Linear system theory and semidefinite representations of atomic norms

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Atomic norm

$$\|y\|_{\mathcal{A}} = \inf \{\sum_{k=1}^{m} |x_k| \ \mid \ y = \sum_{k=1}^{m} x_k a_k, \ \ a_1, \ldots, a_m \in \mathcal{A}\}$$

- $\mathcal{A}$ is a set (dictionary) of elements (atoms) in $\mathbf{C}^n$ or $\mathbf{R}^n$
- a nonnegative, positively homogeneous, convex function; satisfies

$$\|sa\|_{\mathcal{A}} \leq 1 \ \text{ for all } a \in \mathcal{A}, \quad |s| = 1$$

- the greatest function with these properties
- generalizes vector 1-norm ($\mathcal{A} = \{e_1, \ldots, e_n\}$), matrix trace norm, etc.

(Chandrasekharan, Recht, Parrilo, Willsky 2012)

Regularization with atomic norm

$$\text{minimize} \quad f(y) + \|y\|_{\mathcal{A}}$$

- f a convex function (possibly an indicator)
- equivalent to

$$\begin{array}{ll} \text{minimize} & f(y) + \sum_{k=1}^{m} |x_k| \\ \text{subject to} & \sum_{k=1}^{m} x_k a_k = y \\ & a_1, \ldots, a_m \in \mathcal{A} \end{array}$$

  unknowns are variable y, parameters x_k, a_k, m of the decomposition

- extends Lasso, (noisy) basis pursuit, . . . to non-finite sets $\mathcal{A}$

Complex exponentials

$$\mathcal{A} = \{\frac{1}{\sqrt{n}}(1, e^{\mathrm{j}\omega}, e^{\mathrm{j}2\omega}, \ldots, e^{\mathrm{j}(n-1)\omega}) \mid \omega \in [0, 2\pi)\}$$

- atomic norm $\|y\|_{\mathcal{A}}$ is minimum of $\sum_k |x_k|$ subject to

$$y = \frac{1}{\sqrt{n}} \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ e^{\mathrm{j}\omega_1} & e^{\mathrm{j}\omega_2} & \cdots & e^{\mathrm{j}\omega_m} \\ \vdots & \vdots & & \vdots \\ e^{\mathrm{j}(n-1)\omega_1} & e^{\mathrm{j}(n-1)\omega_2} & \cdots & e^{\mathrm{j}(n-1)\omega_m} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_m \end{array}\right]$$

- $\|y\|_{\mathcal{A}}$ is optimal value of semidefinite program in $X \in \mathbf{H}^n$, $z \in \mathbf{R}$

$$\begin{array}{ll} \text{minimize} & (\operatorname{tr} X + z)/2 \\ \text{subject to} & \left[\begin{array}{cc} X & y \\ y^H & z \end{array}\right] \succeq 0, \quad X \text{ is Toeplitz} \end{array}$$

(Candès, Fernandez-Granda 2013; Tang, Bhaskar, Shah, Recht 2013; Yang, Xie 2015)

Atomic norm regularization

$$
\begin{array}{ll}
\text{minimize} & f(y) + \sum_{k=1}^{m} |x_k| \\
\text{subject to} & \frac{1}{\sqrt{n}} \left[\begin{array}{ccc} 1 & \cdots & 1 \\ e^{\mathrm{j}\omega_1} & \cdots & e^{\mathrm{j}\omega_m} \\ \vdots & & \vdots \\ e^{\mathrm{j}(n-1)\omega_1} & \cdots & e^{\mathrm{j}(n-1)\omega_m} \end{array} \right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_m \end{array} \right] = y
\end{array}
$$

variables: y, parameters x_k, ω_k, m of decomposition

Convex formulation

$$
\begin{array}{ll}
\text{minimize} & f(y) + (\operatorname{tr} X + z)/2 \\
\text{subject to} & \left[\begin{array}{cc} X & y \\ y^H & z \end{array} \right] \succeq 0, \quad X \text{ is Toeplitz}
\end{array}
$$

applications include superresolution, ‘gridless’ compressed sensing

Matrix extension

$$\begin{array}{ll} \text{minimize} & f(Y) + \sum_{k=1}^{m} \|x_k\|_2 \\ \text{subject to} & \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & \cdots & 1 \\ e^{\mathrm{j}\omega_1} & \cdots & e^{\mathrm{j}\omega_m} \\ \vdots & & \vdots \\ e^{\mathrm{j}(n-1)\omega_1} & \cdots & e^{\mathrm{j}(n-1)\omega_m} \end{bmatrix} \begin{bmatrix} x_1^H \\ \vdots \\ x_m^H \end{bmatrix} = Y \end{array}$$

variables: matrix Y, parameters x_k, ω_k, m of decomposition

Convex formulation

$$\begin{array}{ll} \text{minimize} & f(Y) + (\operatorname{tr} X + \operatorname{tr} Z)/2 \\ \text{subject to} & \begin{bmatrix} X & Y \\ Y^H & Z \end{bmatrix} \succeq 0, \quad X \text{ is Toeplitz} \end{array}$$

(Li, Chi 2014; Yang, Xie 2014)

This talk

semidefinite representation of atomic norm for sets

$$\mathcal{A} = \{a \in \mathbf{C}^n \mid (\lambda G - F)a = 0,\ \lambda \in \mathcal{C},\ \|a\|_2 = 1\}$$

- $\lambda G - F$ is a $p \times n$ matrix pencil
- $\mathcal{C}$ is a segment of a line or circle in the complex plane
- other normalizations than $\|a\|_2 = 1$ are possible

Example: $\mathcal{C}$ is the unit circle; F and G are the $(n-1) \times n$ matrices

$$F = \left[\begin{array}{cc} 0 & I_{n-1} \end{array}\right], \qquad G = \left[\begin{array}{cc} I_{n-1} & 0 \end{array}\right]$$

$\mathcal{A}$ is set of vectors of complex exponentials

$$a = c\,(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}) \qquad \text{with } \lambda = e^{\mathrm{j}\omega}, \quad |c| = 1/\sqrt{n}$$

Other examples

State-space linear system model: $\mathcal{C}$ is unit circle or imaginary axis

$$\lambda G - F = \left[\begin{array}{cc} \lambda I - A & B \end{array} \right] \qquad \text{(size } n_{\mathrm{s}} \times (n_{\mathrm{s}} + m))$$

$\mathcal{A}$ contains vectors

$$\left[\begin{array}{c} (\lambda I - A)^{-1} B u \\ u \end{array} \right], \qquad u \in \mathbf{C}^m, \qquad \lambda \in \mathcal{C}$$

Orthogonal polynomials: $\mathcal{C}$ is interval of real axis

$$\lambda G - F = \left[\begin{array}{cc} \lambda I_{n-1} - J & -\beta e_{n-1} \end{array} \right], \qquad J \text{ tridiagonal}$$

$\mathcal{A}$ contains vectors

$$(p_0(\lambda), p_1(\lambda), \ldots, p_{n-1}(\lambda)), \qquad \lambda \in \mathcal{C}$$

for polynomials p_k defined by 3-term recursion with coefficients in J, β

Outline

- Introduction
- Carathéodory-type matrix decompositions
- Semidefinite representation of atomic norm
- Examples

Carathéodory decomposition

an $n \times n$ positive semidefinite Toeplitz matrix X can be decomposed as

$$\begin{aligned} X &= \sum_{k=1}^{m} \theta_k \begin{bmatrix} 1 \\ e^{\mathrm{j}\omega_k} \\ e^{\mathrm{j}2\omega_k} \\ \vdots \\ e^{\mathrm{j}(n-1)\omega_k} \end{bmatrix} \begin{bmatrix} 1 \\ e^{\mathrm{j}\omega_k} \\ e^{\mathrm{j}2\omega_k} \\ \vdots \\ e^{\mathrm{j}(n-1)\omega_k} \end{bmatrix}^H \\ &= \sum_{k=1}^{m} \theta_k \begin{bmatrix} 1 & e^{-\mathrm{j}\omega_k} & \cdots & e^{-\mathrm{j}(n-1)\omega_k} \\ e^{\mathrm{j}\omega_k} & 1 & \cdots & e^{-\mathrm{j}(n-2)\omega_k} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\mathrm{j}(n-1)\omega_k} & e^{\mathrm{j}(n-2)\omega_k} & \cdots & 1 \end{bmatrix} \end{aligned}$$

with $\theta_k > 0$

- terms in sum are extreme rays of the cone of p.s.d. Toeplitz matrices
- next: extensions from papers on Kalman-Yakubovich-Popov lemma (starting with Rantzer 1996)

Quadratic matrix equation

let U, V be $n \times m$ matrices that satisfy

$$UU^H = VV^H$$

- from singular value decompositions $U = P\Sigma Q_1^H$, $V = P\Sigma Q_2^H$:

$$U = V\Lambda \qquad \text{with } \Lambda = Q_2 Q_1^H \text{ unitary}$$

- from Schur decomposition $\Lambda = Q \operatorname{diag}(\lambda) Q^H$:

$$UQ = VQ \operatorname{diag}(\lambda)$$

with Q unitary and $|\lambda_1| = \cdots = |\lambda_m| = 1$

Carathéodory decomposition of p.s.d. Toeplitz matrix

- $n \times n$ matrix X is Toeplitz if $FXF^H = GXG^H$ where

$$F = \left[\begin{array}{cc} 0 & I_{n-1} \end{array}\right], \qquad G = \left[\begin{array}{cc} I_{n-1} & 0 \end{array}\right]$$

- factor X as $X = YY^H$; apply previous result to $U = FY$, $V = GY$:

$$FYQ = GYQ\,\mathrm{diag}(\lambda) \quad \text{with } Q \text{ unitary,} \quad |\lambda_1| = \cdots = |\lambda_m| = 1$$

- columns a_1, . . . , a_m of YQ give decomposition

$$X = \sum_{k=1}^{m} a_k a_k^H, \qquad Fa_k = \lambda_k G a_k, \qquad |\lambda_k| = 1$$

vectors a_k have the form $a_k = c_k(1, \lambda_k, \cdots, \lambda_k^{n-1})$ with $\lambda_k = e^{\mathrm{j}\omega_k}$

General quadratic equation

suppose $\Phi \in \mathbf{H}^2$ with $\det \Phi < 0$, and U, V are $n \times m$ matrices with

$$\Phi_{11}UU^H + \Phi_{21}UV^H + \Phi_{12}VU^H + \Phi_{22}VV^H = 0$$

- there exist unitary Q, vectors μ, ν with

$$UQ\,\mathrm{diag}(\nu) = VQ\,\mathrm{diag}(\mu), \qquad \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix}^H \Phi \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix} = 0, \qquad (\mu_k, \nu_k) \neq 0$$

- last condition restricts $\lambda_k = \mu_k/\nu_k$ to circle or line in complex plane

Φ:	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \mathrm{j} \\ -\mathrm{j} & 0 \end{bmatrix}$
λ:	unit circle	imaginary axis	real axis

Quadratic matrix equation and inequality

let U and V be $n \times m$ matrices that satisfy

$$UV^H + VU^H = 0, \qquad UU^H \preceq VV^H$$

- there exists an $m \times m$ matrix Λ with

$$U = V\Lambda, \qquad \Lambda + \Lambda^H = 0, \qquad \Lambda\Lambda^H \preceq I$$

(Iwasaki, Meinsma, Hara 2000)

- Schur decomposition $\Lambda = Q \operatorname{diag}(\lambda) Q^H$ gives

$$UQ = VQ \operatorname{diag}(\lambda), \qquad \operatorname{Re} \lambda_i = 0, \qquad |\lambda_i| \leq 1$$

- efficiently computed using singular value and Schur decompositions

Quadratic matrix equation and inequality

suppose $\Phi, \Psi \in \mathbf{H}^2$ with $\det \Phi < 0$, and U, V are $n \times m$ matrices with

$$\Phi_{11}UU^H + \Phi_{21}UV^H + \Phi_{12}VU^H + \Phi_{22}VV^H = 0$$
$$\Psi_{11}UU^H + \Psi_{21}UV^H + \Psi_{12}VU^H + \Psi_{22}VV^H \preceq 0$$

- then there exist unitary Q, vectors μ, ν with $(\mu_k, \nu_k) \neq 0$, such that

$$UQ\,\mathrm{diag}(\nu) = VQ\,\mathrm{diag}(\mu)$$

  and

$$\begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix}^H \Phi \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix} = 0 \qquad \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix}^H \Psi \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix} \leq 0$$

- last two conditions restrict $\lambda_k = \mu_k/\nu_k$ to segment of circle or line

(Iwasaki and Hara 2003, Pipeleers and V. 2013)

Generalized Carathéodory decomposition

suppose X is positive semidefinite and satisfies

$$\Phi_{11}FXF^H + \Phi_{21}FXG^H + \Phi_{12}GXF^H + \Phi_{22}GXG^H = 0$$
$$\Psi_{11}FXF^H + \Psi_{21}FXG^H + \Psi_{12}GXF^H + \Psi_{22}GXG^H \preceq 0$$

- then it can be decomposed as

$$X = \sum_{k=1}^{m} a_k a_k^H$$

where $(\mu_k G - \nu_k F)a_k = 0$ for nonzero (μ_k, ν_k) that satisfy

$$\begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix}^H \Phi \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix} = 0, \qquad \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix}^H \Psi \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix} \leq 0,$$

- follows from previous result with $U = FY$, $V = GY$ where $X = YY^H$

Example

$$F = \begin{bmatrix} 0 & I \end{bmatrix}, \quad G = \begin{bmatrix} I & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & -e^{\mathrm{j}\alpha} \\ -e^{-\mathrm{j}\alpha} & 2\cos\beta \end{bmatrix}$$

if X is positive semidefinite and satisfies

$$FXF^H - GXG^H = 0 \qquad (X \text{ is Toeplitz})$$

$$-e^{\mathrm{j}\alpha}FXG^H - e^{-\mathrm{j}\alpha}GXF^H + 2(\cos\beta)GXG^H \preceq 0$$

then it can be factored as

$$X = \sum_{k=1}^{m} |c_k|^2 \begin{bmatrix} 1 \\ e^{\mathrm{j}\omega_k} \\ \vdots \\ e^{\mathrm{j}(n-1)\omega_k} \end{bmatrix} \begin{bmatrix} 1 \\ e^{\mathrm{j}\omega_k} \\ \vdots \\ e^{\mathrm{j}(n-1)\omega_k} \end{bmatrix}^H, \qquad |\omega_k - \alpha| \le \beta$$

Outline

- Introduction
- Carathéodory-type matrix decompositions
- **Semidefinite representation of atomic norm**
- Examples

Atoms

$$\mathcal{A} = \{a \in \mathbf{C}^n \mid (\mu G - \nu F)a = 0,\ (\mu,\nu) \in \overline{\mathcal{C}},\ \|a\|_2 = 1\}$$

- $\lambda G - F$ is a $p \times n$ matrix pencil
- $\overline{\mathcal{C}}$ is set of nonzero solutions (μ, ν) of

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix}^H \Phi \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0, \qquad \begin{bmatrix} \mu \\ \nu \end{bmatrix}^H \Psi \begin{bmatrix} \mu \\ \nu \end{bmatrix} \leq 0$$

  Φ, Ψ are Hermitian 2×2 matrices with $\det \Phi < 0$

- $\mathcal{C} = \{\lambda = \mu/\nu \mid (\mu,\nu) \in \overline{\mathcal{C}}\}$ is segment of line or circle in $\mathbf{C}$
- $(\mu, 0) \in \overline{\mathcal{C}}$ denotes point at infinity if $\mathcal{C}$ is an unbounded line segment

Semidefinite representation of atomic norm

$$
\begin{array}{ll}
\text{minimize} & f(Y) + \sum_{k=1}^{m} \|x_k\|_2 \\
\text{subject to} & \sum_{k=1}^{m} a_k x_k^H = Y \\
& a_1, \ldots, a_m \in \mathcal{A}
\end{array}
$$

SDP formulation (with variables Y, X, Z)

$$
\begin{array}{ll}
\text{minimize} & f(Y) + (\operatorname{tr} X + \operatorname{tr} Z)/2 \\
\text{subject to} & \left[\begin{array}{cc} X & Y \\ Y^H & Z \end{array}\right] \succeq 0 \\
& \Phi_{11} F X F^H + \Phi_{21} F X G^H + \Phi_{12} G X F^H + \Phi_{22} G X G^H = 0 \\
& \Psi_{11} F X F^H + \Psi_{21} F X G^H + \Psi_{12} G X F^H + \Psi_{22} G X G^H \preceq 0
\end{array}
$$

Outline of proof: SDP as relaxation

$$\begin{array}{ll} \text{minimize} & f(Y) + \sum_{k=1}^{m} \|x_k\|_2 \\ \text{subject to} & \sum_{k=1}^{m} a_k x_k^H = Y \\ & a_1, \ldots, a_m \in \mathcal{A} \end{array}$$

if Y, x_k, a_k are feasible in this problem, then

$$\left[\begin{array}{cc} X & Y \\ Y^H & Z \end{array}\right] = \sum_{k=1}^{r} \|x_k\|_2 \left[\begin{array}{c} a_k \\ x_k/\|x_k\|_2 \end{array}\right] \left[\begin{array}{c} a_k \\ x_k/\|x_k\|_2 \end{array}\right]^H$$

is feasible in the SDP, with

$$\frac{1}{2}(\operatorname{tr} X + \operatorname{tr} Z) = \sum_{k=1}^{m} \|x_k\|_2$$

Outline of proof: exactness of relaxation

$$
\begin{array}{ll}
\text{minimize} & f(Y) + (\operatorname{tr} X + \operatorname{tr} Z)/2 \\
\text{subject to} & \begin{bmatrix} X & Y \\ Y^H & Z \end{bmatrix} \succeq 0 \\
& \Phi_{11} F X F^H + \Phi_{21} F X G^H + \Phi_{12} G X F^H + \Phi_{22} G X G^H = 0 \\
& \Psi_{11} F X F^H + \Psi_{21} F X G^H + \Psi_{12} G X F^H + \Psi_{22} G X G^H \preceq 0
\end{array}
$$

- feasible X of rank m can be decomposed as

$$
X = \sum_{k=1}^{m} \theta_k a_k a_k^H, \qquad \theta_1, \ldots, \theta_m > 0, \qquad a_1, \ldots, a_m \in \mathcal{A}
$$

- first constraint implies there exist x_1, . . . , x_m such that

$$
Y = \sum_{k=1}^{m} a_k x_k^H, \qquad Z \succeq \sum_{k=1}^{m} \frac{1}{\theta_k} x_k x_k^H, \qquad \frac{1}{2}(\operatorname{tr} X + \operatorname{tr} Z) \geq \sum_{k=1}^{m} \|x_k\|_2
$$

Duality

Dual SDP (variables W, P, Q)

$$\begin{array}{ll} \text{maximize} & -f^*(W) \\ \text{subject to} & \left[\begin{array}{c} F \\ G \end{array}\right]^H (\Phi \otimes P - \Psi \otimes Q) \left[\begin{array}{c} F \\ G \end{array}\right] + WW^H \preceq I \\ & Q \succeq 0 \end{array}$$

Equivalent dual

$$\begin{array}{ll} \text{minimize} & -f^*(W) \\ \text{subject to} & \|W^H a\|_2 \leq 1 \quad \text{for all } a \in \mathcal{A} \end{array}$$

follows from generalized Kalman-Yakubovich-Popov lemma

(Iwasaki and Hara 2005)

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- Semidefinite representation of atomic norm
- **Examples**

Notation

in the examples we will use sets of the form

$$
\begin{aligned}
\mathcal{A} &= \{\frac{1}{\sqrt{n}}(1, e^{\mathrm{j}\omega}, e^{\mathrm{j}2\omega}, \ldots, e^{\mathrm{j}(n-1)\omega}) \mid |\omega - \alpha| \le \beta\} \\
&= \{a \in \mathbf{C}^n \mid (\lambda G - F)a = 0,\ \lambda \in \mathcal{C},\ \|a\|_2 = 1\}
\end{aligned}
$$

- matrices $G = \begin{bmatrix} I & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & I \end{bmatrix}$
- $\mathcal{C}$ is segment of the unit circle defined by

$$
\begin{bmatrix} \lambda \\ 1 \end{bmatrix}^H \Phi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0, \qquad \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^H \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \le 0
$$

with

$$
\Phi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} 0 & -e^{\mathrm{j}\alpha} \\ -e^{-\mathrm{j}\alpha} & 2\cos\beta \end{bmatrix}
$$

Structured matrix completion

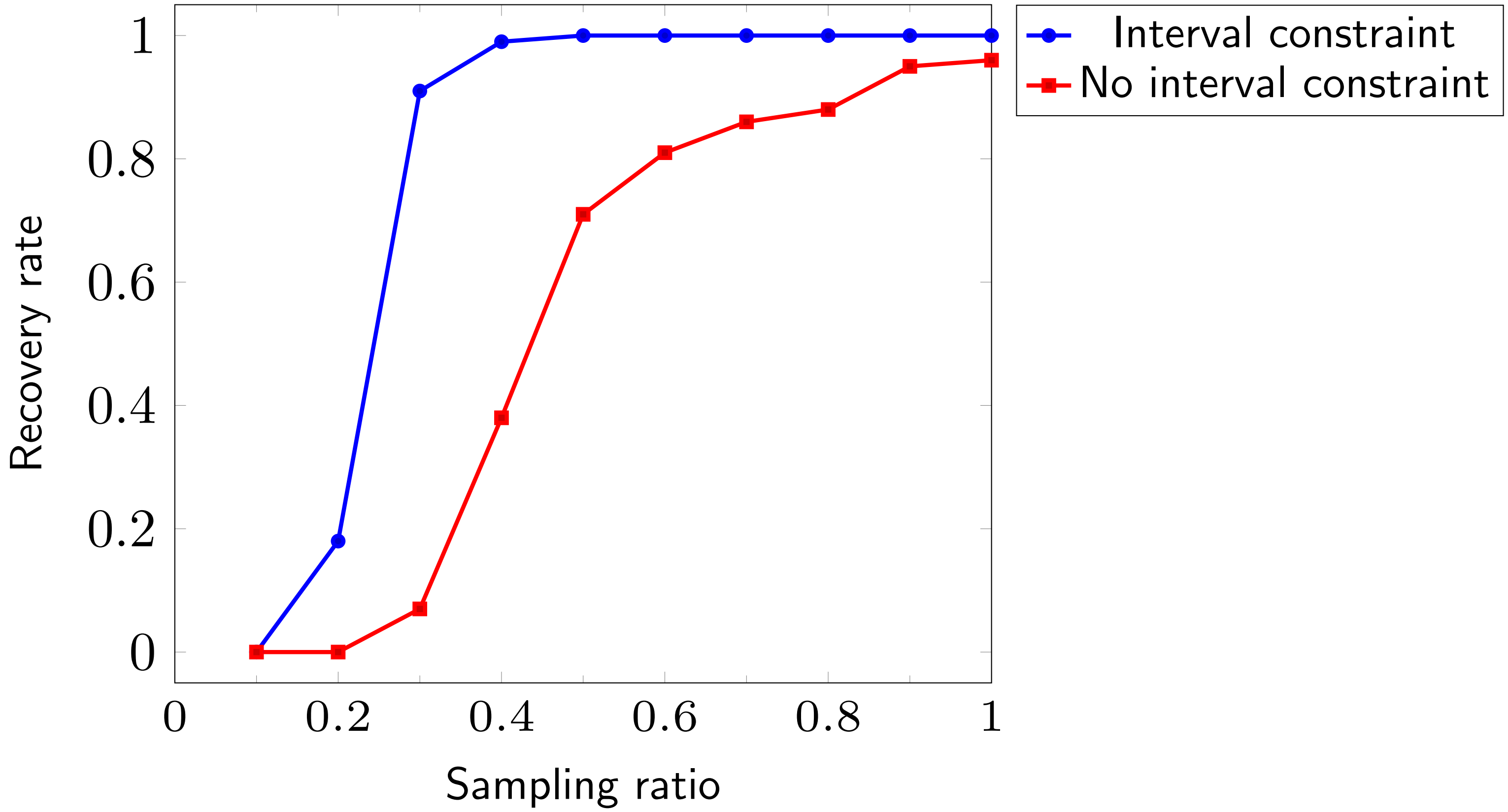
$$\begin{array}{ll} \text{minimize} & f(Y) + \sum_{k=1}^{m} \|x_k\|_2 \\ \text{subject to} & \sum_{k=1}^{m} a_k x_k^H = Y, \quad a_1, \ldots, a_m \in \mathcal{A} \end{array}$$

- f is indicator for $\{Y \in \mathbf{C}^{n \times n} \mid Y_{ij} = C_{ij}, \ (i,j) \in S\}$
- $\mathcal{A} = \{(1, e^{\mathrm{j}\omega}, \ldots, e^{\mathrm{j}(n-1)\omega}) \mid |\omega| \leq \beta\}$

SDP formulation

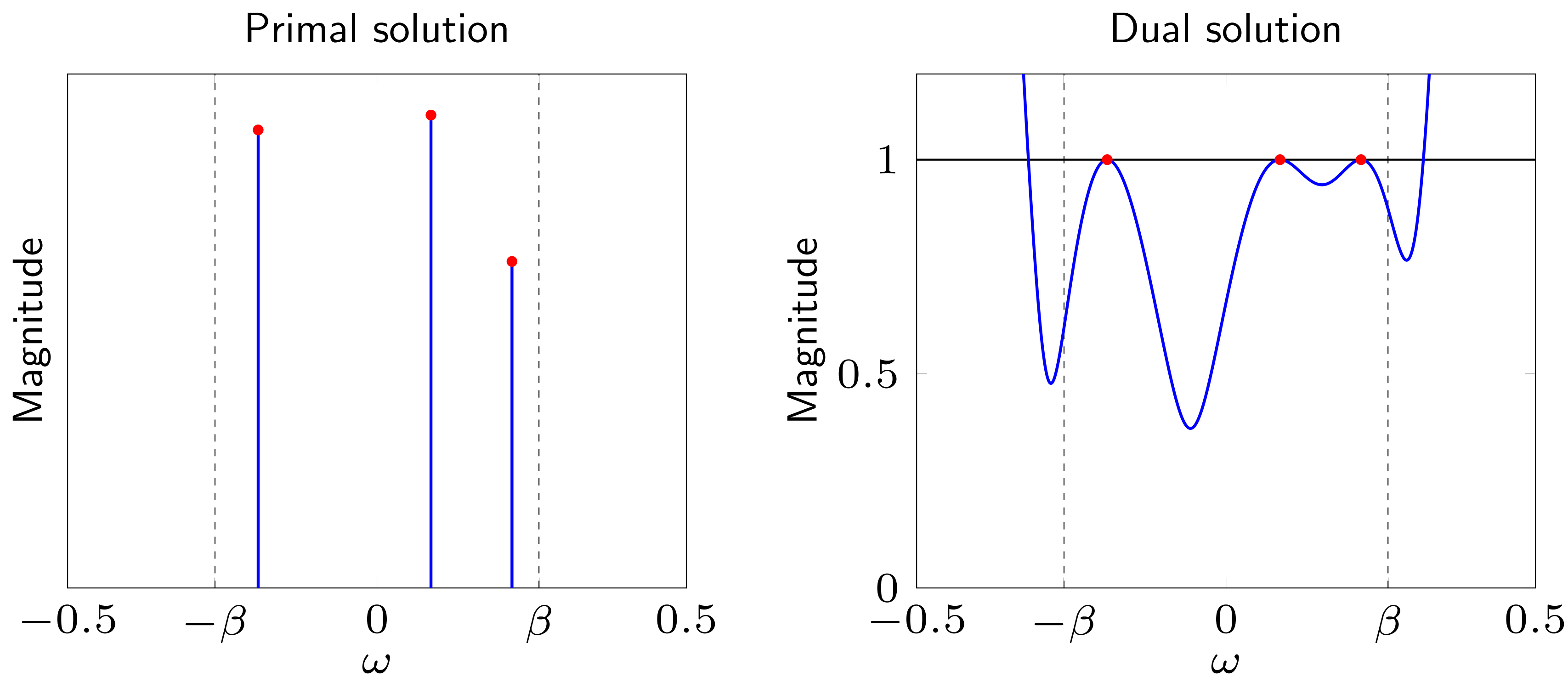
$$\begin{array}{ll} \text{minimize} & f(Y) + (\operatorname{tr} X + \operatorname{tr} Z)/2 \\ \text{subject to} & \left[\begin{array}{cc} X & Y \\ Y^H & Z \end{array}\right] \succeq 0 \\ & X \text{ is Toeplitz} \\ & -FXG^H - GXF^H + 2(\cos\beta)GXG^H \preceq 0 \end{array}$$

Example



- exact 30×30 matrix $C = AB$ has rank 3
- $A_{ik} = \exp\left(\mathrm{j}(i-1)\omega_k\right)$, three frequencies with $|\omega_k| \leq \beta = \pi/12$
- entries of B generated from standard normal distribution

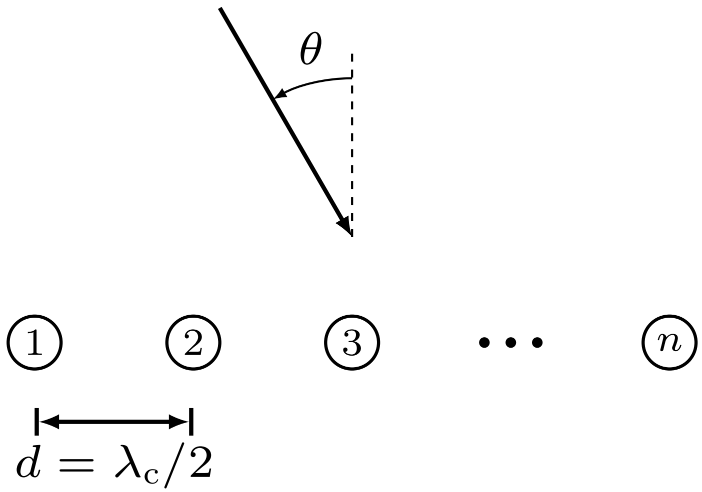
Example



left figure shows $\|W^H a\|_2$ for optimal solution of dual problem

$$\begin{array}{ll} \text{maximize} & -f^*(W) \\ \text{subject to} & \|W^H a\|_2 \leq 1 \text{ for al } a \in \mathcal{A} \end{array}$$

Linear sensor array



- m signals s_k arriving from angles θ_k
- linear array of n sensors
- p randomly chosen sensors are used

- output of sensor i is

$$y_i = \sum_{k=1}^{m} d_i(\omega_k) s_k e^{-\mathrm{j}(i-1)\omega_k} \qquad \text{where } \omega_k = \pi \sin\theta_k$$

- two types of sensors, detecting signals in $[-\pi/2, \pi/6]$ or $[-\pi/6, \pi/2]$:

$$d_i(\omega) = \begin{cases} 1 & \text{for } \theta \in [-\pi/2, \pi/6] \text{ or } [-\pi/6, \pi/2], \text{ respectively} \\ 0 & \text{otherwise} \end{cases}$$

Atomic norm formulation

$$\begin{array}{ll} \text{minimize} & \|y_1\|_{\mathcal{A}_1} + \|y_2\|_{\mathcal{A}_2} + \|y_3\|_{\mathcal{A}_3} \\ \text{subject to} & (y_1 + y_2)_{I_1} = b_1 \\ & (y_2 + y_3)_{I_2} = b_2 \end{array}$$

- variables y_1, y_2, y_3 are n-vectors
- three sets $\mathcal{A}_j$, for three sectors $\theta \in [-\frac{\pi}{2}, -\frac{\pi}{6}]$, $[-\frac{\pi}{6}, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{2}]$:

$$\mathcal{A}_j = \{(1, e^{\mathrm{j}\omega}, e^{\mathrm{j}2\omega}, \ldots, e^{\mathrm{j}(n-1)\omega}) \mid |\omega - \alpha_j| \leq \beta_j\}, \quad j = 1, 2, 3$$

- index sets I_1 and I_2 contain indices of used sensor outputs of type 1, 2
- b_1 and b_2 are measurements (assumed exact for simplicity)

Semidefinite formulation

Expanded atomic norm formulation

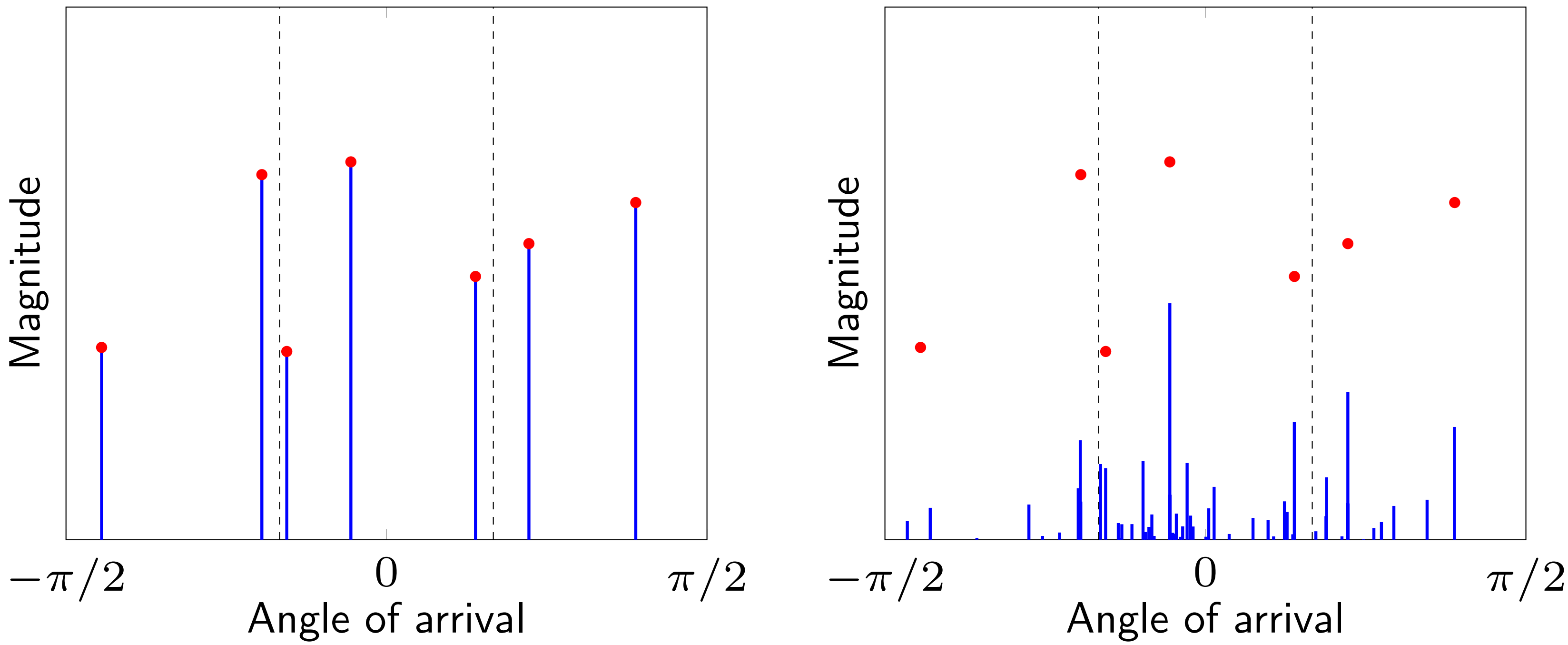
$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{j=1}^{3}\sum_{k=1}^{m_j} |x_{jk}| \\
\text{subject to} & \displaystyle\sum_{k=1}^{m_j} x_{jk} a_{jk} = y_j, \quad a_{jk} \in \mathcal{A}_j, \quad j = 1,2,3 \\
& (y_1 + y_2)_{I_1} = b_1, \quad (y_2 + y_3)_{I_2} = b_2
\end{array}
$$

Equivalent SDP

$$
\begin{array}{ll}
\text{min.} & \displaystyle\sum_{j=1}^{3} (\operatorname{tr} X_j + z_j)/2 \\
\text{s.t.} & \begin{bmatrix} X_j & y_j \\ y_j^H & z_j \end{bmatrix} \succeq 0, \quad X_j \text{ is Toeplitz}, \quad j = 1,2,3 \\
& -e^{-\mathrm{j}\alpha_j} F X_j G^H - e^{\mathrm{j}\alpha_j} G X_j F^H + 2\cos\beta_j G X_j G^H \preceq 0, \;\; j = 1,2,3 \\
& (y_1 + y_2)_{I_1} = b_1, \quad (y_2 + y_3)_{I_2} = b_2
\end{array}
$$

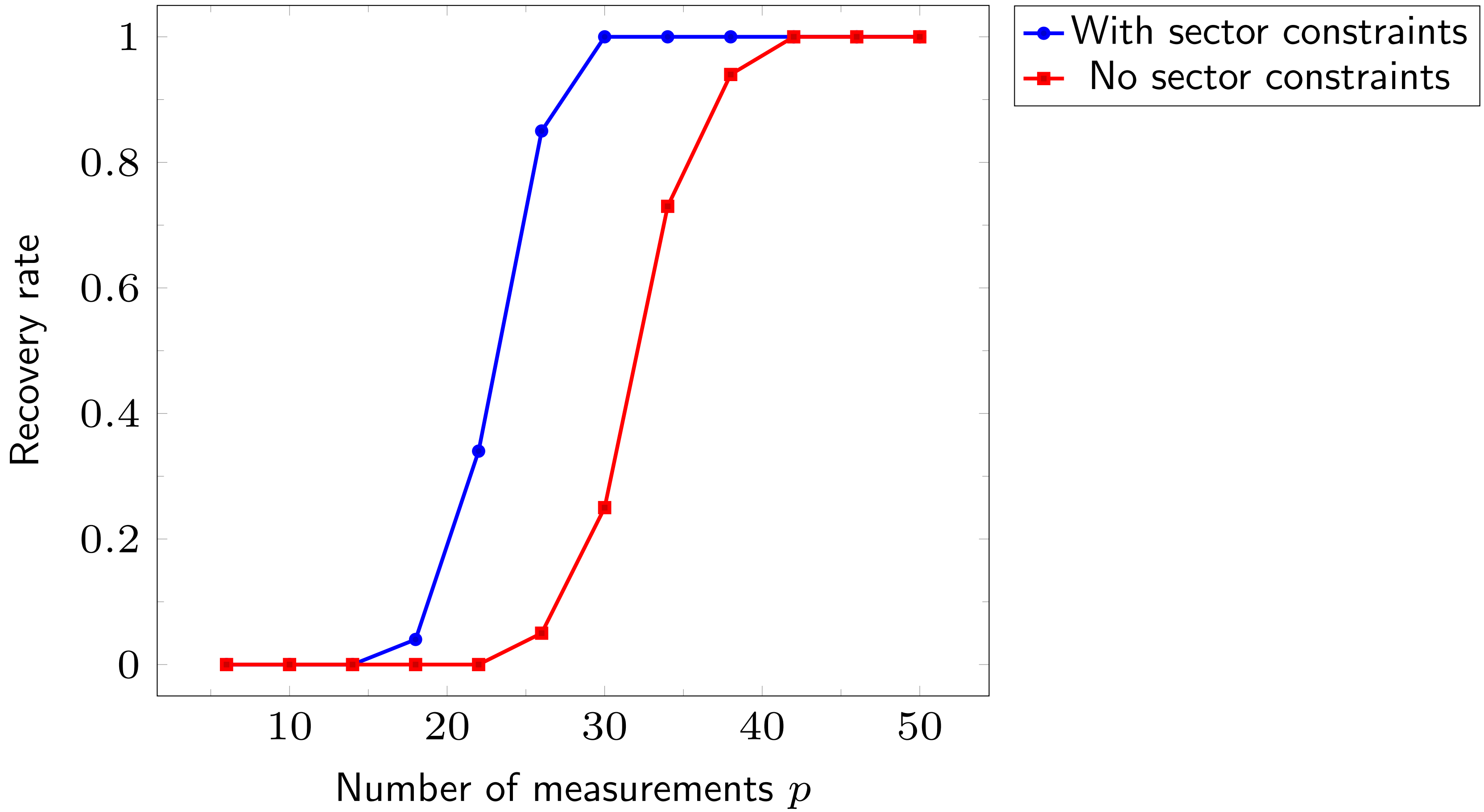
Example

$n = 500$ sensors, 20 sensor used of each type, 7 sources



- red: exact solution
- blue (left): solution from SDP with sector information
- blue (right): solution from SDP omitting sector constraints

Exact recovery



$n = 50$ sensors; 7 sources; p sensor measurements used

Conclusion

semidefinite representation of atomic norm associated with sets

$$\mathcal{A} = \{a \mid (\lambda G - F)a = 0,\ \lambda \in \mathcal{C},\ \|a\|_2 = 1\}$$

- $\mathcal{C}$ is segment (interval) of circle or line in the complex plane
- can be extended to unions of intervals

  (Krein, Nudelman 1977; Faybusovich 2006; Pipeleers, Iwasaki, Hara 2014)
- based on results for (generalized) KYP lemma, $i.e.$, matrix pencil

$$\lambda G - F = \left[\begin{array}{cc} \lambda I - A & B \end{array}\right]$$