Regularity of Solutions of Delay Differential Equations: $C^\infty$ versus Analytic

John Mallet-Paret
Division of Applied Mathematics
Brown University

June 25, 2016
In memory of Professor George R. Sell
Joint work with Roger Nussbaum
Delay Differential Equations

\[
\dot{x}(t) = f(t, x(t), x(t - r_1), \ldots, x(t - r_N))
\]

Initial condition (perhaps)

\[
x(\theta) = \phi(\theta), \quad \theta \in [-R, 0], \quad \text{some } R \geq r_k.
\]

Delays

\[
r_k \geq 0 \text{ (constant), or}
\]

\[
r_k = r_k(t) \geq 0 \text{ (nonautonomous variable), or}
\]

\[
r_k = r_k(x(t)) \geq 0 \text{ (state dependent).}
\]

Many other more complicated possibilities: distributed delays, implicitly defined delays, \ldots
Dynamical Systems Framework

Phase space $X = C[-R, 0]$ works well for constant delays. Extensive development by Jack Hale and many co-workers:

- local linearization
- Floquet theory
- invariant manifolds
- (finite-dimensional) attractors

Variable/state-dependent delays: fundamental work by Hartung, Krisztin, Walther, Wu.

For non-constant delays much remains to be done (e.g., smoothness of stable manifold).
Regularity of Solutions

\[ \dot{x}(t) = f(t, x(t), x(t - r_1), \ldots, x(t - r_N)) \]

Suppose \( x(t) \) is a bounded solution defined for all \( t \in \mathbb{R} \) (e.g., a periodic solution or more generally a solution on the attractor). If \( f \) and \( r_k \) are \( C^\infty \) smooth, then so is \( x(t) \).

What if \( f \) and \( r_k \) are analytic?

**Theorem (Nussbaum).** If each \( r_k > 0 \) is a constant, and \( f \) is analytic and independent of \( t \), then \( x(t) \) is analytic in \( t \).

But in general the answer is not so clear.

\[ \dot{x}(t) = \sin(t^2)x(t - 1) \quad \text{or} \quad \dot{x}(t) = e^{it^2}x(t - 1) \]

There exists a solution for \( t \in \mathbb{R} \) with \( x(-\infty) = 1 \). It is \( C^\infty \), but we don’t know whether or not it is analytic.
Some Examples

\[ \dot{x}(t) = -f(x(t - 1)) \]

If \( xf(x) > 0 \) for \( x \neq 0 \), \( f'(0) > \frac{\pi}{2} \), and \( f \) is appropriately bounded, then there exists a “slowly oscillating periodic solution,” which is part of a global compact attractor.

\[ \sigma \dot{x}(t) = -x(t) - f(x(t - 1)) \]

Similar conclusion with \( f \) as above, except \( f'(0) > 1 \) and \( \sigma > 0 \) sufficiently small.

Replace \( x(t - 1) \) with \( x(t - r) \) above, where \( r = r(x(t)) \) for appropriate \( r(\cdot) \), for similar results.
\[
\sigma \dot{x}(t) = -x(t) - kx(t - r),
\]

\[
\sigma > 0, \quad k > 1, \quad r(x(t)) = 1 + x(t).
\]

For \(\sigma\) small the periodic solution is \(C^\infty\), but analyticity is unknown.

For a given \(C^\infty\) solution \(x(t)\) we distinguish two sets:

\[
\mathcal{A} = \{ t_0 \mid x(t) \text{ is analytic for } t \text{ in some neighborhood of } t_0 \},
\]

\[
\mathcal{N} = \mathbb{R} \setminus \mathcal{A}.
\]

Note that \(\mathcal{A} \subseteq \mathbb{R}\) is open and \(\mathcal{N} \subseteq \mathbb{R}\) is closed.
\[
\dot{x}(t) = \alpha(t)x(t) + \beta(t)x(\eta(t)), \quad \eta(t) = t - r(t)
\]

Here \(\alpha(t)\), \(\beta(t)\), and \(r(t)\) are \(2\pi\)-periodic and analytic.

If \(x(t)\) is a \(2\pi\)-periodic solution, it can happen that both \(A \neq \emptyset\) and \(N \neq \emptyset\).

The sets \(A\) and \(N\) are intimately related to the dynamics of the “history map” \(\eta : S^1 \rightarrow S^1\).
Theorem. Consider the equation

\[ \dot{x}(t) = \alpha(t)x(t) + \beta(t)x(\eta(t)) \]

where \( \alpha(t) \), \( \beta(t) \), and \( \eta(t) \) are analytic for all \( t \in \mathbb{R} \). Assume that \( x(t) \) is a solution for all \( t \in \mathbb{R} \). Then

\[ \eta(\mathcal{N}) \subseteq \mathcal{N}, \quad \eta(\mathcal{A} \setminus \mathcal{M}) \subseteq \mathcal{A}, \]

where \( \mathcal{M} \subseteq \mathbb{R} \) is the set of local minima and maxima of \( \eta(t) \).

Proof. First observe that \( x(t) \) is \( C^\infty \) everywhere.

Take any \( t_0 \in \mathbb{R} \) and let \( t_1 = \eta(t_0) \). Assume that \( t_1 \in \mathcal{A} \), that is, \( x(t) \) is analytic in a neighborhood of \( t = t_1 \). Then \( x(\eta(t)) \) is analytic near \( t = t_0 \). Regarding \( \beta(t)x(\eta(t)) \) as a known forcing term in the differential equation, we conclude that \( x(t) \) is analytic near \( t = t_0 \), hence \( t_0 \in \mathcal{A} \).
Now suppose \( x(t) \) is analytic in a neighborhood of \( t = t_0 \). From the differential equation it follows that \( x(\eta(t)) \) is also analytic near \( t = t_0 \).

Write \( \eta(t) = t_1 + ((t - t_0)\theta(t))^m \) for \( t \) near \( t_0 \), and some odd \( m \geq 1 \), where \( \theta(t_0) \neq 0 \). Introducing the new variable \( s = (t - t_0)\theta(t) \), we see that \( x(t_1 + s^m) \) is analytic in \( s \) near \( s = 0 \), and thus

\[
x(t) = \sum_{j=0}^{\infty} x_j s^j = \sum_{j=0}^{\infty} x_j (t - t_1)^{j/m}
\]

as a convergent series for \( t \) near \( t_1 \). However, because \( x(t) \) is \( C^\infty \), it follows that \( x_j = 0 \) if \( j \) is not divisible by \( m \). Therefore,

\[
x(t) = \sum_{k=0}^{\infty} x_{mk} (t - t_1)^k,
\]

and so \( x(t) \) is analytic near \( t_1 \). ///
A Class of Integral Equations

\[ \nu x(t) = \int_{\eta(t)}^{t} x(s) \, ds, \quad \eta(t) = t - r(t) \]

Here \( \nu \neq 0 \) and \( r : \mathbb{R} \to \mathbb{R} \) with \( r(t + 2\pi) = r(t) \geq 0 \). Any solution of this equation also satisfies

\[ \nu \dot{x}(t) = x(t) + \dot{\eta}(t)x(\eta(t)). \]

The quantity \( \nu \) will appear as an eigenvalue of the above integral operator.
Standing Assumptions

\[ r : \mathbb{R} \to \mathbb{R} \text{ is continuous}, \]

\[ r(t) \geq 0, \quad r(t + 2\pi) = r(t), \]

\[ \eta(t) = t - r(t), \]

for all \( t \in \mathbb{R}. \)

We shall later also assume \( r(t) \) is analytic for all \( t. \)
Integral Operator

\[(Lx)(t) = \int_{\eta(t)}^{t} x(s) \, ds, \quad x \in X,\]

\[X = \{x : \mathbb{R} \to \mathbb{R} \mid \text{continuous and } 2\pi \text{ periodic}\}\]

Then \(L : X \to X\) is a positive operator (with respect to the cone of nonnegative functions).

Krein-Rutman implies there exists \(\nu > 0\) and \(x \in X \setminus \{0\}\), with \(x \geq 0\), such that

\[Lx = \nu x\]

if and only if the spectral radius equals \(\text{rad}(L) > 0\). And if so, one can take \(\nu = \text{rad}(L)\).
Theorem. The spectral radius is positive, \( \text{rad}(L) > 0 \), if and only if

\[
\inf_{s \geq t} \eta(s) < t
\]

for every \( t \in \mathbb{R} \).

Remark. If \( \eta(t) < t \) (that is, \( r(t) > 0 \)) for every \( t \), then (\( * \)) holds and \( \text{rad}(L) > 0 \). In this case the eigenfunction is unique.
**Sketch of Proof.** Suppose \((\ast)\) holds for every \(t\). Using \((\ast)\) we obtain points

\[
t_0 < t_1 < t_2 < \cdots < t_m \equiv t_0 \pmod{2\pi}
\]

such that

\[
t_k \in (\eta(t_{k+1}), t_{k+1}).
\]

It follows that if \(x \geq 0\) and \(x(t_k) > 0\), then \((Lx)(t_{k+1}) > 0\).

Taking \(x \geq 0\) to be a function with small bumps at the points \(t_k\), it follows that

\[
Lx \geq cx \quad \text{for some } c > 0.
\]

This implies (upon iterating) that \(\|L^n\| \geq c^n\), and thus \(\text{rad}(L) \geq c > 0\).
Now suppose (⋆) is false for some $t$ but that $\text{rad}(L) > 0$.

By Krein-Rutman there exists a nontrivial $x \in X$, with $x \geq 0$, such that $Lx = \nu x$ for some $\nu > 0$.

Then for any $\tau \geq t$ we have $t \leq \eta(\tau) \leq \tau$, and so

$$\nu |x(\tau)| \leq \int_{\eta(\tau)}^{\tau} |x(s)| \, ds \leq \int_{t}^{\tau} |x(s)| \, ds.$$  

Gronwall implies $x(t) \equiv 0$ identically, a contradiction.///
We now come to a main result on the analyticity set $\mathcal{A}$.

**Theorem.** In addition to the standing assumptions (periodicity and nonnegativity) on $r(t)$, assume that

$r(t)$ is analytic in $t$,

$r(t_*) = 0$ for some $t_*$, and

$\text{rad}(L) > 0$.

Then the set of analyticity $\mathcal{A}$ is a nonempty open set with infinitely many connected components (mod $2\pi$). The set of nonanalyticity $\mathcal{N}$ is uncountable. Further, under a “stretching” condition on $\eta$ the set $\mathcal{N}$ has empty interior and no isolated points, and is thus a generalized Cantor set.
An example of a system satisfying the above conditions is given by

\[ r(t) = \rho(1 - \cos t), \quad \rho > \rho_0. \]

If \( \rho = n\pi \) for an integer \( n \), the sets \( A \) and \( N \) can be described precisely and \( N \) is a Cantor set.
Steps in the Proof

Study invariant intervals \( l = [a, b] \), namely \( \eta(l) \subseteq l = \text{compact} \)
\( l \) invariant \( \implies x(t) = 0 \) for all \( t \in l \), thus \( \text{int}(l) \subseteq A \)
\( l \) invariant \( \implies \text{len}(l) = b - a < 2\pi \)
Possible to have \( l \subseteq J \) both invariant, with \( l \neq J \)
Each invariant \( l \) is contained in a **maximal** invariant \( J \)
The maximal intervals are pairwise disjoint
\( l = [a, b] \) maximal \( \implies \eta(a) = \eta(b) = a \)
There are finitely many maximal intervals, and at least one
\( l = [a, b] \) maximal implies:
\[ x(t) \not\equiv 0 \text{ in } [a - \varepsilon, a] \text{ or } [b, b + \varepsilon] \text{ for any } \varepsilon, \text{ thus } a, b \in N \]
\[ [a - \varepsilon, a] \cap N \text{ and } [b, b + \varepsilon] \cap N \text{ are uncountable for any } \varepsilon \]
Uncountability of $\mathcal{N}$

Suppose $I = [a, b]$ is the only maximal interval of $\eta$.

Denote $I_k = [a + 2\pi k, a + 2\pi (k + 1)]$. Then for large $\nu$ we have
\[
\eta^\nu(I_k) \supseteq I_k \quad \text{and} \quad \eta^\nu(I_{k+1}) \supseteq I_k.
\]

For any $t_0 \in \mathbb{R}$ let
\[
S(t_0) = \{ t \in \mathbb{R} \mid \eta^\mu(t) = t_0 \pmod{2\pi} \text{ for some } \mu \geq 1 \}.
\]

Then the closure $\overline{S(t_0)}$ is uncountable.

Do this with $t_0 = a \in \mathcal{N}$. Then $\overline{S(a)} \subseteq \mathcal{N}$ is uncountable.

Iterate the points in $\overline{S(a)}$ backwards to get them in a neighborhood of $a \pmod{2\pi}$, and of $b$. 
Components of \( \mathcal{A} \)

Again suppose \( I = [a, b] \) is the only maximal interval.

There exists some point \( c \in A \) with \( c \in (b - 2\pi, a) \).

Iterate \( c \) backward to get arbitrarily close to \( a \). Then \( a \) is a limit point (to the left) of points in \( \mathcal{A} \), and of points in \( \mathcal{N} \).

Thus \( \mathcal{A} \) has infinitely many components near \( a \) (and near \( b \)).
\[ \mathcal{N} \text{ Can Have Isolated Points} \]

\( \eta(t) \) is near \( t - 2\pi n \) over some interval

\( \eta(t_0) = t_0 - 2\pi n \) and \( |\dot{\eta}(t_0)| < 1 \implies t_0 \in \mathcal{A} \)

\( \eta(t_0) = t_0 - 2\pi n \) and \( |\dot{\eta}(t_0)| > 1 \implies \text{generically } t_0 \in \mathcal{N} \)
Nonanalyticity at a Point

\[ \dot{x}(t) = \alpha(t)x(t) + \beta(t)x(\eta(t)), \quad \eta(t) = t - r(t) \]

\(\alpha(t), \beta(t), r(t)\) analytic and \(2\pi\)-periodic

Assume that

\[ \eta(t_0) = t_0, \quad |\dot{\eta}(t_0)| > 1. \]

An analytic Hartman-Grobman transformation gives

\[ \dot{y}(t) = \tilde{\alpha}(t)y(t) + \tilde{\beta}(t)y(\mu t), \quad |\mu| > 1 \]
\[ \dot{y}(t) = \tilde{\alpha}(t)y(t) + \tilde{\beta}(t)y(\mu t), \quad |\mu| > 1 \]

Expand (formal) Taylor series to get

\[ y(t) = \sum_{k=0}^{\infty} y_k t^k, \quad y_k = \left( \frac{\mu^{k(k-1)/2} \tilde{\beta}(0)^k}{k!} \right) w_k, \]

with \( \lim_{k \to \infty} w_k = w_\infty \) finite.

**Theorem.** \( w_\infty = 0 \) if and only if there exists an analytic solution in a neighborhood of \( t_0 \).
Sketch of Proof. For simplicity assume \( \tilde{\alpha}(t) \) is identically zero, so

\[
\dot{y}(t) = \tilde{\beta}(t)y(\mu t), \quad |\mu| > 1,
\]

with

\[
y(t) = \sum_{k=0}^{\infty} y_k t^k, \quad \tilde{\beta}(t) = \sum_{k=0}^{\infty} \beta_k t^k.
\]

The series for \( \tilde{\beta}(t) \) is assumed to converge, but the series for \( y(t) \) is formal. Assuming \( \beta_0 \neq 0 \), we scale the coefficients \( y_k \) as

\[
y_k = \left( \frac{\mu^{k(k-1)/2} \beta_0^k}{k!} \right) w_k.
\]
Substituting into the differential equation gives a recursion for $w_k$, namely

$$w_{k+1} = w_k + \sum_{j=0}^{k-1} \frac{\beta_{k-j}}{\beta_0} \left( \frac{k!}{\mu^{k(k+1)/2} \beta_0^k} \right) \left( \frac{\mu^{j(j+1)/2} \beta_j^j}{j!} \right).$$

Note that $\mu^{k(k+1)/2}$ is much larger than $k!$. This implies the terms in the summation are very small, and ensures the finite limit

$$\lim_{k \to \infty} w_k = w_\infty.$$
Can $\mathcal{N}$ have nonempty interior?

Answer unknown, but if so it would be very interesting: An interval where the solution is everywhere $C^\infty$ but nowhere analytic.
\( \mathcal{N} \) Can Be a Cantor Set

\[ \eta(t) = t - n\pi(1 - \cos t) \]

Then there is a maximal interval \( I = [0, \tau] \) for some \( \tau \in (0, \frac{\pi}{2}) \)

There is also its symmetric “twin” \( I' = [\pi - \tau, \pi] \) which is invariant mod \( 2\pi \).

Although \( x(t) \) is nonzero in \( I' \), it is nonetheless analytic in the interior.

But the endpoints of \( I' \) are not points of analyticity. Thus

\[ (0, \tau), (\pi - \tau, \pi) \subseteq \mathcal{A}, \quad 0, \tau, \pi - \tau, \pi \in \mathcal{N} \]
Take any other interval (connected component) of $\mathcal{A}$, say

$$J = (a, b) \subseteq \mathcal{A}, \quad a, b \in \mathcal{N}$$

Consider the iterates $\eta^k(J)$. Either

$$\eta^k(J) = \text{int}(I) \quad \text{or} \quad \eta^k(J) = \text{int}(I')$$

for some $k$, or else

$$\eta^k(J) \cap I = \eta^k(J) \cap I' = \emptyset \quad \text{for all } k \quad (***)$$

But (***) is impossible due to the stretching condition:

There exist $k_1 < k_2 < k_3 < \ldots$ such that

$$\text{len}(\eta^{k_{i+1}}(J)) > 2\text{len}(\eta^{k_i}(J))$$