Ancient Solutions to Geometric Flows

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Ancient and Eternal Solutions

- We will discuss ancient or eternal solutions to geometric parabolic partial differential equations.

- These are special solutions which exist for all time $-\infty < t < T$ where $T \in (-\infty, +\infty]$.

- They appear as blow up limits near a singularity.

- Understanding ancient and eternal solutions often sheds new insight to the singularity analysis.
In this talk we will address:

- the classification of ancient solutions to parabolic partial differential equations, with emphasis to geometric flows: Mean Curvature flow, Ricci flow and Yamabe flow.

- methods of constructing new ancient solutions from the gluing of two or more solitons (self-similar solutions).

- new techniques and future research directions.
Ancient and Eternal solutions

- **Definition:** A solution \( u(\cdot, t) \) to a parabolic equation is called ancient if it is defined for all time \(-\infty < t < T, \ T < +\infty\).

- Ancient solutions typically arise as blow up limits at a type I singularity.

- **Definition:** A solution \( u(\cdot, t) \) to a parabolic equation is called eternal if it is defined for all \(-\infty < t < +\infty\).

- Eternal solutions as blow up limits at a type II singularity.
Solitons (self-similar solutions) are typical examples of ancient or eternal solutions and often models of singularities.

A well known technique introduced by R. Hamilton (1995) has been widely used to characterize as solitons the eternal solutions to geometric flows which attain a space-time curvature maximum.

Such solutions typically appear as carefully chosen blow up limits near type II singularities.

Its proof relies on a clever combination of the strong maximum principle and Li-Yau type differentiable Harnack estimates.
However, there exist other ancient or eternal solutions which are not solitons.

These, often may be visualized as obtained from the gluing as $t \to -\infty$ of two or more solitons.

Obtaining more information about such solutions, often leads to the better understanding of the singularities.

Objective: How to construct such solutions and how to characterize them among all ancient or eternal solutions.
Liouville’s theorem for the heat equation on manifolds

Let $M^n$ be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(M^n) \geq 0$.

Yau (1975): Any positive harmonic function $u$ on $M^n$ must be constant.

This is the analogue of Liouville’s Theorem for harmonic functions on $\mathbb{R}^n$.

Question: Does the analogue of Yau’s theorem hold for positive solutions of the heat equation

$$u_t = \Delta u \quad \text{on } M^n?$$

Answer: No. Example $u(x, t) = e^{x_1 + t}, x = (x_1, \cdots, x_n)$ on $M^n := \mathbb{R}^n$. 

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A Liouville type theorem for the heat equation

Souplet - Zhang (2006): Let $M^n$ be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(M^n) \geq 0$.

(a) If $u$ be a positive ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that

$$u(p, t) = e^{o(d(p) + \sqrt{|t|})} \text{ as } d(p) \to \infty$$

then $u$ is a constant.

(b) If $u$ be an ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that

$$u(p, t) = o(d(p) + \sqrt{|t|}) \text{ as } d(p) \to \infty$$

then $u$ is a constant.

Proof: By using a local gradient estimate of Li-Yau type on large appropriately scaled parabolic cylinders.
Consider positive solutions $u > 0$ of the semilinear elliptic equation

$$\Delta u + f(u) = 0, \quad \text{on } \mathbb{R}^n.$$ 

Well known example related to the Yamabe problem is $f(u) = u^{\frac{n+2}{n-2}}$.

Gidas, Ni and Nirenberg (1979): Solutions $u > 0$ with mild decay condition as $|x| \to +\infty$ are rotationally symmetric.

Many related important subsequent results including those by Cafarelli, Gigas and Spruck and Berestycki and Nirenberg.
We will next discuss parabolic Liouville type results related to the blow up analysis of solutions $u(\cdot, t) \in H^1(\mathbb{R}^n)$ of equation

$$\text{(SL)} \quad u_t = \Delta u + |u|^p \quad \text{on} \quad \mathbb{R}^n \times (0, T), \quad 1 < p < \frac{n + 2}{n - 2}.$$ 

We will assume for simplicity that $u > 0$.

Semilinear equations similar to (SL) appear in the blow up analysis of geometric flows.

In particular in neckpinches of solutions to the Ricci flow or the Mean Curvature flow.

Also in the analysis of the blow downs as $t \to -\infty$ of ancient solutions.

We say that the solution $u$ of (SL) blows up in finite time $T$ if

$$\lim_{t \to T} \|u(t)\|_{H^1(\mathbb{R}^n)} = +\infty.$$
The rescaled semi-linear heat equation

- **Self-similar scaling:**

\[ w(y, \tau) = (T-t)^{1 \over p-1} u(x, t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad \tau = -\log(T-t). \]

- **Giga - Kohn (1985):** \( \|w(\tau)\|_{L^\infty(\mathbb{R}^n)} \leq C, \ \tau > -\log T. \)

- The rescaled solution satisfies the equation

\[ (*) \quad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p. \]

- **Objective:** To analyze the blow up behavior of \( u \) one needs to understand the long time behavior of \( w \) as \( \tau \to +\infty. \)

- This is closely related to the classification of bounded eternal solutions of \( (\ast) \).
Consider bounded positive eternal solutions of equation

\[(\star)\quad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.\]

Steady states: \(w = 0\) or \(w = \kappa\), with \(\kappa := (p - 1)^{-\frac{1}{p-1}}\).

Giga - Kohn: The monotonicity of the Lyapunov functional implies: \(\lim_{\tau \to \pm \infty} w(\cdot, \tau) = \text{steady state}\).

ODE solutions: \(\phi(\tau) = \kappa (1 + e^\tau)^{-\frac{1}{p-1}}\).

Theorem (Giga - Kohn ’87 and Merle - Zaag ’98)
If \(w\) is bounded positive eternal solution of \((\star)\) then

\(w = 0\) or \(w = \kappa\) or \(w(\tau) = \phi(\tau - \tau_0)\).
Giga and Kohn: The monotonicity of the Lyapunov functional implies: If \( \lim_{\tau \to \pm \infty} w(\cdot, \tau) = 0 \) or \( \lim_{\tau \to \pm \infty} w(\cdot, \tau) = \kappa \), then \( w \equiv 0 \) or \( w \equiv \kappa \) respectively.

Merle - Zaag: Classified the orbits \( w(\cdot, t) \) that connect \( \lim_{\tau \to -\infty} w(\cdot, \tau) = \kappa \) with \( \lim_{\tau \to +\infty} w(\cdot, \tau) = 0 \).

This involves the careful analysis of the projections of \( w(\cdot, t) \) as \( \tau \to -\infty \) into the positive, null and negative eigenspaces of the linearized operator at \( w = \kappa \).

Elliptic analogues: Similar in spirit with the elliptic results by Gidas-Ni-Nirenberg '79 and Berestycki-Nirenberg '91 but without using the moving plane method.

Merle - Zaag (2000): Generalized the result to the vectorial case (in particular without the sign assumption).
Although *Liouville type results* often appear with respect to *elliptic equations*, there are not many such results available in the *parabolic setting*.

G. Koch, N. Nadirashvili, G. Seregin and V. Sverak (2009):
(i) *Liouville type* result for ancient *bounded* solutions \( u(x, t) \) of the 2-dim Navier Stokes equations.

(ii) Also, similar result for bounded, *axi-symmetric with no swirl* solutions \( u(x, t) \) of the 3-dim Navier Stokes equations.
Let $\Gamma_t$ be a family of closed curves which is an embedded solution to the Curve shortening flow, i.e. the embedding $F : \Gamma_t \to \mathbb{R}^2$ satisfies

$$\frac{\partial F}{\partial t} = -\kappa \nu$$

with $\kappa$ the curvature of the curve and $\nu$ the outer normal.

- Gage (1984); Gage and Hamilton (1996); Grayson (1987): the CSF shrinks $\Gamma_t$ to a round point.

- Problem: Classify the ancient compact embedded solutions to the Curve shortening flow.
The curvature $\kappa$ of $\Gamma_t$ evolves, in terms of its arc-length $s$, by

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

**Definition:** $\Gamma_t$ is **type I** if $\limsup_{t \to -\infty} \sqrt{|t|} \max_{\Gamma_t} \kappa(\cdot, t) < \infty$. Otherwise, $\Gamma_t$ is of **type II**.

**Type I solution:** the contracting circles.

**Type II solution:** the Angenent ovals (paper clips). These are ancient convex solutions in closed form which are not solitons.
The Classification of Ancient Convex solutions to the CSF

- The Angenent ovals (paper clips) as $t \to -\infty$ may be visualized as two grim reaper solutions glued together.

**Theorem (D., Hamilton, Sesum - 2010)**
The only ancient convex solutions to the CSF are the contracting spheres or the Angenent ovals.

**Proof:** It is based on various monotonicity formulas and the fact that at its singular time any solution becomes circular with very sharp rates of convergence.
Question: Do they exist non convex compact embedded solutions to the curve shortening flow?

Angenent (2011): Presents a YouTube video of an ancient solution to the CSF built out from one Yin-Yang spiral and one Grim Reaper.

S. Angenent: is currently working on a rigorous construction of these solutions.
The Mean curvature flow

Let $F : N^n \times (-\infty, T) \rightarrow \mathbb{R}^{n+1}$ be a smooth ancient compact and convex solution of the Mean curvature flow

\[
\frac{\partial F}{\partial t} = -H \nu
\]

where $H(p, t)$ denotes the Mean curvature of the surface $M_t := F(\cdot, t)(N^n)$ at the point $F(p, t)$.

The only ancient compact and convex self-similar solutions to the MCF are the contracting spheres.

Problem: Understand ancient compact solutions $M_t$ of the Mean curvature flow.
Weimin Sheng and Xu-Jia Wang; Ben Andrews: Introduced an $\alpha$-noncollapsed condition.

$$B = B \frac{\alpha}{H(p)}$$

Definition. We call an ancient oval to be any ancient compact and $\alpha$-noncollapsed solution to MCF that is not self-similar.

Haslhofer & Kleiner (2013): Ancient ovals are convex at each instant $t$.

One may construct various examples of ancient compact and collapsed solutions of the MCF.
Ancient MCF ovals

- **White (2003); Haslhofer & Hershkovits (2013):** Establish the existence of ancient ovals with $O(k) \times O(l)$ symmetry. We will refer to them as **White ancient ovals**.

- **Angenent (2012):** Establishes the formal matched asymptotics of the White ancient ovals as $t \rightarrow -\infty$. He shows that are small perturbations of **ellipsoids**.

- **Problem:** Show the uniqueness of ”White” ovals.
Unique asymptotics of Ancient MCF ovals

- S. Angenent, D., and N. Sesum (2015): All ancient ovals with $O(1) \times O(n)$ symmetry have unique asymptotics as $t \to -\infty$.

- In addition we establish their precise asymptotic description at $t = -\infty$ as formally predicted by Angenent.

- It follows from our analysis that as $t \to -\infty$:

  \[
  \text{diam}(t) \approx \sqrt{8|t| \log|t|} \quad \text{and} \quad H_{\text{max}}(t) \approx \frac{\sqrt{\log|t|}}{\sqrt{2|t|}}.
  \]

- The proof involves: the analysis of the linearized operator at the cylinder, Huisken’s monotonicity formula and carefully constructed barriers at the intermediate region.
Uniqueness of Ancient MCF ovals

- **Work in progress**: to establish such asymptotics in the non-symmetric case.

- **Next Step**: Establish the uniqueness of the Ancient ovals.

- **Conjecture 1**: The Ancient ovals with $O(l) \times O(k)$ symmetry are uniquely determined by their asymptotics at $t \to -\infty$.

- **Hence**: they are unique (up to dilation and translation in rescaled time).

- **Conjecture 2**: All Ancient ovals are $O(l) \times O(k)$ symmetric.
Ancient compact solutions to the Ricci flow

We will discuss next ancient solutions to the Ricci flow on a compact manifold $M^n$.

The Ricci flow of a Riemannian metric $g_{ij}$ on a complete manifold $M^n$ is the evolution

$$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}.$$ 

by the Ricci curvature of the metric $g_{ij}$ on $M^n$.

R. Hamilton (1981): introduced this flow in his seminal work as an analytical and geometric tool to approach the resolution of the geometrization conjecture of William Thurston.


Many many other important works!
Consider an ancient solution of the Ricci flow

\[
\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}
\]

on a compact manifold \( M^2 \) which exists for all time \(-\infty < t < T\) and becomes singular at time \( T \).

In dim 2, we have \( R_{ij} = \frac{1}{2} R g_{ij} \), where \( R \) is the scalar curvature.

Hamilton (1988), Chow (1991): After re-normalization, the metric becomes spherical at \( t = T \).

Problem: Provide the classification of all ancient compact solutions.
Ancient compact solutions to the 2-dim Ricci flow

- We choose \( g_{S^2} = d\psi^2 + \cos^2 \psi \ d\theta^2 \) a parametrization of the limiting spherical metric.

- If \( g(\cdot, t) = u(\cdot, t) \ g_{S^2} \), then the (RF) becomes equivalent to:

\[
    u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (-\infty, T).
\]

- It is natural to consider the pressure function \( v = u^{-1} \) which evolves by

\[
    v_t = v \Delta_{S^2} v - |\nabla v|^2 + 2v^2.
\]

- **Definition:** An ancient solution is called of type I if:

\[
    \limsup_{t \to -\infty} (|t| \max_{S^2} R(\cdot, t)) < \infty.
\]

Otherwise, it is called of type II.
Examples of compact solutions on $S^2$

- **Type I** solution: the contracting spheres.

- **Type II** solution: the King-Rosenau solution given $\forall \mu > 0$ by:

  $$v(\psi, t) = \mu [\coth(2\mu(T - t)) - \tanh(2\mu(T - t)) \sin^2 \psi]$$

  As $t \to -\infty$ the King-Rosenau solution looks like two cigar solitons glued together.

- The King-Rosenau solutions are not solitons. For this reason it is difficult to capture them.
The classification result

**Theorem:** (D., Hamilton, Sesiun - 2012)

An *ancient* solution to the Ricci flow on $S^2$ is either one of the contracting spheres or one of the King-Rosenau solutions.

**Proof:** The proof of this result involves:

- a *monotonicity formula* and uniform *a priori* $C^{1,\alpha}$ *estimates* that allow us to pass to the limit as $t \to -\infty$.
- *geometric arguments* that allow us to classify the backward limit as $t \to -\infty$.
- *maximum principle* arguments that allow us to characterize the King-Rosenau solutions among type II solutions.
- an *isoperimetric inequality* that allow us to characterize the contracting spheres among type I solutions.
The characterization of King solutions

- To capture the King solutions we consider the scaling invariant nonotone quantity

$$Q(x, y, t) := \bar{v} \left[ (\bar{v}_{xxx} - 3\bar{v}_{xxy})^2 + (\bar{v}_{yyy} - 3\bar{v}_{xyy})^2 \right]$$

where $\bar{v} := \bar{u}^{-1}$ is the pressure in plane coordinates.

- Using complex variable notation $z = x + iy$, this quantity is nothing but

$$Q = \bar{v} |\bar{v}_{zzz}|^2.$$

- The quantity $Q$ is well defined.

- It turns out that $Q \equiv 0$ implies that $\bar{v}$ is one of the King solutions.

- To establish that $Q \equiv 0$ we prove that:
  
  i. $Q_{\text{max}}(t)$ is decreasing in $t$ (by considering its evolution equation), and
  
  ii. $\lim_{t \to -\infty} Q_{\text{max}}(t) = 0.$
3-dim Ricci flow: The analogue of the 2-dim King-Rosenau solutions have been shown to exist by G. Perelman. They are not given in closed form, they are type II and k-noncollapsed.

Other collapsed compact solutions in closed form have been found by V.A. Fateev in a paper dated back to 1996.

Conjecture: The only k-noncollapsed ancient and compact solutions to the 3-dim Ricci flow are the contracting spheres and the Perelman solutions.

Brendle, Huisken & Sinestrari (2011): Present a pinching curvature condition that characterizes the ancient compact solutions to the 3-dim Ricci as contracting spheres.
We will conclude by discussing ancient solutions \( g = g_{ij} \) of the Yamabe flow on \( S^n, \ n \geq 3 \).

This is the evolution of metric \( g(\cdot, t) \) conformally equivalent to the standard metric on \( S^n \) by

\[
\frac{\partial g}{\partial t} = -R g \quad \text{on} \quad -\infty < t < T
\]

where \( R \) denotes the scalar curvature of \( g \).

The Yamabe flow may be viewed as the higher dimensional analogue of the 2-dim Ricci flow.

**Question:** Is it possible to provide the classification of all such ancient solutions?
Let \((M^n, g_0),\ n \geq 3\) be a compact manifold without boundary. The scalar curvature \(R\) of a metric \(g = v^{\frac{4}{n-2}} g_0\) conformal to \(g_0\) is given by

\[
R = -v^{-\frac{n+2}{n-2}} \left( c_n \Delta g_0 v - R_0 v \right)
\]

where \(R_0\) denotes the scalar curvature of \(g_0\).

R. Hamilton (1989): introduced the Yamabe flow as a parabolic approach to resolve the Yamabe problem.

S. Brendle (2007): convergence of the normalized flow to a metric of constant scalar curvature (up to a mild technical assumption for \(\dim n \geq 6\)).

Previous important works: Hamilton '89, Chow '92, Ye '94, Schwetlick-Struwe '2003.
Ancient solutions to the Yamabe flow on $S^n$

- Let $g = \nu^{\frac{4}{n-2}} g_{S^n}$ be an ancient solution to the Yamabe flow, which is conformal to the standard metric on $S^n$.

- The function $\nu$ evolves by the fast diffusion equation

$$
(\nu^{\frac{n+2}{n-2}})_t = \Delta_{S^n} \nu - c_n \nu \quad \text{on } S^n \times (-\infty, T).
$$

- Let $g = \bar{\nu}^{\frac{4}{n-2}} g_{\mathbb{R}^n}$ after stereographic projection. Then,

$$
(\bar{\nu}^{\frac{n+2}{n-2}})_t = \Delta \bar{\nu} \quad \text{on } \mathbb{R}^n \times (-\infty, T).
$$

- **Definition:** An ancient solution is called of **type I** if:

$$
\limsup_{t \to -\infty} \left( |t| \max_{S^n} |Rm| (\cdot, t) \right) < \infty.
$$

Otherwise, it is called of **type II**.
J. King (1993): discovered non-self similar type I ancient compact solutions to the (YF) on $S^n$ in closed form.

**King solutions:** $g = \hat{v}_K(\cdot, t)^\frac{4}{n-2} g_{\mathbb{R}^n}$, where

$$\hat{v}_K(x, t) = \left( \frac{a(t)}{1 + 2b(t)|x|^2 + |x|^4} \right)^{\frac{n-2}{4}}, \quad x \in \mathbb{R}^n.$$

As $t \to -\infty$ they converge (after rescaling) to two Barenblatt type self-similar solutions (shrinking solitons) joined by a long cylindrical neck.

$t \approx -\infty$
Problem:
Understand the ancient solutions $g = v^{\frac{4}{n-2}} g_{S^n}$ to the Yamabe flow, conformal to the standard metric on $S^n$.

This is equivalent to studying ancient positive and smooth solutions $v : S^n \times (-\infty, T) \rightarrow \mathbb{R}$ of the fast diffusion equation

$$(v^{\frac{n+2}{n-2}})_t = \Delta_{S^n} v - c_n v.$$ 

Question 1:
Are the contracting spheres and the King solutions the only examples of type I ancient solutions?

Question 2:
Are there any type II ancient solutions?
**Question:** Are the spheres and the King solutions the only examples of type I ancient solutions?

**Recent work:** (D., del Pino, J. King and N. Sesum - 2015)
There exist infinite many other type I ancient solutions.

As $t \to -\infty$ they look as two traveling waves moving away from each other with a given speed.

We refer to them as ancient converging traveling waves.
Complete traveling waves with cylindrical behavior

- We look for special rotationally symmetric solutions of the (YF) expressed in cylindrical coordinates $g = v^{\frac{4}{n-2}} g_{cyl}$.

- $v(x, \tau)$ satisfies (up to a type I rescaling) the equation:

  $$(*) \quad \left(v^{\frac{n+2}{n-2}}\right)_\tau = v_{xx} - v + v^{\frac{n+2}{n-2}}.$$

- $\forall \lambda \geq 1$, there exist traveling waves $v_{\lambda, h} := v_{\lambda}(x - \lambda \tau + h)$ of $(*)$ with behavior $v_\lambda \approx 1 - C_\lambda e^{-\gamma_\lambda x}$, as $x \to +\infty$.

- They correspond to complete non-compact conformally flat solitons of the (YF) with cylindrical behavior at infinity.

- Also the space independent solutions $\xi_k(\tau)$ of $(*)$ correspond to the cylindrical solution.

- Theorem: (D., J. King and N. Sesum) $L^1$ stability of the traveling wave solutions $v_{\lambda, h}$.
Theorem: (D., del Pino, J. King and N. Sesum)  
For any \((\lambda, \lambda', h, h', k) \in \mathbb{R}^5\) such that \(\lambda, \lambda' > 1, \ k \geq 0\), there exists an ancient solution \(v_{\lambda, \lambda', h, h', k}\) of (*) given in terms of the pressure function \(f := v^q, \ q := -\frac{4}{n-2}\) by:

\[
v_{\lambda, \lambda', h, h', k} \approx v_{\lambda, h}^q(x, \tau) + \xi_k(\tau)^q + v_{\lambda', h'}^q(-x, \tau)
\]

which defines a smooth ancient type I solution of the Yamabe flow on \(S^n \times (-\infty, T)\) with nonnegative Ricci curvature.

Proof: By the construction of precise ancient barriers.

Hamel and Nadirashvili (1999): Constructed such ancient solutions for the KPP equation \(u_t = u_{xx} + f(u), \ x \in \mathbb{R}\).
Question: Are there any type II ancient solutions to (YF) ?

D., del Pino and Sesum (2013):
We construct a class of ancient solutions of the Yamabe flow on $S^n$ which (after re-normalization) converge as $t \to -\infty$ to a tower of n-spheres. They are rotationally symmetric.

The curvature operator in these solutions changes sign and they are of type II.

Our construction also holds for any number of bubbles.
Our construction can be viewed as a parabolic analogue of the gluing technique which has been used in the past in various geometric elliptic settings.

We refer to the pioneering works of Kapouleas ’90 -’95 and also to the fundamental works of Mazzeo, Pacard, Pollack, Ulhenbeck and del Pino, Dolbeault, Musso among others.

Parabolic gluing methods could be potentially used for the construction of ancient solutions in other geometric flows.

In fact, we refer to the recent work of Brendle & Kapouleas (2014) where they use gluing techniques to construct a new ancient compact solution to the 4-dim Ricci flow.
Conclusion

- We discussed ancient solutions to geometric parabolic PDE.
- Typical examples are either solitons or other special solutions obtained from the gluing as $t \to -\infty$ of solitons.
- The classification of ancient solutions often contributes to the better understanding of the formation of singularities.
- The only existing classification results heavily rely on knowing the exact form of these ancient solutions.
- Future research direction: develop new techniques that allow us to characterize and construct other types of ancient or eternal solutions.