# On Erdős - Ko - Rado for random hypergraphs

recent (25-30 years?) theme:

when* does [thm X] hold*

<u>in a random setting</u>?

<u>E.g.</u> Thm X's:

Ramsey (R)

Turán (T)

Szemerédi (S)

* for what $p$

* probably

Names (some)

Frankl - Rödl (R, T)

Babai - Simonovits - Spencer (T)

Rödl - Ruciński (R)

Kohayakawa - Łuczak - Rödl
Conlon - Gowers
Schacht
} (S, T)

Brightwell - Panagiotou - Steger
De Marco - Kahn
} (T)

Balogh - Morris - Samotij
Saxton - Thomason
} (!)

## "sparse random" Erdős-Ko-Rado

$n > 2k$, $\mathcal{K} = \binom{[n]}{k}$, in $\mathcal{K}$:

- clique: intersecting family
- trivial: $\cap \neq \emptyset$ [star, princ. int'g fam]

BBM*: $\mathcal{H} \subseteq \mathcal{K}$ is (strongly) EKR if

every largest clique is a star

$\boxed{\text{EKR thm: } \mathcal{K} \text{ is EKR}}$

* Balogh, Bohman, Mubayi

Q: when is $\boxed{\mathcal{H}} = \mathcal{H}_k(n,p)$ (probably) EKR?

what you think

* not an increasing property $\xrightarrow{\text{let's say}}$

"threshold": $\min p_0 \ni$

$\Pr(\mathcal{H}_k(n,p) \text{ is EKR}) \geq \frac{1}{2} \quad \forall p \geq p_0$

(but usually look for EKR a.s.)

"Hidden param":

$$k = k(n)$$

$$p = p(n)$$

$$\mathcal{H} = \mathcal{H}_k(n,p) \qquad (= \mathcal{H}(n))$$

$$\text{etc.} \qquad (= \text{etc.}(n))$$

[ when is $\mathcal{H}_k(n,p)$ EKR? ]

Vague: change at $k \approx \sqrt{n}$

extreme: $n = 2k+1 \longrightarrow p_0 \gtrsim 3/4$

Hilton-Milner (H-M) fam:

$\text{star}(x) \cup \{A\}$ $(A \not\ni x)$

Q (BBM):

$n = 2k+1 \longrightarrow p_c < .99$ ?

& more gen'ly ...

[ Q (BBM): $n = 2k+1$ $p_c < .99$ ? ]

Thm 1 (Hamm-K) $\exists\, \varepsilon > 0 \ni$

$$\mathcal{H}_k(\underbrace{2k+1}_{n}, \underbrace{1-\varepsilon}_{p}) \text{ is EKR (a.s.)}$$

a.a.s.

GUESS: H-M the bottleneck ("obstruction"?)

e.g. (conj) $p_c \to 3/4$

e.g. $n = 2k + 2$: conj $p_c \asymp k^{-1}$

(don't know $o(1)$)

Remk: For pf of thm 1

main issue is cliques $\bar{w}$ small max deg.

(not H-M & such . . .)

Smaller k: H-M the bottleneck?

. . . NO . . .

(smaller $k$) <u>notation:</u> $\boxed{\mathcal{H} = \mathcal{H}_k(n,p)}$

$\boxed{\varphi} = \binom{n-1}{k-1} p = \mathbb{E}\deg.$ [more natural...]

$\varphi_0 = \binom{n-1}{k-1} p_0$

$\boxed{m} = \mathbb{E}|\mathcal{H}| = \varphi n/k$

$\boxed{q} = \Pr(A \cap B \neq \emptyset) \approx 1 - e^{-k^2/n}$ [$A, B$ indept unif $k$-sets]

given $\varphi$:

$\boxed{\Lambda(t)}$ $(= \Lambda_\varphi(t)) = \binom{m}{t} q^{\binom{t}{2}}$

$\Pr(A_1, \ldots, A_t \text{ clique})$ if $\{A_i \cap A_j \neq \emptyset\}$'s indept

$$\left[\ \boxed{\varphi} = \binom{n-1}{k-1} p\,; \qquad \boxed{m} = \mathbb{E}\,|\mathcal{H}|\,; \qquad \boxed{q} = \Pr(A \cap B \neq \emptyset)\,; \quad \boxed{\Lambda(t)} = \binom{m}{t}\, q^{\binom{t}{2}}\ \right]$$

think: $\Lambda(t)$ : $\mathbb{E}\,|\{\text{generic } t\text{-cliques}\}|$ (??)

if so then "EKR a.s." should need

$$\boxed{\Lambda(\Delta) < o(1) \text{ a.s.}} \quad (*)$$

$\Delta = \Delta_{\mathcal{H}}$ (always)

[ if $(*)$ fails then EKR should fail w.p.p. (??) ]

[ with pos. prob.

[ $\boxed{\Lambda(\Delta) < o(1) \text{ a.s.}}$ — $\circledast$ ]

<u>Thm 2</u> (H-K) For $\boxed{k < \sqrt{\left(\frac{1}{4}-\varepsilon\right) n \log n}}$ — (A)

$\ln$

$\circledast \Longrightarrow \mathcal{H}$ is EKR (a.s.)

$\boxed{\text{"}\Longleftarrow\text{" ?}}$

essentially yes $\left\{ \begin{array}{l} \text{for (A)} \\ \text{in gen'l ?} \quad \text{(probably)} \end{array} \right.$

watch $\Delta \leq 2$

$$\underline{\Lambda'(t)} := \begin{cases} 0 & \text{if } t \leq 2 \\ \Lambda(t) \left(= \binom{m}{t} q^{\binom{t}{2}}\right) & \text{o'wise} \end{cases}$$

$\Lambda'_q(t)$

**Thm 2'** (H-K) For $k < \sqrt{\left(\frac{1}{4} - \varepsilon\right) n \log n}$

$$\boxed{\mathcal{H} \text{ is EKR a.s.} \iff \Lambda'(\Delta) < o(1) \text{ a.s.}}$$

GUESS:

▸ $\circledast \Longrightarrow$ EKR up to $k \approx \sqrt{\frac{1}{2} n \log n}$

(vs $\boxed{k \lesssim \sqrt{\frac{1}{4} n \log n}}$ — A )

▸ then H-M

Optimistic Conj $\forall k$ if $\varphi \vDash \circledast$ and a.s. each $\mathcal{H}_x$ is a max'l clique of $\mathcal{H}$ then $\mathcal{H}$ sat. EKR a.s.

Recall:

**Thm 1** (Hamm-K) $\exists \varepsilon > 0 \ni$

$$\mathcal{H}_k(\underbrace{2k+1}_{n}, \underbrace{1-\varepsilon}_{p}) \text{ is EKR a.s.}$$

Proof mumbles $\longrightarrow$

# Setting up:

$$\mathbb{X} = \mathcal{H}_k(n, p)$$

$$\left(\mathcal{K} = \binom{[n]}{k}\right) \quad K_x = \text{star}(x)$$

<u>Natural:</u> $\mathcal{M} = \{$ max'l nontriv cliques in $\mathcal{K}\}$

<u>want:</u>

$$\boxed{\max_{g \in \mathcal{M}} |\mathbb{X} \cap g| < \max_{x} |\mathbb{X} \cap K_x|}$$

$K_x$'s are maximum cliques (in $\mathcal{K}$)

— size $M = \binom{n-1}{k-1} \approx 4^k$

E.g. we cannot say: $\exists\, t$ s.t. a.s.

$$\max_{g \in \mathcal{M}} |\mathbb{X} \cap g| < t \le \max_{x} |\mathbb{X} \cap K_x|$$

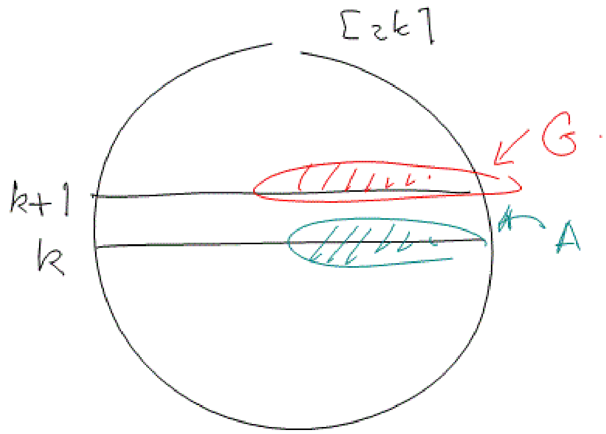
$\longrightarrow$ need direct comparison of

$|\mathbb{X} \cap g|$, $|\mathbb{X} \cap K_x|$ for approp. $x$.

... slightly long story ...

MAIN [after a while] $\quad A \subseteq \binom{[2k]}{k} =: \Gamma_k$

closed*




$G = \nabla(A)$ upper shadow

$\subseteq \binom{[2k]}{k+1} =: \Gamma_{k+1}$

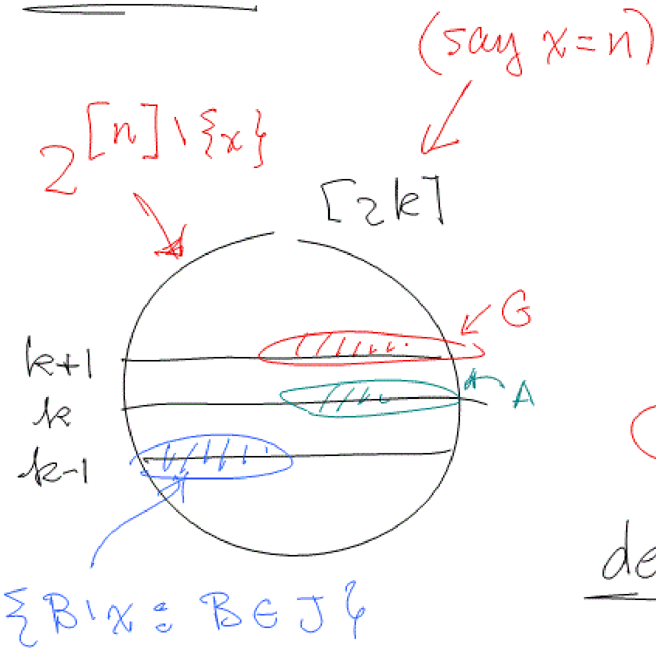
$\mathbb{X} = (\Gamma_k \cup \Gamma_{k+1})_p$

want: $|\mathbb{X} \cap G| > |\mathbb{X} \cap A|$ $\forall$ relevant $A$.

* closed : $x \in \Gamma_k,\ \nabla(x) \subseteq \nabla(A) \Rightarrow x \in A$

$\longrightarrow$ $G$ det. $A$ $\left(= \{x : \nabla(x) \subseteq G\}\right)$

## whence:




$A = g \setminus \mathcal{K}_x \quad (\subseteq \Gamma_k)$

$J = \mathcal{K}_x \setminus g$

$G = J^c = \{[n] \setminus T : T \in J\}$

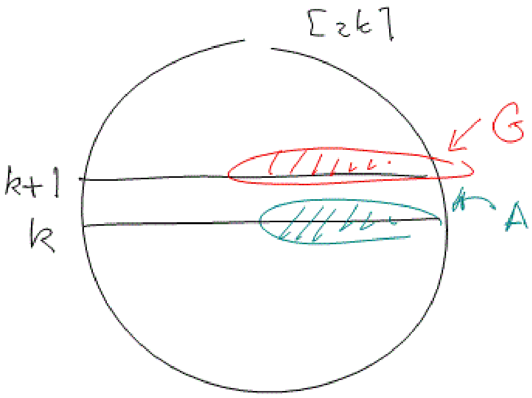
define $\mathbb{X} \cap G = (\mathbb{X} \cap J)^c$

$\longrightarrow$

$$\underbrace{|\mathbb{X} \cap \mathcal{K}_x| - |\mathbb{X} \cap g|}_{\parallel} = \boxed{|\mathbb{X} \cap G| - |\mathbb{X} \cap A|}$$

$$|\mathbb{X} \cap (\mathcal{K}_x \setminus g)| - |\mathbb{X} \cap (g \setminus \mathcal{K}_x)|$$

$> 0$ ?




$$\boxed{\begin{array}{l}|\mathcal{X} \cap G| > |\mathcal{X} \cap A| \\ \qquad \forall \text{ relevant } A\end{array}} \ ?$$

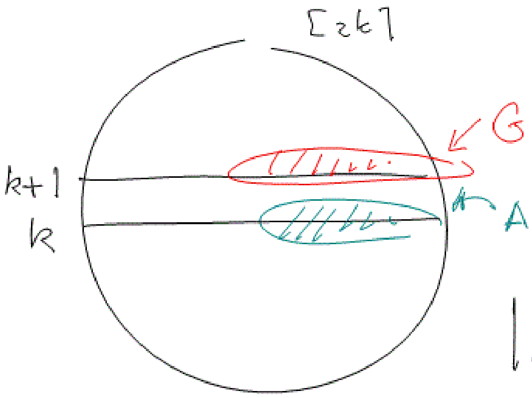
$|A| = \boxed{a}$

$|G| = \boxed{g} = (1+\boxed{\delta})\, a$

smaller $\delta$ is harder

Kruskal-Katona + Frankl $\to$

WMA $\delta$ not too small: $\boxed{\delta > c/k}$




$$\boxed{|\mathcal{X}\cap G| > |\mathcal{X}\cap A| \quad \forall \text{ relevant } A} \; ?$$

$|A| = a$, $|G| = g = (1+\delta)a$

(think: $\delta = c/k$)

OKAY if

$|\mathcal{X}\cap G|, |\mathcal{X}\cap A|$ off by $< \delta a p / 3$ say.

from their $\mathbb{E}$'s

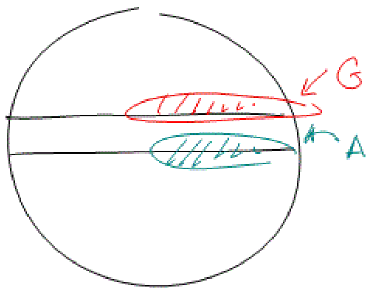
union bound?

Certainly not

$$\# \text{ of } A\text{'s} \approx \binom{|\Gamma_k|}{a} \approx \exp\left[ a \log \frac{|\Gamma_k|}{a} \right]$$

$$\Pr(\mathbb{X} \cap A \text{ bad}) \approx \exp\left[ -\delta^2 a \right]$$

what did I tell you?




... build $A, G$ ... e.g.:

$$A = \underbrace{(A \setminus A^*_{R(A)})}_{\text{small}} \sqcup \underbrace{(A^*_{R(A)}}_{\substack{\text{large but}\\ \text{not too many}}} \setminus \underbrace{(A^*_{R(A)} \setminus A)}_{\text{small}})$$

$$\left[\ A^*_R = S'_R \setminus (S'_R \setminus A^*_R) \sqcup (A^*_R \setminus S'_R)\right.$$

$$\vdots \qquad \rfloor$$

... via <u>successive approximations</u>

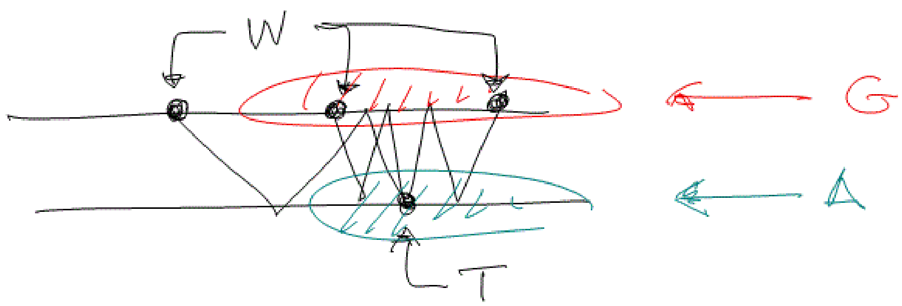
("Sapozhenko's method") $\longrightarrow$

talking points: ① basic idea comes up in many places (e.g containers)

② "tradeoff" makes it v. nat. in our sit.

e.g. <u>1st approx to G</u>:

$T \subseteq A$: $\quad W = W_T = N^3(T) \cap \Gamma_{k+1}$




<u>want</u>:
- $W \approx G$
- $T$ small $\to$ few choices

[eventually: $(\#\text{ of } T\text{'s}) \cdot \Pr(X \cap W \text{ bad})$]

(if I haven't run out of time $\to$)

<u>Claim</u>: $\exists\, T \subseteq A$ s.t.

(T1) $|T| < C \boxed{a k^{-3}} k^{\delta} \log k$    [$\delta$ fixed, small (crucial but we won't see why)]

(T2) $|W_T \,\Delta\, G| < C \boxed{\delta a} k^{\delta} \log k$

small? think $\delta \simeq 1/k$

<u>sketch</u>:

$T$ random (of course) $\longleftrightarrow$ $C k^{-3+\delta} \log k$.

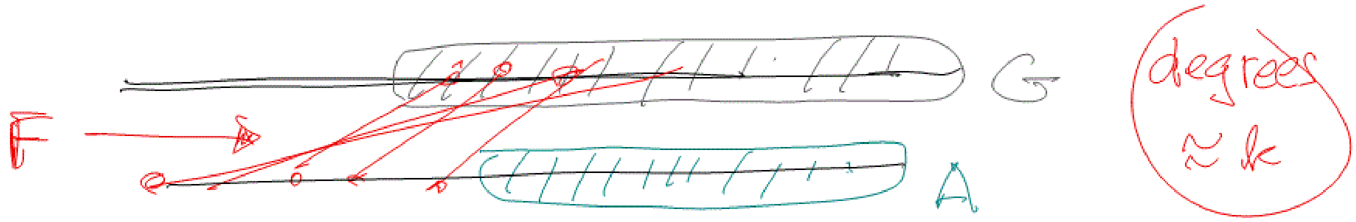
$( \longrightarrow (T1) \checkmark )$

$$\Big[\; T = A_q \qquad q \approx k^{-3+\varrho} \qquad \left(C\,\boxed{k^{-3}}{}^{+\varrho} \log k\right)$$

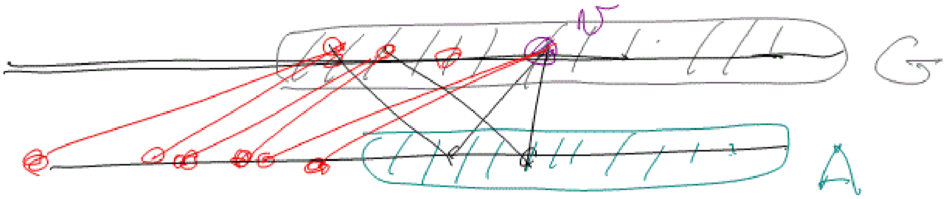
<u>want</u>

$$\left.\begin{matrix} |G \setminus W| \\ |W \setminus G| \end{matrix}\right\} \lesssim \delta a k^{\varrho} \qquad \left(C\,\boxed{\delta a}\,k^{\varrho} \log k\right) \;\Big]$$

<u>a key point:</u>




$$\underbrace{|\nabla(G, \Gamma_k \setminus A)|}_{F} \lesssim \underline{\delta} a k \qquad (\text{vs. } ak)$$




$$\left| F = \nabla(G, \Gamma_k \setminus A) \right| \lesssim \delta a k$$

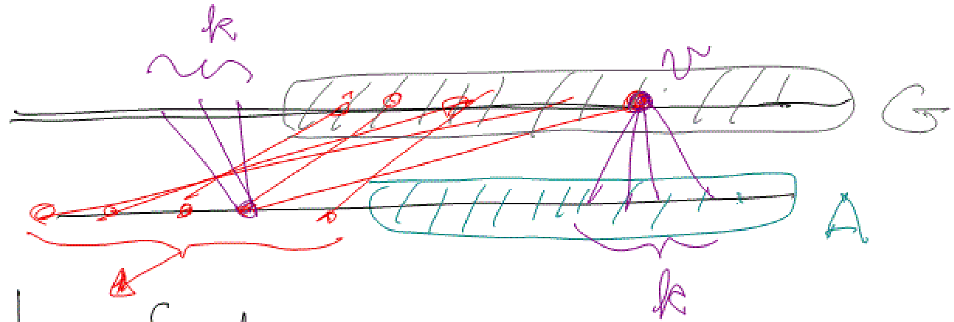
G\W : for most $v \in G$

$$|N^3(v) \cap A| \approx k^3 \quad (\text{e.g. } \gtrsim k^3/2)$$

($v$ violates $\Rightarrow$ heavy use of $F$)

$\longrightarrow$ $\Pr(v \notin W)$ small

W \ G:




$|F| \lesssim \delta a k$

$$\Pr(v \in N(T)) \approx k^{-2+\varsigma}$$

typ: $|\nabla(N(T), \Gamma_k \setminus A)| \lesssim \delta a k^{-1+\varsigma}$

$\longrightarrow \quad |W \setminus G| \lesssim \delta a k^{\varsigma}$

(as promised) "☒"

$$\left[\begin{array}{l}(T1)\quad |T| < C\,\boxed{a k^{-3}}^{+\delta}\,\log k \;=: t \\ (T2)\quad |W_T \,\Delta\, G| < C\,\boxed{\delta a}\,k^{\delta}\,\log k\end{array}\right]$$

$\circledast$ $\Pr\left( X \cap W \text{ bad} \right) \lesssim \exp\left[-\delta^2 a\right]$

(as before ; note $|W| \approx a$ )

$\triangleright$ # of $W$'s $\leq$ $\underline{\#\text{ of } T\text{'s}}$ $\lesssim \binom{|\Gamma_k|}{t}$

$$\approx \exp\left[ C\,\boxed{a k^{-3}}^{+\delta}\,\log k\,\log\left(|\Gamma_k|/t\right)\right]$$

$k^{-2}$ here is $\underline{\text{no good}}$

Thank You