Self-similar point vortices and confinement of vorticity

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Dedicated to the memory of George Sell
Joint work with Carlo Marchioro (Rome).

2D incompressible Euler equation

\[ \frac{\partial}{\partial t} u + u \cdot \nabla u = -\nabla p, \quad \text{div } u(t, \cdot) = 0 \]

Perfect incompressible fluid flow:

- \( u : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is the velocity field;
- \( p : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is the pressure.

The vorticity

\[ \omega = \partial_1 u_2 - \partial_2 u_1 \]

verifies a transport equation

\[ \frac{\partial}{\partial t} \omega + u \cdot \nabla \omega = 0 \]
Velocity can be recovered from vorticity via the Biot-Savart law

\[ u(t, x) = \frac{1}{2\pi} \int_{\Omega} \frac{(x - y)^\perp}{|x - y|^2} \omega(t, y) \, dy \]

Vorticity is transported by the flow. If

\[ \partial_t \Psi = u(t, \Psi) \]
\[ \Psi_0 = Id \]

then

\[ \omega(t, \Psi(t, \cdot)) = \omega_0(\cdot). \]

Incompressibility \iff divergence free velocity \iff \( \Psi(t, \cdot) \) preserves measure.

The \( L^p \) norms of the vorticity are constant in time.
Global well-posedness
- classical solutions: Wolibner (1933)

Global existence of weak solutions:
- $L^1 \cap L^p$: DiPerna and Majda (1986);
- Vortex sheets (vorticity is a positive measure): Delort (1991).

**Question:**
What is the large time behavior of the solutions?

The velocity is bounded in space and time.

$\text{supp } \omega_0$ compact $\Rightarrow$ $\text{supp } \omega(t, \cdot)$ compact and bounded by $O(t)$.

**Question:**
How large the support of the vorticity can become?
To gain insight, one can look at a discrete version of the Euler system.

**Point-vortex model** (Helmholtz 1867): the vorticity is a sum of Dirac masses:

\[
\omega = \sum_{j=1}^{n} m_j \delta_{z_j}
\]

moving according to the following system of ODEs:

\[
z_j'(t) = \sum_{k \in \{1,\ldots,n\} \setminus j} m_k \frac{(z_j - z_k)\perp}{2\pi |z_j - z_k|^2} \quad \forall 1 \leq j \leq n.
\]

The point-vortex model is a good approximation of the Euler system (Marchioro and Pulvirenti).
Important facts about the point vortex system:

- Positive masses \( m_j \implies \) bounded positions \( z_j \) (conservation of the moment of inertia \( \sum_{i=1}^{n} m_j |z_j|^2 \)).

- Two point vortices with opposite masses will translate with constant speed (travelling wave solutions).

- There are examples of four point vortices whose diameter spreads linearly in time. Total mass is zero.

- Self-similar point vortices: the configuration of point vortices have a spiral motion, evolving by rotation and dilation of order \( O(\sqrt{t}) \). The total mass is non zero.

It appears that at least three cases must be distinguished.

- Single signed masses;
- Total mass zero;
- Total mass non zero, not single signed masses.
Examples of smooth solutions:

- radial vorticity $\rightarrow$ stationary solution
- rotating ellipse
- steady vortex pairs (the Batchelor couple), Lamb 1945. Travelling wave solution.
Previous results on large time behavior:

- Single-signed vorticity
  - Confinement of the support of the vorticity:
    - Marchioro (94): $t^{\frac{1}{3}}$.
    - D.I., T. Sideris, P. Gamblin (99), Serfati (99): $(t \log t)^{\frac{1}{4}}$.

- Vorticity of vanishing total mass:
  - D.I., T. Sideris, P. Gamblin (99): smooth version of the four point vortices spreading as $O(t)$.

The diameter of the support of the vorticity grows like $O(t)$. 
The vorticity can go fast to infinity through the steady vortex pair mechanism.

Let us rescale the vorticity at the scale $x \simeq t^\alpha$:

$$\tilde{\omega}_\alpha(t, x) = t^{2\alpha} \omega(t, t^\alpha x), \quad \alpha \leq 1.$$  

This is bounded in $L^1$. Weak limits in the sense of measures exist as $t \to \infty$.

A weak limit of $\tilde{\omega}_\alpha$ as $t \to \infty$ ignores the “steady vortex pairs” the vorticity might generate.

But a weak limit of $|\tilde{\omega}_\alpha|$ would include the “steady vortex pairs” part of the vorticity.
\[ \tilde{\omega}_\alpha(t, x) = t^{2\alpha} \omega(t, t^\alpha x) \]


- If \( \alpha > \frac{1}{2} \), then

  \[ \tilde{\omega}_\alpha(t, x) \rightharpoonup m\delta_0, \quad m = \int \omega_0 \]

  as \( t \to \infty \).

- If the weak limit of \( |\omega_1| \) exists as \( t \to \infty \), then it must be a sum of Dirac masses.

First part is a weak confinement of the *imbalance* between the positive and negative parts of vorticity.

In the case of point vortices, the theorem fails for \( \alpha = \frac{1}{2} \).

**Question:**

Does the result fail for \( \alpha = \frac{1}{2} \) when the vorticity is smooth too?
A configuration of point vortices is called self-similar if

$$z_j(t) = \omega(t)z_j(0) \quad \forall j \in \{1, \ldots, n\}$$

and if it verifies the point-vortex system:

$$z_j'(t) = \sum_{k \in \{1, \ldots, n\} \setminus j} m_k \frac{(z_j - z_k) \perp}{2\pi|z_j - z_k|^2} \quad \forall 1 \leq j \leq n.$$ 

A direct calculation shows that \(\omega(t) = \sqrt{at + 1}e^{ib\frac{\ln(at+1)}{a}}\) so

$$z_j(t) = \sqrt{at + 1}e^{ib\frac{\ln(at+1)}{a}}z_j(0)$$

where \(a\) and \(b\) are real numbers and \(a > 0\).
Self-similar point vortices

Such self-similar configurations are known to exist:

- $n = 3$: Groebli 1877
- $n = 4, 5$: Novikov and Sedov 1979
- Arbitrary $n$: study by O’Neil 1987

We would like to construct a smooth version of these self-similar vortices.

One possible approach would be to start close to a configuration of self-similar vortices and show confinement around the point-vortices better than $t^{\frac{1}{2}}$.

Two problems must be dealt with:

- Instability of the self-similar configurations;
- Prove the confinement itself.

Here we will only deal with confinement issues. We will study a toy model.
Let

- \((m_j, z_j)_{1 \leq j \leq n}\) be a self-similar configuration of point-vortices;
- \(\omega_0\) be a smooth non-negative vorticity of mass \(m_1\) concentrated near \(z_1(0)\):

\[
\omega_0 \geq 0, \quad \int \omega_0 = m_1, \quad \text{supp} \omega_0 \subset D(z_1(0), \varepsilon)
\]

where \(\varepsilon\) is sufficiently small.

We assume that the vorticity is transported by the velocity field generated by itself and by the other point-vortices:

\[
\partial_t \omega + (u + F) \cdot \nabla \omega = 0
\]

where

\[
u(x) = \int \frac{(x - y)^\perp}{2\pi |x - y|^2} \omega(y) \, dy \quad \text{and} \quad F(x) = \sum_{j=2}^{n} m_j \frac{(x - z_j)^\perp}{2\pi |x - z_j|^2}
\]

We call this the toy model. It is not the well-known vortex-wave model. The instability issue is removed from the model.
We would like to know if confinement occurs for the toy model. More precisely:

**Definition:** We say that the toy model has the confinement property if for \( \varepsilon \) sufficiently small there exist some \( \beta < \frac{1}{2} \) and some constant \( C > 0 \) such that

\[
\text{supp} \omega(t, \cdot) \subset D(z_1(t), C(1 + t)^\beta) \quad \forall t \geq 0.
\]

Observe that

\[
F(z) = \sum_{j=2}^{N} m_j \frac{(z - z_j)}{2\pi |z - z_j|^2} = \sum_{j=2}^{N} \frac{im_j}{2\pi(z - z_j)}
\]

is antiholomorphic. We can derive with respect to \( \bar{z} \) to obtain

\[
\frac{\partial F}{\partial z}(z_1) = \frac{e^{-2ib \ln(at+1)}}{at + 1} \nu \quad \text{where} \quad \nu = \sum_{j=2}^{N} \frac{im_j}{2\pi(z_1(0) - z_j(0))^2}
\]

is a complex number which does not depend on time.
So
\[ \left| \frac{\partial F}{\partial z}(z_1) \right| = \frac{|v|}{at + 1}. \]

Let
\[ \alpha = \frac{|v|}{a} \] so that \[ \left| \frac{\partial F}{\partial z}(z_1) \right| \sim \frac{\alpha}{t} \quad \text{as } t \to \infty \]

We prove the following theorem:

**Theorem** (C. Marchioro, D.I., 2016)
If \( \alpha < \frac{1}{2} \) then the toy model has the confinement property.

**Remarks**
- Long-time confinement properties around point-vortices are interesting independently of the large time behavior issues mentioned before.
- Our toy model is in fact the Euler equation with a particular forcing. We can prove a confinement result for a general forcing but with a more stringent assumption (which would translate to \( \alpha < \frac{1}{4} \)).
We prove that if
\[ \text{supp } \omega(t, \cdot) \subset D(z_1(t), M(1 + t)\beta) \quad \forall t \in [0, T] \]
then
\[ \text{supp } \omega(t, \cdot) \subset D(z_1(t), \frac{M}{2}(1 + t)\beta) \quad \forall t \in [0, T] \]
Let
\[ I(t) = \int_{\mathbb{R}^2} |x - z_1|^2 \omega(t, x) \, dx \]
the moment of inertia with respect to \( z_1 \).

**First step:** Prove a bound for \( I \) (better than \( O(t^{2\beta}) \)):

**Lemma:** We have that \( I(t) \leq C\varepsilon^2(1 + t)^{2\alpha} \).
Proof of the Lemma: We differentiate \( I(t) = \int |x - z_1|^2 \omega(t, x) \, dx \):

\[
I'(t) = \int_{\mathbb{R}^2} |x - z_1|^2 \partial_t \omega(t, x) \, dx + 2 \int_{\mathbb{R}^2} z_1'(t) \cdot (z_1 - x) \omega(t, x) \, dx
\]

\[
= -\int_{\mathbb{R}^2} |x - z_1|^2 \text{div}((u + F)\omega) \, dx + 2 \int_{\mathbb{R}^2} z_1'(t) \cdot (z_1 - x) \omega(t, x) \, dx
\]

\[
= \int_{\mathbb{R}^2} \nabla(|x - z_1|^2) \cdot (u + F)\omega \, dx + 2 \int_{\mathbb{R}^2} z_1'(t) \cdot (z_1 - x) \omega(t, x) \, dx
\]

\[
= 2 \int_{\mathbb{R}^2} (x - z_1) \cdot (u + F)\omega \, dx + 2 \int_{\mathbb{R}^2} z_1'(t) \cdot (z_1 - x) \omega(t, x) \, dx
\]

\[
= 2 \int_{\mathbb{R}^2} (F(z_1) - F(x)) \cdot (z_1 - x) \omega(t, x) \, dx
\]

\[
\leq 2I \sup_{|x - z_1| \leq M(1 + t)^\beta} \frac{|F(x) - F(z_1)|}{|x - z_1|}
\]

\[
\leq 2I \left( \frac{\alpha}{1 + t} + C(1 + t)^{\beta - \frac{3}{2}} \right)
\]

We conclude that

\[
I(t) \leq I(0)e^{2\alpha \log(1+t)} + C \int_0^t (1 + s)^{\beta - \frac{3}{2}} \leq C \varepsilon^2 (1 + t)^{2\alpha}.
\]
Second step: Prove estimates for higher moments of the vorticity. Let

\[ I_n(t) = \int_{\mathbb{R}^2} |x - z_1|^n \omega(t, x) \, dx \]

Proposition: We have that \( I_n(t) \leq (Cn\varepsilon)^{\frac{n}{3}} (1 + t)^{(\alpha+1)\frac{n}{3}} \).

Proof of the Proposition: Along the same lines as in the lemma with the additional difficulty of estimating the velocity term that no longer vanishes. The first step allows to overcome this difficulty.
End of the proof. Let

\[ R(t) = \max_{x \in \text{supp } \omega(t)} |x - z_1| = |X - z_1| \]

Then

\[ R' = \partial_t |X - z_1| = \frac{X - z_1}{|X - z_1|} \cdot (\partial_t X - z_1') = \frac{X - z_1}{|X - z_1|} \cdot (u(t, X) + F(X) - F(z_1)) \]

But

\[ \frac{X - z_1}{|X - z_1|} \cdot (F(X) - F(z_1)) \leq R\left( \frac{\alpha}{1 + t} + C(1 + t)^{\beta - \frac{3}{2}} \right) \]  \[ \text{OK} \]

\[ \frac{X - z_1}{|X - z_1|} \cdot u(t, X) = \frac{X - z_1}{|X - z_1|} \cdot \int_{\mathbb{R}^2} \frac{(X - y)^\perp}{2\pi |X - y|^2} \omega(t, y) \, dy \]

\[ = \int_{|y - z_1| > \frac{R}{2}} \cdots + \int_{|y - z_1| < \frac{R}{2}} \cdots \leq \frac{C\varepsilon}{R^2} (1 + t)^{\alpha} + \frac{C\varepsilon^3}{R^n} (1 + t)^{\frac{n(\alpha + 1)}{3}} \]  \[ \text{OK} \]
Is the condition $\alpha < \frac{1}{2}$ necessary for confinement?

Assume that $\omega$ itself a point vortex $\omega = m_1 \delta_{\tilde{z}_1}$ evolving in the toy model. Assume by absurd that $\alpha \geq \frac{1}{2}$ and that $\tilde{z}_1 \in D(z_1(t), M(1 + t)^\beta)$ for some $\beta < \frac{1}{2}$. We look for a contradiction.

Let

$$f(t) = \tilde{z}_1 - z_1 = O((1 + t)^\beta)$$

Then

$$\bar{f}'(t) = \bar{F}(\tilde{z}_1) - \bar{F}(z_1) = (\tilde{z}_1 - z_1) \frac{\bar{F}(\tilde{z}_1) - \bar{F}(z_1)}{\tilde{z}_1 - z_1} \equiv uf$$

where

$$u = \frac{\partial \bar{F}}{\partial z}(z_1) + \frac{\bar{F}(\tilde{z}_1) - \bar{F}(z_1)}{\tilde{z}_1 - z_1} - \frac{\partial \bar{F}}{\partial z}(z_1) = \frac{\partial \bar{F}}{\partial z}(z_1) + O((1 + t)^{\beta - \frac{3}{2}})$$

so

$$|u| \sim \frac{\alpha}{t} \quad \text{as } t \to \infty.$$
Is the condition $\alpha < \frac{1}{2}$ necessary for confinement?

\[
\bar{f}'(t) = uf \quad \text{with} \quad |u| \sim \frac{\alpha}{t} \quad \text{as} \ t \to \infty.
\]

The ODE for $f$ can be written under the form

\[
f' = Af \quad \text{where} \quad A = \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix} \quad u = x + iy
\]

The matrix $A$ is symmetric and the eigenvalues are $\pm |u|$. The largest eigenvalue of $A$ equivalent to $\frac{\alpha}{t}$ as $t \to \infty$. This may suggest a growth for the solution like $t^\alpha$ as $t \to \infty$. This would be a contradiction.

This is not a proof that the condition $\alpha < \frac{1}{2}$ is necessary for confinement to occur (far from being one, in fact).

But it does show that our method is limited to the case $\alpha < \frac{1}{2}$. 
Can the condition $\alpha < \frac{1}{2}$ be satisfied?

Three point-vortices form a self-similar system if

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 0$$

and

$$\frac{|z_1 - z_2|^2}{m_3} + \frac{|z_2 - z_3|^2}{m_1} + \frac{|z_1 - z_3|^2}{m_2} = 0$$

One can compute

$$\alpha = \frac{|m_2 + m_3||z_2 - z_3|^2|m_2(z_1 - z_3)^2 + m_3(z_1 - z_2)^2|}{2|m_2m_3(|z_1 - z_2|^2 + |z_1 - z_3|^2)(z_1 - z_2) \cdot (z_1 - z_3) + (m_2^2 + m_3^2)(|z_1 - z_2|^2|z_1 - z_3|^2)}$$

Choose $|z_2 - z_3|$ of order 1 and $|z_1 - z_2|$ and $|z_1 - z_3|$ large.