Nonlocal evolution equations, SQG

Peter Constantin

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Collaborators

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- confined matter, unconfined fields: electroconvection

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C-Majda-Tabak: analogies to 3D Euler.

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- quasilinear, critical: no room

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5. C-Tarfulea-Vicol: nonlinear maximum principle, small Hölder exponent.

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- Global Hölder bounds for critical SQG in bounded domains (Ignatova, C)

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$-\Delta$ is a positive selfadjoint operator in $L^2(\Omega)$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. The ground state w_1 is positive and

$$c_0 d(x) \le w_1(x) \le C_0 d(x),$$

where

$$d(x) = dist(x, \partial\Omega)$$

Functional Calculus

$$(-\Delta)^{\alpha} f = \sum_{j=1}^{\infty} \lambda_j^{\alpha} f_j w_j$$

with

$$f_j = \int_{\Omega} f(y) w_j(y) dy$$

and for $f \in \mathcal{D}\left((-\Delta)^{\alpha}\right) = \{f \mid \ (\lambda_j^{\alpha} f_j) \in \ell^2(\mathbb{N})\}$. We denote by

$$\Lambda_D = (-\Delta)^{\frac{1}{2}}$$

It is well-known and easy to show that (Kato conjecture, trivial case)

$$\mathcal{D}\left(\Lambda_D\right) = H_0^1(\Omega).$$

Indeed, for $f \in \mathcal{D}\left(-\Delta\right)$ we have

$$\|\nabla f\|_{L^2(\Omega)}^2 = \int_{\Omega} f\left(-\Delta\right) f dx = \|\Lambda_D f\|_{L^2(\Omega)}^2.$$

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valid for $0 \leq \alpha < 1$. Representation

$$(\Lambda_D^{2\alpha} f)(x) = \left((-\Delta)^{\alpha} f\right)(x) = c_{\alpha} \int_0^{\infty} \left[f(x) - e^{t\Delta} f(x)\right] t^{-1-\alpha} dt$$

for $f \in \mathcal{D}\left((-\Delta)^{\alpha}\right) = \mathcal{D}\left((-\Lambda_D)^{2\alpha}\right)$.

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Electric field determined by charge density:

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$\Omega \subset \mathbb{R}^2 \times \{0\}$ (thin film): Fractional Laplacian emerges.

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Solved by

$$\Phi(x,z) = \Phi_0(x) + \begin{cases} e^{-z\Lambda_D}\Lambda_D^{-1}q, & z > 0, \\ e^{z\Lambda_D}\Lambda_D^{-1}q, & z < 0 \end{cases}$$

Permittivity

$$\epsilon E = (-\partial_1\Phi, -\partial_2\Phi, 0)_{|\,\Omega}$$

Global Regularity for Electroconvection in 2D Bounded Domains

Theorem

(C, Elgindi, Ignatova, Vicol) Let $\Omega \subset \mathbb{R}^2$ open, bounded, with smooth boundary. Let $u_0 \in [H_0^1(\Omega) \cap H^2(\Omega)]^2$ be divergence-free. Let $q_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. Then the electroconvection system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - q \nabla \Lambda_D^{-1} q - q \nabla \Phi_0, \\ div u = 0, \\ \partial_t q + u \cdot \nabla q + \Lambda_D q = 0 \end{cases}$$

with homogeneous Dirichlet boundary conditions for both u and q has global unique strong solutions,

$$u \in L^\infty(0, T; [H_0^1(\Omega) \cap H^2(\Omega)]^2) \cap L^2(0, T; H^{\frac{5}{2}}(\Omega)^2),$$

$$q \in L^\infty(0, T; W_0^{1,4}(\Omega) \cap H^2(\Omega)) \cap L^2(0, T; H^{\frac{5}{2}}(\Omega)),$$

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(C, Ignatova) Let $\theta_0\in L^2(\Omega)$ and let $T>0$. There exists a weak solution of critical SQG,

$$\theta\in L^{\infty}(0,T;L^2(\Omega))\cap L^2\left((0,T;\mathcal{D}\left(\Lambda_D^{\frac{1}{2}}\right)\right)$$

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Local existence of smooth solutions: OK as well.

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$$\sup_{d(x)\ge\ell,\ |h|\le\frac{\ell}{16},\ t\ge0} \frac{|\theta(x+h,t)-\theta(x,t)|}{|h|^\alpha} \le \|\theta_0\|_{C^\alpha} + \Gamma\ell^{-\alpha}\|\theta_0\|_{L^\infty(\Omega)}$$

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Higher interior regularity and global existence follow.

Elements of the proof

- Gaussian bounds for heat kernel; cancellation due to translation invariance effective for small time.

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- Gaussian bounds for heat kernel; cancellation due to translation invariance effective for small time.
- Nonlinear maximum principle (lower bound for Λ_D) giving smoothing and a strong boundary repulsion damping effect.
- Good cutoff χ and bound for the commutator $[\delta_h, \Lambda_D]$ away from boundary; (the most expensive item, fighting boundary repulsion)
- Finite difference bounds for Riesz transforms using the nonlinear max principle bound in its finite difference variant.

Bounds for heat kernel

We use precise upper and lower bounds for the kernel $H_D(t, x, y)$ of the heat operator,

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These are as follows (Davies, Q.S Zhang): There exists a time $T > 0$ depending on the domain Ω and constants c, C, k, K, depending on T and Ω such that

$$c \min\left(\frac{w_1(x)}{|x-y|}, 1\right) \min\left(\frac{w_1(y)}{|x-y|}, 1\right) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{kt}} \leq$$
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holds for all $0 \leq t \leq T$. Moreover

$$\frac{|\nabla_x H_D(t,x,y)|}{H_D(t,x,y)} \leq C \begin{cases} \frac{1}{d(x)}, & \text{if } \sqrt{t} \geq d(x), \\ \frac{1}{\sqrt{t}}\left(1 + \frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t} \leq d(x) \end{cases}$$

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Symmetry of bounds for the gradient

Note that, in view of

$$H_D(t,x,y)=\sum_{j=1}^{\infty} e^{-t\lambda_j} w_j(x) w_j(y)$$

elliptic regularity estimates and Sobolev embedding which imply uniform absolute convergence of the series (if $\partial\Omega$ is smooth enough), we have that

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for positive t, where we denoted by ∂_1^β and ∂_2^β derivatives with respect to the first spatial variables, respectively the second spatial variables.

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$$\frac{|\nabla_y H_D(t,x,y)|}{H_D(t,x,y)} \le C \begin{cases} \frac{1}{d(y)}, & \text{if } \sqrt{t} \ge d(y), \\ \frac{1}{\sqrt{t}} \left(1 + \frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t} \le d(y) \end{cases}$$

Additional bounds; translation invariance effect

$$|\nabla_x \nabla_x H_D(x,y,t)| \le C t^{-1-\frac{d}{2}} e^{-\frac{|x-y|^2}{\tilde{K} t}}$$

holds for $t \le c d(x)^2$ and $0 < t \le T$.

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holds for $t \le cd(x)^2$ and $0 < t \le T$. This follows from previous bounds. Important additional bounds we need are

$$I_1(x,t) = \int_\Omega |(\nabla_x + \nabla_y) H_D(x,y,t)|\, dy \le Ct^{-\frac{1}{2}} e^{-\frac{d(x)^2}{\tilde{K}t}}$$

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$$\int_0^T t^{-\frac{k}{2}} I_j(x,t) dt \le d(x)^{2-j-k}$$

for $j = 1, 2$ and $k \ge 0$.

The convex damping inequality

Proposition

(C, Ignatova) Let Ω *be a bounded domain with smooth boundary, let* $0 < s < 2$*. There exists a constant* C *depending on the domain and on* s *such that for every* Φ*, a* C^2 *convex function satisfying* $\Phi(0) = 0$*, and every* $f \in C_0^{\infty}(\Omega)$

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$$\Phi'(f)\Lambda_D^s f - \Lambda_D^s(\Phi(f)) \geq \frac{C}{d(x)^s}\left(f(x)\Phi'(f(x)) - \Phi(f(x))\right)$$

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Dramatically different from $\mathbb{R}^d$!

The nonlinear bound for derivatives

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$$f\Lambda_D^s f - \frac{1}{2}\Lambda_D^s f^2 \ge c\|q\|_{L^\infty}^{-s}|f_d|^{2+s}$$

holds pointwise in Ω, with

$$|f_d(x)| = \begin{cases} |f(x)| & \text{if } |f(x)| \ge C\|q\|_{L^\infty(\Omega)}\frac{1}{d(x)}, \\ 0 & \text{if } |f(x)| \le C\|q\|_{L^\infty(\Omega)}\frac{1}{d(x)}, \end{cases}$$

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$$|f_d(x)| = \begin{cases} |f(x)| & \text{if } |f(x)| \ge C\|q\|_{L^\infty(\Omega)}\frac{1}{d(x)}, \\ 0 & \text{if } |f(x)| \le C\|q\|_{L^\infty(\Omega)}\frac{1}{d(x)}, \end{cases}$$

Proof: nontrivial, uses precise bounds on the heat kernel and

$$f\Lambda_D^s f - \frac{1}{2}\Lambda_D^s f^2 \ge \frac{c_s}{2}\int_0^\infty t^{-1-\frac{s}{2}}dt\int_\Omega H_D(t,x,y)(f(x)-f(y))^2 dy$$

Good cutoff

Lemma

Let Ω be a bounded domain with C^2 boundary. For $\ell > 0$ small enough (depending on Ω) there exist cutoff functions χ with the properties: $0 \le \chi \le 1$, $\chi(y) = 0$ if $d(y) \le \frac{\ell}{4}$, $\chi(y) = 1$ for $d(y) \ge \frac{\ell}{2}$, $|\nabla^k \chi| \le C\ell^{-k}$ with C independent of ℓ and

$$\int_\Omega \frac{(1-\chi(y))}{|x-y|^{d+j}} dy \le C \frac{1}{d(x)^j}$$

and

$$\int_\Omega |\nabla \chi(y)| \frac{1}{|x-y|^d} \le C \frac{1}{d(x)}$$

hold for $j \ge 0$ and $d(x) \ge \ell$. We will refer to such χ as a "good cutoff".

Nonlinear bound, finite differences

Theorem

Let Ω be a bounded domain with smooth boundary. There exists a constant C such that, for every $f \in C_0^\infty(\Omega)$

$$D(f) = f\Lambda_D f - \frac{1}{2}\Lambda_D f^2 \geq \frac{C}{d(x)} f^2(x)$$

holds for all $x \in \Omega$. Let $\chi \in C_0^\infty(\Omega)$ be a good cutoff with scale $\ell > 0$ and let

$$f(x) = \chi(x)(\delta_h q(x)) = \chi(x)(q(x+h) - q(x)).$$

Then

$$(f\Lambda_D f)(x) - \frac{1}{2}(\Lambda_D f^2)(x) \geq \gamma_1 |h|^{-1} \frac{|f_d(x)|^3}{\|q\|_{L^\infty}} + \gamma_1 \frac{f^2(x)}{d(x)}$$

holds pointwise in Ω when $|h| \leq \frac{\ell}{16}$, and $d(x) \geq \ell$ with

$$|f_d(x)| = |f(x)|, \qquad \text{if } |f(x)| \geq M\|q\|_{L^\infty(\Omega)} \frac{|h|}{d(x)}.$$

Commutator

Let χ be a good cutoff.

Lemma

There exists a constant Γ_0 such that the commutator

$$C_h(\theta) = \chi\delta_h\Lambda_D\theta - \Lambda_D(\chi\delta_h\theta)$$

obeys

$$|C_h(\theta)(x)| \le \Gamma_0 \frac{|h|}{d(x)^2}\|\theta\|_{L^\infty(\Omega)}$$

for $d(x) \ge \ell$, $|h| \le \frac{\ell}{16}$.

Finite difference of Riesz transform

Lemma

Let χ be a good cutoff, and let u be defined by

$$u = R_D^{\perp}\theta.$$

Then

$$|\delta_h u(x)| \le C\left(\sqrt{\rho D(f)(x)} + \|\theta\|_{L^\infty}\left(\frac{|h|}{d(x)} + \frac{|h|}{\rho}\right) + |\delta_h\theta(x)|\right)$$

holds for $d(x) \ge \ell$, $\rho \le cd(x)$, $f = \chi\delta_h\theta$ and with C a constant depending on Ω.

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holds for $d(x) \geq \ell$, $\rho \leq cd(x)$, $f = \chi\delta_h\theta$ and with C a constant depending on Ω.

This gives a bound on $|h|^{-1}|\delta_h u(x)|$ which costs $D(f)$.

Idea of proof of Hölder bound

Good cutoff, and equation for $\delta_h\theta$ imply:

$$\frac{1}{2}L_\chi\left(\delta_h\theta\right)^2 + D(f) + (\delta_h\theta)C_h(\theta) = 0$$

with

$$L_\chi g = \partial_t g + u\cdot\nabla_x g + \delta_h u\cdot\nabla_h g + \Lambda_D(\chi^2 g).$$

and

$$D(f) \geq \gamma_1|h|^{-1}\|\theta\|_{L^\infty}^{-1}|(\delta_h\theta)_d|^3 + \gamma_1(d(x))^{-1}|\delta_h\theta|^2$$

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Multiply by $|h|^{-2\alpha}$ with $\epsilon = \alpha\|\theta_0\|_{L^\infty}$ small. Obtain:

$$L_\chi\left(\frac{\delta_h\theta(x)^2}{|h|^{2\alpha}}\right) + \frac{\gamma_1}{4d(x)}\left(\frac{\delta_h\theta(x)^2}{|h|^{2\alpha}} - \Gamma_1\ell^{-2\alpha}\|\theta\|_{L^\infty}^2\right) \le 0.$$

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- Electroconvection and SQG: free boundaries.