On the minimum degree of minimal Ramsey graphs

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joint work with
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IMA

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Introduction

- G is r-Ramsey for H, denoted by $G \to (H)_r$, if any r-coloring of $E(G)$ contains a monochromatic copy of H
- $\mathcal{R}_r(H) = \{G : G \to (H)_r\}$
- **Ramsey's Theorem**: $\mathcal{R}_r(H) \neq \emptyset$ for every H
  - Ramsey number $R_r(H)$ = smallest n such that $K_n \to (H)_r$
  - $2^{k/2+o(k)} \leqslant R_2(K_k) \leqslant 2^{2k+o(k)}$
  - $2^{\Omega(r)} \leqslant R_r(K_3) \leqslant 2^{O(r \ln r)}$

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GOAL: Learning about $\mathcal{R}_r(H)$, that is, learning about $\mathcal{M}_r(H)$

Their number and extremal properties I

Question of Nešetřil: Is $|\mathcal{M}_r(K_k)| = \infty$?

For which graphs is $\mathcal{M}_r(H)$ finite?

Burr, Erdős, Faudree, Rousseau, Schelp, Nešetřil, Rödl, Sheehan, Łuczak, Ruciński, ...

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Theorem (Rödl-Siggers, 2008)
For all $k \geqslant 3, r \geqslant 2$ there is a constant $c > 1$ such that there are at least c^{n^2} non-isomorphic graphs $G \in \mathcal{M}_r(K_k)$ on at most n vertices.

Their number and extremal properties II

- $\min_{G \in \mathcal{M}_r(H)} v(G) = R_r(H)$, Ramsey number
- $\min_{G \in \mathcal{M}_r(H)} e(G) = \hat{R}_r(H)$, size Ramsey number
  (Erdős-Faudree-Rousseau-Schelp, 1976, ...)
- $\min_{G \in \mathcal{M}_r(H)} \omega(G) = \omega(H)$ (Folkmann, Nešetřil-Rödl, '70s)

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Question of Nešetřil: Is $|\mathcal{M}_r(K_k)| = \infty$?

Burr, Erdős,and Lovász (1976): systematic study of extremal properties of various graph parameters in $\mathcal{M}_2(K_k)$.
$\min_{G \in \mathcal{M}_r(H)} \chi(G)$
$\min_{G \in \mathcal{M}_r(H)} \Delta(G)$
$\max_{G \in \mathcal{M}_r(H)} \Delta(G)$
$\min_{G \in \mathcal{M}_r(H)} \kappa(G)$

$$s_r(H) := \min_{G \in \mathcal{M}_r(H)} \delta(G)$$

A simple lower bound on $s_r(H)$

Proposition (Fox-Lin)

$s_r(H) \geqslant r(\delta(H) - 1) + 1$

Proof.

- G: r-Ramsey minimal for H, $\deg_G(v) = \delta(G) = s_r(H)$
- there is an r-coloring χ of $G - v$ without m.c. H
- every extension of χ to $\{uv : u \in N(v)\}$ must yield a m.c. H
- if $\delta(G) \leqslant r(\delta(H) - 1)$, then the extension with at most $\delta(H) - 1$ edges in each color would not yield a copy of H

Simple lower bound is tight for example for

- complete bipartite graphs (Fox-Lin, 2006)
- trees, even cycles, bi-regular bipartite graphs, ...
  (Sz.-Zumstein-Zürcher, 2010)
- 3-connected bipartite, ... (Fox-Grinshpun-Liebenau, 2014+)

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Theorem (Burr-Erdős-Lovász, 1976): $s_2(K_k) = (k-1)^2$

Question

How large is $s_r(K_k)$ *for* $r > 2$*?*

BEL: $s_2(K_k) \geqslant (k-1)^2$

Proof:

- Take a graph $G \in \mathcal{M}_2(K_k)$ with $\delta(G) < (k-1)^2$ and a vertex v of min-degree
- Take a m.c. K_k-free 2-coloring χ of $E(G - v)$

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- Take a m.c. K_k-free 2-coloring χ of $E(G-v)$
- Let $U \subseteq N(v)$ be the union of the vertices in a maximal vertex disjoint family of m.c. `red` K_{k-1} inside $N(v)$
- Extend χ by coloring $uv \in E(G)$ `green` if $u \in U$ and `red` if $u \in N(v) \setminus U$

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- Extend χ by coloring $uv \in E(G)$ green if $u \in U$ and red if $u \in N(v) \setminus U$
- There is no m.c. K_k in the extension, so $G \not\to (K_k)_2$, contradiction

Main Results

Erdős-Rogers function (1962)

$$f_{s,t}(n) := \min\{\alpha_s(G) : G \text{ is } K_t\text{-free}\},$$

where $\alpha_s(G)$ is the size of the largest K_s-free subset

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Pseudotheorem (Fox, Grinshpun, Liebenau, Person, Sz. 2014+)

For every $k \geqslant 3$, every $r \geqslant r_0$,

$$c_k r (f_{k-1,k}(r))^2 \leqslant s_r(K_k) \leqslant C_k r (f_{k-1,k}(r))^2 \log^{8k} r$$

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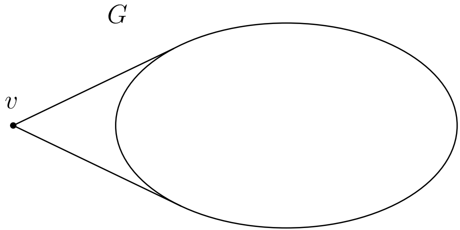
Theorem (Fox, Grinshpun, Liebenau, Person, Sz. 2014+)

For $k \geqslant 3$, $r \geqslant 3$,

$$s_r(K_k) \leqslant 8(k-1)^6 r^3.$$

Lower bound with more information



- Take $G \in \mathcal{M}_r(K_k)$ with $\delta(G) = s_r(K_k)$ and vertex v of min-degree
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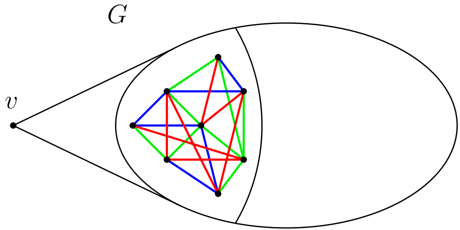
Lower bound with more information



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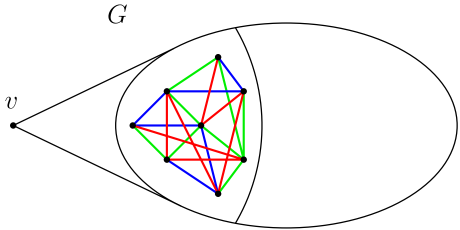

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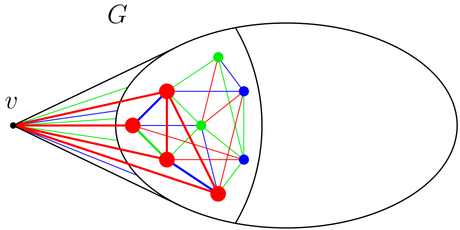

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(P2) every r-vertex coloring of $N(v)$ must contain K_{k-1} in G_i on vertices of color i, for some i. (i.e., strongly m.c. K_k)

- $P_r(k-1) := \min n$ such that there is a color pattern $G_1 \dot\cup \ldots \dot\cup G_r$ on $[n]$ that satisfies (P1) and (P2).

Passing to $P_r(k-1)$

- $P_r(k-1) = \min n$ such that there is a color pattern $G_1 \dot\cup \ldots \dot\cup G_r$ on $[n]$ that satisfies

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- By the above:

  Claim $s_r(K_k) \geqslant P_r(k-1)$

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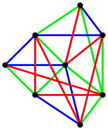
- By the above:

  Claim $s_r(K_k) \geqslant P_r(k-1)$
- *Remark:* $P_2(k-1) \geqslant (k-1)^2$

How about $s_r(K_k) \leqslant P_r(k-1)$?

$s_r(K_k) \leqslant P_r(k-1)$:

Fix color pattern $G_1 \dot\cup \ldots \dot\cup G_r$ on V, $|V| = P_r(k-1)$, with (P1)+(P2).



Goal: Construct r-Ramsey-minimal graph B' with $G_1 \dot\cup \ldots \dot\cup G_r$ as the induced neighborhood of a min-degree vertex

Wishful thinking: what if ...

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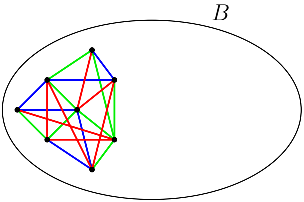

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(i) $B \not\to (K_k)_r$ and

(ii) any K_k-free r-coloring of B "produces" $G_1 \dot\cup \ldots \dot\cup G_r$

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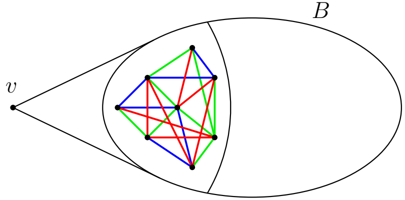

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Then just add a vertex v with $N(v) = V$, so $\delta(B + v) = P_r(k-1)$

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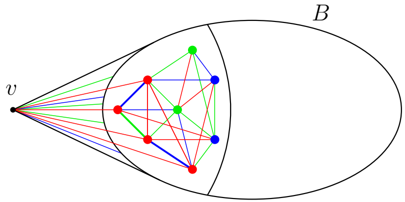

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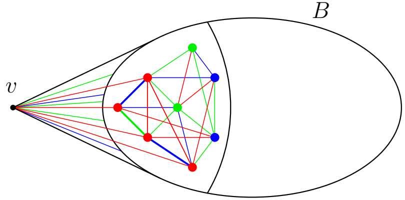

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Then just add a vertex v with $N(v) = V$, so $\delta(B+v) = P_r(k-1)$

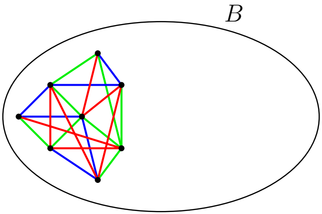
- $B + v \to (K_k)_r$ (by (ii) and definition of $P_r(k-1)$) and
- $\delta(B') \leqslant P_r(k-1)$ for any $B' \subseteq B + v$ with $B' \to (K_k)_r$ (by (i))

BEL gadgets

BEL gadget

Given H, and a *color pattern* $G_1 \dot\cup \ldots \dot\cup G_r$ such that $H \not\subseteq G_i$, for all i. If H is 3-connected, then there exists a graph B such that

- $G_1 \dot\cup \ldots \dot\cup G_r$ is an induced subgraph of B,
- $B \not\to H$, and
- in any m.c. H-free coloring of B, all subgraphs G_i are m.c. and any two have different colors.



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- Burr-Erdős-Lovász (1976): $H = K_k$ and $r = 2$
- Burr-Nešetřil-Rödl (1984): H is 3-connected and $r = 2$
- Rödl-Siggers (2008): H is 3-connected, arbitrary r

The new task

So

Theorem (FGLPSz, 2014+)

For every $r \geqslant 2$ and $k \geqslant 3$,

$$s_r(K_k) = P_r(k-1).$$

Now: bound on $P_r(k)$?

Recall: $P_r(k) = \min n$ such that there is a color pattern $G_1 \dot{\cup} \ldots \dot{\cup} G_r$ on vertex set $[n]$ that satisfies

(P1) $K_{k+1} \not\subseteq G_i$ for all i, and

(P2) every r-coloring of $[n]$ must contain a s.m.c. K_k

A recurrence for $P_r(k)$

- Let $G_1 \dot\cup \ldots \dot\cup G_r$ be a color pattern on $[P_r(k)]$ that satisfies

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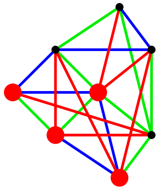
  (P2) every r-coloring of $[P_r(k)]$ must contain a s.m.c. K_k
- Let I_i be a largest K_k-free subset in G_i. Then for the induced color pattern $\dot\cup_{j \neq i} G_j$ on $[P_r(k)] \setminus I_i$
  (P1) $K_{k+1} \not\subseteq G_j$ for all $j \neq i$, and
  (P2) every $(r-1)$-coloring of $[P_r(k) \setminus I_i]$ must contain a s.m.c. K_k



$G_1 \dot\cup \ldots \dot\cup G_r$

$$\Rightarrow \quad P_r(k) \geqslant P_{r-1}(k) + \max_i |I_i|$$

A lower bound on $P_r(k)$

Erdős-Rogers function $f_{s,t}(n) = \min\{\alpha_s(G) : G \text{ is } K_t\text{-free}\}$ where $\alpha_s(G)$ is the size of the largest K_s-free subset

Proposition (FGLPSz)

$s_r(K_{k+1}) = P_r(k) \geqslant P_{r-1}(k) + f_{k,k+1}(P_r(k))$

- Ajtai-Komlós-Szemerédi 1980; Shearer 1983: $f_{2,3}(n) = \Omega\left(\sqrt{n \ln n}\right)$
- Krivelevich 1994, Shearer 1995 (Dudek-Mubayi 2013): $f_{k,k+1}(n) = \Omega\left(\sqrt{\frac{n \ln n}{\ln \ln n}}\right)$

Results

Pseudotheorem (Fox, Grinshpun, Liebenau, Person, Sz. 2014+)

$$s_r(K_k) = \Omega\left(r(f_{k-1,k}(r))^2\right)$$

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Theorem (Fox, Grinshpun, Liebenau, Person, Sz. 2014+)

There is a constant $c > 0$ such that for all $r \geqslant 2$, we have

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For all $k \geqslant 4$ there is a constant $c = c(k)$ such that for all $r \geqslant 2$

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Results

Pseudotheorem (Fox, Grinshpun, Liebenau, Person, Sz. 2014+)

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Theorem (Fox, Grinshpun, Liebenau, Person, Sz. 2014+)

For all $k \geqslant 4$ there are constants $c = c(k)$ and $C = C(k)$ such that for all $r \geqslant 2$

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Upper bound on $P_r(k)$ — wishful thinking

- Take $n := \Theta\left(\left(\frac{\mathrm{id}}{f_{k,k+1}}\right)^{-1}(r)\right)$ vertices
- Take a K_{k+1}-free graph G with $\alpha_k(G) = f_{k,k+1}(n)$
- Create a color pattern $G_1 \dot\cup \cdots \dot\cup G_r$ with $r = \left\lfloor \frac{n}{f_{k,k+1}(n)} \right\rfloor$ disjoint copies of it.

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Then

(P1) $K_{k+1} \not\subseteq G_i$ for all i, and

(P2) every r-coloring of $[n]$ must contain a s.m.c. K_k.

(since some color i occurs at least $\frac{n}{r} > \alpha_k(G_i)$ times)

Wishful Corollary

$$s_r(K_k) = \Theta\left(r(f_{k-1,k}(r))^2\right)$$

Upper bounds on $P_r(k)$ — reality

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$P_r(K_3)$ — reality

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- Iterative triangle-free process???

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Problem

Construct BEL gadgets for 2-connected H.

Ramsey equivalence

H and G are Ramsey equivalent if $\mathcal{M}_2(H) = \mathcal{M}_2(G)$

Proposition (SzZZ) $K_k + K_2$ and K_k are Ramsey equivalent for $k \geqslant 4$

Conjecture (SzZZ) K_k and $K_k \cdot K_2$ are Ramsey equivalent (for large enough k)

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Theorem (FGLPSz, 2014)

$s_2(K_k \cdot K_2) = k - 1$,, that is, $K_k \cdot K_2$ and K_k are **not** *Ramsey equivalent.*

Corollary (FGLPSz, 2014)

Any graph G which is Ramsey equivalent to K_k must be the disjoint union of K_k and a graph of smaller clique-number

What else is monochromatic?

$f(k,t)$ is the largest f such that K_k and $K_k + f \cdot K_t$ are Ramsey-equivalent

Remark. $f(k,k) = 0$, $f(k,1) = R(K_k) - k$.

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Theorem (FGLPSz, 2014)

For $k > t \geqslant 3$,

$$f(k,t) \leqslant \frac{R(k,k-t+1) - 1}{t}.$$

Open Problems

Problem

Determine $f(k,t)$.

Conjecture

K_k *is Ramsey equivalent to* $K_k + K_{k-1}$ *for* $k \geqslant 4$.