

$$\lambda + a - \kappa e^{-\lambda \tau} = 0,$$

has only countably many zeros $\lambda_j, j \in \mathbb{N}_0$, that can be ordered with respect to decreasing real parts so that

$$\text{Re} \lambda_j \geq \text{Re} \lambda_{j+1} \quad \forall j \in \mathbb{N}_0.$$  

The values $\lambda_j$ appear in conjugate complex pairs.
For convenience, we take $\Omega = (0, 1)$ and consider the case of the linear local Pyragas feedback.

**Pyragas feedback equation**

\[
\begin{align*}
\partial_t y(t) - \Delta y(t) + (a + \kappa) y(t) &= \kappa y(t - \tau), \quad t > 0, \\
\partial_n y(t) &= 0, \quad t > 0, \\
y(\cdot, s) &= \Phi(\cdot, s) \quad s \in [-\tau, 0].
\end{align*}
\]
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\begin{align*}
\partial_t y(t) - \Delta y(t) + (a + \kappa) y(t) &= \kappa y(t - \tau), & t > 0, \\
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y(\cdot, s) &= \Phi(\cdot, s) & s \in [-\tau, 0].
\end{align*}
\]

**Eigenvalues $\mu_n$, normalized eigenfunctions $w_n$:**

\[
\begin{align*}
\left\{ \begin{array}{l}
-\Delta w_n(x) = \mu_n w_n(x), & x \in (0, 1), \\
\partial_x w_n(x) = 0, & x \in \{0, 1\},
\end{array} \right.
\end{align*}
\]

\[
\mu_n = n^2 \pi^2, \quad w_n(x) = \sqrt{2} \cos(n \pi x), \quad n \in \mathbb{N}_0.
\]

**Fourier ansatz:**

\[
y(x, t) = \sum_{n=0}^{\infty} y_n(t) w_n(x).
\]
Family of ODEs with delay

\[
\begin{align*}
\left\{ \begin{array}{l}
y'_n(t) + a_n y_n(t) &= \kappa y_n(t - \tau), \quad t > 0 \\
y_n(s) &= \Phi_n(s), \quad s \in [-\tau, 0],
\end{array} \right.
\end{align*}
\]

where \( a_n = a + \kappa + \mu_n \), \( \Phi_n(s) = \int_0^1 \Phi(x, s) w_n(x) \, dx \).
Family of ODEs with delay

\[
\begin{align*}
\left\{ \begin{array}{c}
y'_n(t) + a_n y_n(t) &= \kappa y_n(t - \tau), \quad t > 0 \\
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\end{align*}
\]

where \( a_n = a + \kappa + \mu_n \), \( \Phi_n(s) = \int_0^1 \Phi(x, s) w_n(x) \, dx \).

Let \( \{ Y_n(t) \} \) denote the family of associated fundamental solutions. Then

\[
Y_n(t) = \sum_{j=0}^{\infty} \frac{e^{\lambda_{n,j} t}}{1 + \kappa e^{-\lambda_{n,j}}} \quad \text{(real function!)}
\]

and

\[
y_n(t) = Y_n(t)\Phi_n(0) + \int_{-\tau}^{0} Y_n(t - s - \tau) \Phi_n(s) \, ds.
\]

Characteristic equations:

\[
\lambda + a_n - \kappa e^{-\lambda \tau} = 0.
\]

Roots:

\( \lambda_{n,j} \in \mathbb{C}, \quad j \in \mathbb{N}_0. \)
Zeros of the characteristic equation

Since, for each $n \in \mathbb{N}_0$, the zeros $\lambda_{n,j}$, $j \in \mathbb{N}_0$, appear in pairs of conjugate complex numbers and are ordered with respect to decreasing real parts, we have w.l.o.g

$$\lambda_{n,j} = \begin{cases} r_{n,j}, & j = 2k + 1 \\ \overline{r_{n,j}}, & j = 2k, \end{cases} \quad k \in \mathbb{N}_0.$$
Zeros of the characteristic equation

Since, for each $n \in \mathbb{N}_0$, the zeros $\lambda_{n,j}$, $j \in \mathbb{N}_0$, appear in pairs of conjugate complex numbers and are ordered with respect to decreasing real parts, we have w.l.o.g

$$\lambda_{n,j} = \begin{cases} r_{n,j}, & j = 2k + 1 \\ \overline{r_{n,j}}, & j = 2k, \end{cases} \quad k \in \mathbb{N}_0.$$

**Lemma**

**Assume that $\kappa < 0$. Then the zeros of the characteristic equations obey**

$$\text{Re } r_{n,j} > \text{Re } r_{n,j+1} \quad \forall j \in \mathbb{N}_0$$

**Proof:** Follows from the characteristic equations by a discussion of the zeros $\nu$ of the equation

$$0 = \nu + c \, e^{\nu \cot \nu} \, \sin \nu.$$  

*(Working) assumption on growth:* \quad $\text{Re } r_{n,j} > \text{Re } r_{n+1,j} \quad \forall n \in \mathbb{N}_0.$
Fourier expansion of $y(\cdot, t)$

We obtain

$$y(x, t) = \sum_{n=0}^{\infty} \left( Y_n(t) \Phi_n(0) + \int_{-\tau}^{0} Y_n(t-s-\tau) \Phi_n(s) \, ds \right) w_n(x), \quad t \geq \tau.$$ 

$$Y_n(t) \sim 2 \text{Re} \frac{e^{r_n,0 \cdot t}}{1 + \kappa e^{-r_n,0}}, \quad t \to \infty,$$

hence

$$y(x, t) \sim 2 \sum_{n=0}^{\infty} \frac{e^{r_n,0 \cdot t}}{1 + \kappa e^{-r_n,0}} \left( \Phi_n(0) + \int_{-\tau}^{0} e^{r_n,0 (-s-\tau)} \Phi_n(s) \, ds \right) w_n(x)$$

$$\sim 2 \frac{e^{r_{0,0} \cdot t}}{1 + \kappa e^{-r_{0,0}}} \left( \Phi_0(0) + \int_{-\tau}^{0} e^{r_{0,0} (-s-\tau)} \Phi_0(s) \, ds \right) w_0(x),$$

if $\neq 0$. 

Let $\Phi$ be in $C^1([-\tau, 0], L^\infty(\Omega))$. If $\kappa < 0$, the growth assumption is fulfilled and

$$\left( \Phi_0(0) + \int_{-\tau}^{0} e^{r_{0,0}(-s-\tau)} \Phi_0(s) \, ds \right) \neq 0,$$

then $y(x, t)$ is oscillatory as $t \to \infty$. 

Ongoing work:
- Discussion of the growth assumption
- Case of "Integral Pyragas"
- Discussion of the nonlinear equation with reaction term $R(y)$.

A different result on oscillation for the nonlinear equation can be found in

Let $\Phi$ be in $C^1([-\tau, 0], L^\infty(\Omega))$. If $\kappa < 0$, the growth assumption is fulfilled and

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- Discussion of the growth assumption
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A different result on oscillation for the nonlinear equation can be found in

Bainov, D.D., Mishev, D.P.

Oscillation theory for neutral differential equations with delay.,

A. Hilger, 1991
Outline

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2. Forward and design problem
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We recall the

**Optimal control problem (P1):**

\[
\min_{g \in C} F(g) := \frac{1}{2} \| y_g - \hat{y} \|^2_{L^2(Q)} + \frac{\nu}{2} \| g \|^2_{L^2(0, T)}
\]

\[
\begin{align*}
\partial_t y(t) - \Delta y(t) + R(y(t)) &= \kappa \int_0^T g(\tau) y(t - \tau) d\tau - \kappa y(t) & \text{in } Q, \\
y(s) &= \Phi(s), & s \in [-T, 0], & \text{in } \Omega \\
\partial_n y(t) &= 0 & \text{in } \Sigma,
\end{align*}
\]

where \( C = \{ g \in L^\infty(0, T) : 0 \leq g(\tau) \leq \beta, \int_0^T g(\tau) d\tau = 1 \} \).
The reduced functional $F$ is differentiable. The necessary condition for an optimal $\bar{g}$ is

$$F'(\bar{g})(g - \bar{g}) \geq 0 \quad \forall g \in C.$$
First order necessary optimality conditions

There holds

\[ F'(g)h = \int_0^T \left[ \int_0^T \int_\Omega \varphi_g(x,\tau)y_g(x,\tau-t)dxd\tau + \nu g(t) \right] h(t) dt. \]
First order necessary optimality conditions

There holds

\[ F'(g)h = \int_0^T \left[ \int_0^T \int_\Omega \varphi g(x, \tau) y g(x, \tau - t) dx d\tau + \nu g(t) \right] h(t) dt. \]

The integral constraint \( \int_0^T g(t) dt = 1 \) is a linear and regular constraint.

Theory of Lagrange multiplier rules in Banach spaces

\begin{itemize}
  \item Theorem (Necessary optimality conditions)
  \end{itemize}

Let \( \bar{g} \in C \) be the optimal kernel for the problem of optimal feedback design. Then there exists a Lagrange multiplier \( \bar{\mu} \in \mathbb{R} \) such that

\[ \int_0^T \left[ \int_0^T \int_\Omega \varphi \bar{g}(x, \tau) y \bar{g}(x, \tau - t) dx d\tau + \bar{\mu} + \nu \bar{g}(t) \right] (g(t) - \bar{g}(t)) dt \geq 0 \ \forall g \in C. \]
Class of step functions $g$

Select $0 \leq t_1 < t_2 \leq T$; $t_2 - t_1 \geq \delta > 0$

$$g(\tau) = \begin{cases} \frac{1}{t_2 - t_1}, & t_1 \leq \tau \leq t_2, \\ 0, & \text{else.} \end{cases}$$

Here, $\kappa$, $t_1$, $t_2$ are our control parameters to be optimized.
Example 1

$$\Omega = (-20, 20), \quad \text{Time interval: } [20, 40]$$

$$\hat{y}(x, t) = 3 \sin(t - \cos(\frac{\pi}{20}(x + 20))),$$

**Desired pattern** $\hat{y}$

**Optimal pattern**

Computed optimal value: $F = 1.320e + 03$, $|\nabla F(\kappa, t_1, t_2)| = 4.5e - 02$
Two equivalent periodic patterns

Periodic patterns should be considered as equivalent, if they just differ by a time shift.

\[
\hat{y} = 3 \sin \left( t - \cos \left( \frac{\pi}{20} (x + 20) \right) \right) \quad w = 3 \sin \left( t + 3 - \cos \left( \frac{\pi}{20} (x + 20) \right) \right)
\]

The right pattern is a simple time shift of the left, but

\[
\frac{1}{2} \int \int_Q (\hat{y} - w)^2 \, dx \, dt \approx 1.4374 \times 10^4
\]
Re-definition of the objective function

Measure the difference of $y$ to the closest time shift of $\hat{y}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$:

### Shifted objective function

$$f(g) := \min_s \iint_Q (y_g(x, t) - \hat{y}(x, t + s))^2 \, dx \, dt.$$
Re-definition of the objective function

Measure the difference of $y$ to the closest time shift of $\hat{y}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$:

**Shifted objective function**

$$f(g) := \min_s \iint_{Q} (y_g(x, t) - \hat{y}(x, t + s))^2 \, dx \, dt.$$ 

Assume that $\hat{y}$ has the period $T/2$, define $Q := \Omega \times [T/2, T]$.

$$\iint_{Q} (y_g(t) - \hat{y}(t + s))^2 = \underbrace{\iint_{Q} y_g^2(t)}_{\text{independent of } s} - 2 \iint_{Q} y_g(t) \hat{y}(t + s) + \underbrace{\iint_{Q} \hat{y}^2(t + s)}_{\text{independent of } s \text{ for periodic } \hat{y}}$$

$$= \iint_{Q} y_g^2 - 2 \iint_{Q} y_g(t) \hat{y}(t + s) + \iint_{Q} \hat{y}^2.$$
Re-definition of the objective function

Measure the difference of $y$ to the closest time shift of $\hat{y} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$:

**Shifted objective function**

$$f(g) := \min_s \iint_Q (y_g(x, t) - \hat{y}(x, t + s))^2 \, dx \, dt.$$  

Assume that $\hat{y}$ has the period $T/2$, define $Q := \Omega \times [T/2, T]$.

$$\iint_Q (y_g(t) - \hat{y}(t + s))^2 = \iint_Q y_g^2(t) - 2 \iint_Q y_g(t) \hat{y}(t + s) + \iint_Q \hat{y}^2(t + s)$$

independent of $s$

$$= \iint_Q y_g^2 - 2 \iint_Q y_g(t) \hat{y}(t + s) + \iint_Q \hat{y}^2.$$  

independent of $s$ for periodic $\hat{y}$

Instead of minimizing the left-hand side, we maximize the red term that is known as **cross correlation**.
A second optimization problem

The maximum of the red term is achieved, if \( y_g \) and \( \hat{y}(\cdot + s) \) are collinear, i.e.

\[
\frac{\int \int_Q y(t) \hat{y}(t + s) dxdt}{\sqrt{\int \int_Q y^2(t) dxdt} \sqrt{\int \int_Q \hat{y}^2(t + s) dxdt}} = 1.
\]
A second optimization problem

The maximum of the red term is achieved, if $y_g$ and $\hat{y}(\cdot + s)$ are collinear, i.e.

$$\frac{\iint_Q y(t) \hat{y}(t + s) dx dt}{\sqrt{\iint_Q y^2(t) dx dt} \sqrt{\iint_Q \hat{y}^2(t + s) dx dt}} = 1.$$ 

Therefore, for time-periodic $\hat{y}$ with period $T/2 > 0$, we consider the following

**Optimal control problem (P2)**

$$\min_{g \in C} f_{\text{corr}}(g) := \min_{s \in [0, T/2]} \left( 1 - \frac{(y_g, \hat{y}(\cdot + s))_{L^2(Q)}}{\|y_g\|_{L^2(Q)} \|\hat{y}(\cdot + s)\|_{L^2(Q)}} \right) \quad (P2)$$
A second optimization problem

The maximum of the red term is achieved, if $y_g$ and $\hat{y}(\cdot + s)$ are collinear, i.e.

$$\frac{\int\int_Q y(t) \hat{y}(t + s) dx dt}{\sqrt{\int\int_Q y^2(t) dx dt} \sqrt{\int\int_Q \hat{y}^2(t + s) dx dt}} = 1.$$ 

Therefore, for time-periodic $\hat{y}$ with period $T/2 > 0$, we consider the following

**Optimal control problem (P2)**

$$\min_{g \in C} f_{corr} (g) := \min_{s \in [0, T/2]} \left( 1 - \frac{(y_g, \hat{y}(\cdot + s))_{L^2(Q)}}{\|y_g\|_{L^2(Q)} \|\hat{y}(\cdot + s)\|_{L^2(Q)}} \right)$$ (P2)

This idea of using the normalized cross correlation is a well-known tool in signal processing. It is also used in quantum control.
1. Pyragas type feedback
2. Forward and design problem
3. Oscillating solutions
4. Optimal feedback for periodic targets
5. Numerical examples
Example 2: Example 1 with shifted objective function

Desired pattern $\hat{y}$

Computed optimal values:

$$f_{\text{corr}} = 0.1229, \ [t_1 = 0], \ t_2 = 3.0031, \ \kappa = -2.4318$$
Example 3: Varying parameter $\kappa, t_1, t_2, s$

$\Omega = (-20, 20)$, time interval: $[20, 40]$, $\hat{y}(0, t) = \sin(t - 1)$, 

$\Phi(x, t) := \frac{1}{2} \left(1 - \tanh \left(\frac{x-vt}{2\sqrt{2}}\right)\right)$, \quad $f = \min_s \frac{1}{2} \int_{T/2}^{T}(y(0, t) - \hat{y}(0, t + s))^2 dt$

$\hat{y}(0, t)$ and computed $y(0, t)$

Computed $y(x, t)$

Optimal solution:

$\kappa = -1.348, t_1 = 0.4974, t_2 = 1.626, f = 1.0118, \text{shift } s = 0.9234$
Example 4: Re-construct a kernel $g$

$\Omega = (-20, 20)$, Time interval: $[20, 40]$,

$\hat{y}$ generated by a step function $g$

Initial condition: $\Phi(x, t) = 2$

Re-constructed $g$

$f_{\text{corr}} = 9.4126 \times 10^{-6}$,

Shift: $|s| < 1 \times 10^{-16}$

$f_{\text{corr}} = 9.4126 \times 10^{-6}$,

Computed $\hat{y}(0, t)$
Example 5

Data as in Example 4, but

\[ \Phi(x, t) := \frac{1}{2} \left( 1 - \tanh \left( \frac{x - vt}{2\sqrt{2}} \right) \right) \]

Optimal \( g \)

\[ f_{\text{corr}} = 1.73 \times 10^{-2}, \]

Shift: \( s = -5.28 \)
Reference:

P. Nestler, E. Schöll, F.T.,
Optimization of nonlocal time-delayed feedback controllers
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P. Nestler, E. Schöll, F.T.,
Optimization of nonlocal time-delayed feedback controllers

Thank you