Bipartite Decomposition of Graphs

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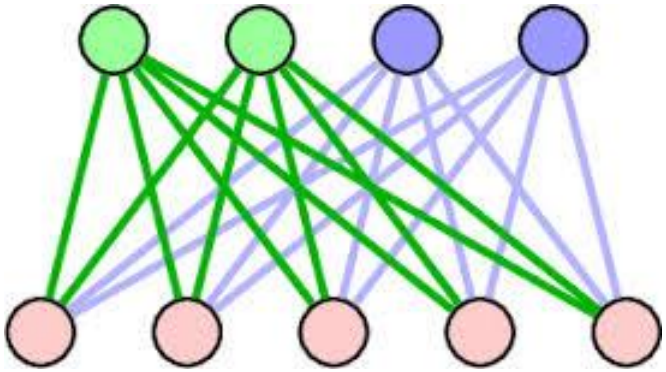

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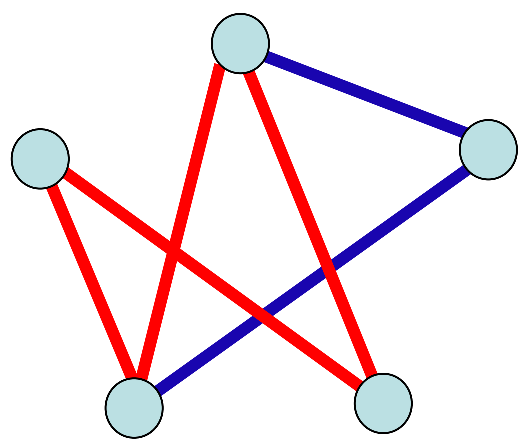
I. Bipartite Decomposition

A bipartite decomposition of a graph G=(V,E) is a family of pairwise edge disjoint complete bipartite subgraphs of G so that every edge of G belongs to exactly one of them.

bc(G)=minimum number of bicliques in such a decomposition



Theorem [Graham-Pollak (72)]: bc(K_n)=n-1

Proofs: Witsenhausen(72), Tverberg (82), Peck(84), Vishwanathan (08)

All apply linear algebra

Proofs give: **If the adjacency matrix A of G has p positive eigenvalues and q negative ones then $bc(G) \geq \max\{p,q\}$**

In particular, $bc(G) \geq (p+q)/2 = rank(A)/2$ (which is easy)

Extension [A (97)]: the minimum number of bicliques of K_n so that each edge belongs to at least 1 of them and at most t of them is

$$\Theta(t\, n^{1/t})$$

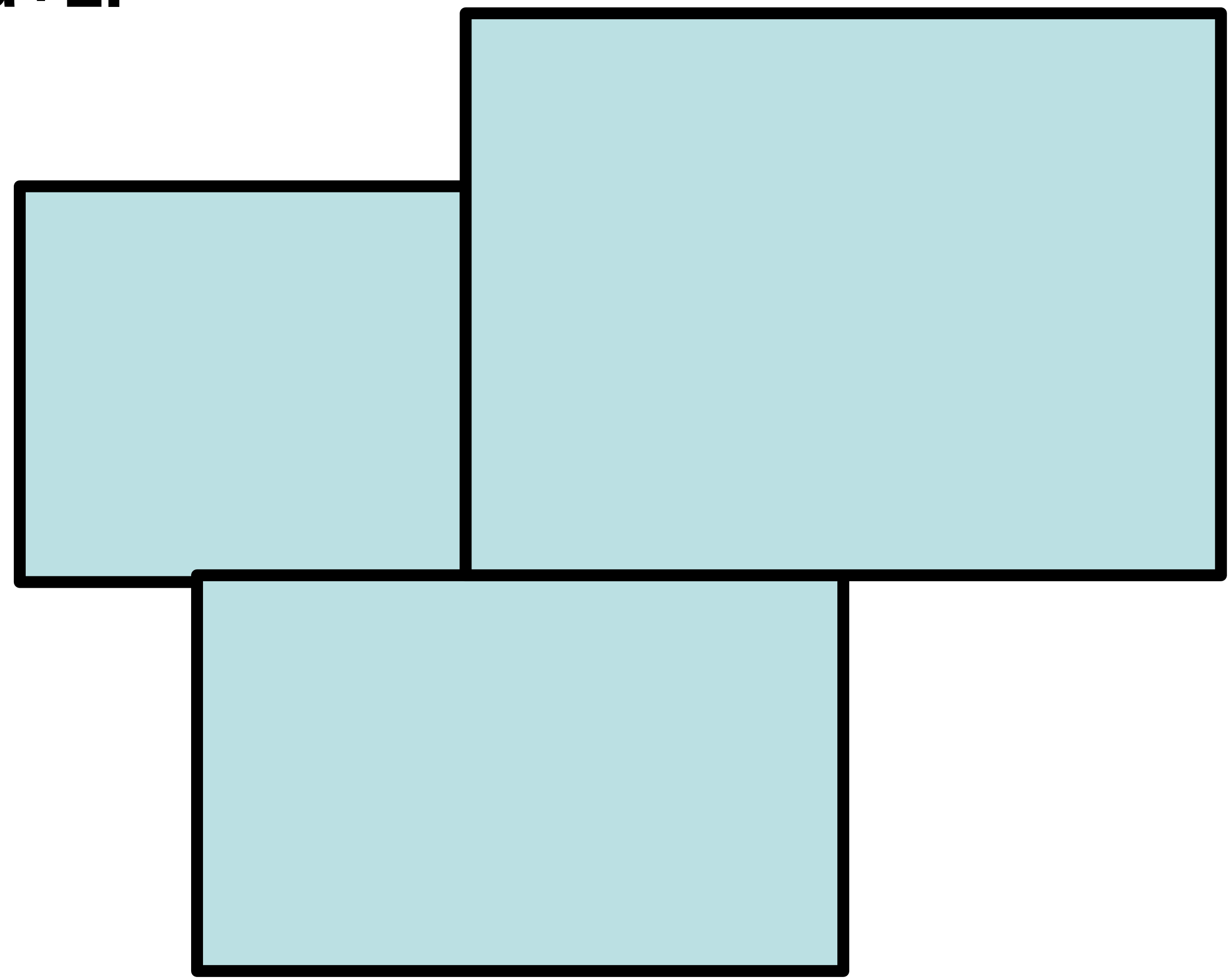
(The precise minimum is not known even for $t=2$).

A geometric interpretation:

A family of d-dimensional convex polytopes is t-neighborly if for any two members A,B of the family $d - t \leq dim(A \cap B) \leq d - 1$.

Zaks (79), A(97): The maximum number of members in a t-neighborly family of d-dimensional aligned boxes is exactly the largest n so that K_n can be covered by d bicliques, with each edge covered at least once and at most t times.

Zaks(79): Thus, the maximum number of aligned d-dimensional boxes with disjoint interiors and with each pair having intersection of dimension d-1 is d+1.



II Related Invariants

By Graham-Pollak, for every G, bc(G) ≥ w(G)-1, where w(G) is the maximum size of a clique in G.

Question [A-Seymour-Saks (90)]: Can we replace clique number by chromatic number, namely, is it true that for every G

$$bc(G) \geq \chi(G) - 1 \ ?$$

Theorem [Huang-Sudakov (12)]: No !

There are graphs G with $bc(G) \leq O(\chi(G)^{\frac{5}{6}})$

Open:

how small can bc(G) be as a function of $\chi(G)$?

Mubayi and Vishwanathan (09), A-Haviv (09):

For every G:

$$bc(G) \geq 2^{\sqrt{2 \log_2 \chi(G)}}$$

Trivially, for a graph G with adjacency matrix A rank(A) ≤ 2 bc(G).

The rank-coloring conjecture [van Nuffelen (76), Fajtlowicz and Graffiti (87)]:

For every graph G, $rank(A) \geq \chi(G) - 1$

Counter-example: A-Seymour(89).

Better counter-examples:
Razborov (92), Raz and Spieker (95), Nisan and Wigderson (95), Kushilevitz (95)

Open:
how small can the rank be as a function of $\chi(G)$?

Conjecture [Lovász and Saks (87)]: There is an $\epsilon > 0$ so that for every graph G with chromatic number χ and adjacency matrix of rank R

$$R \geq 2^{(\log \chi)^{\epsilon}}$$

Equivalently: $\log R \geq (\log \chi)^{\epsilon}$

Best known bounds:

Lovett (14): For every graph with rank R and chromatic number χ: $R \geq \Omega(\,(log\,\chi)^2)$

Kushilevitz (95): There are graphs G with

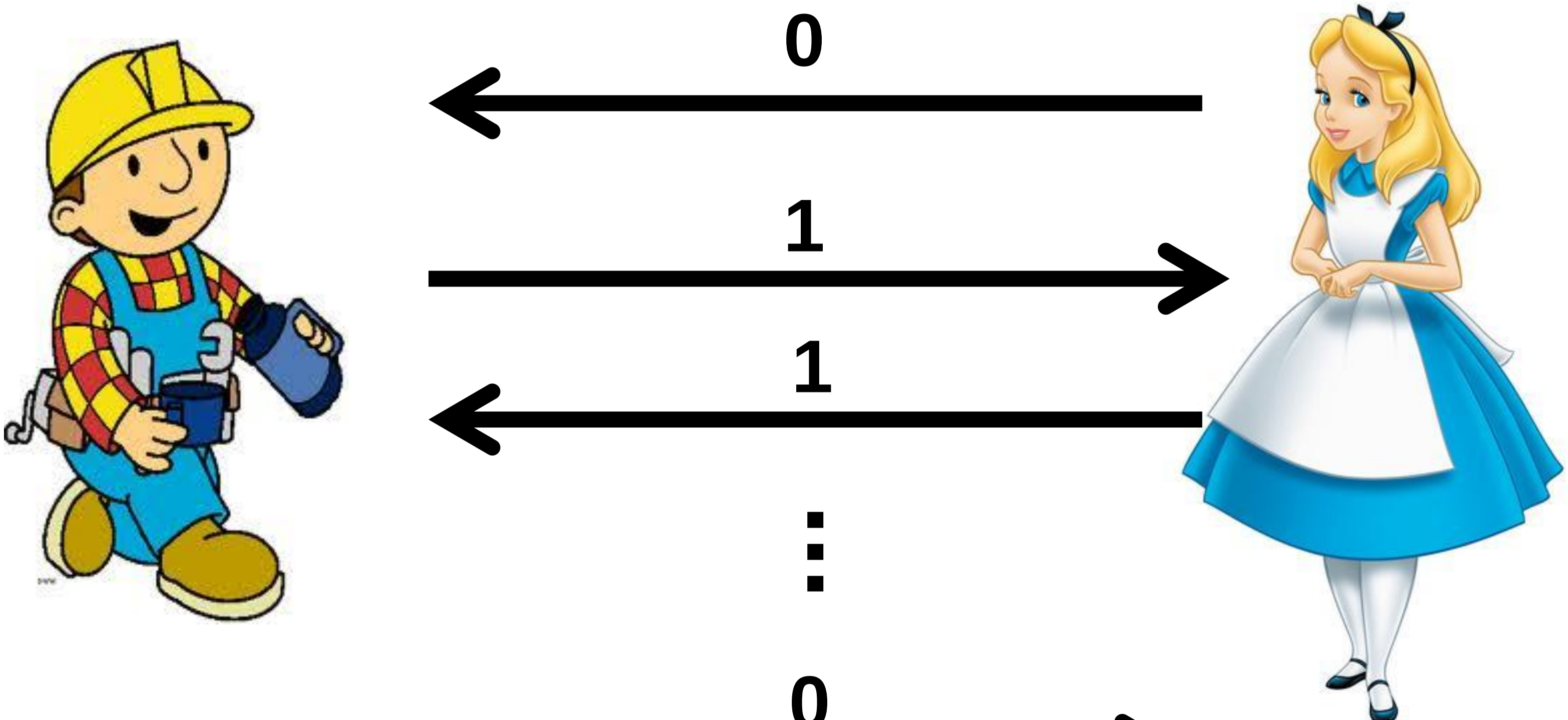
$$R \cdot \quad O(2^{c(\log \chi)^{\log_6 3}})$$

[Equivalently: $\log R \cdot \quad O((\log \chi)^{\log_6 3})$ **]**

Communication Complexity:

Let f: X $\times$ Y$\longrightarrow$ {0,1} be a Boolean function. Alice knows x in X, Bob knows y in Y, and they wish to compute f(x,y) by communicating as few bits as possible among them.

D(f)=the deterministic communication complexity of f = the minimum number of bits enabling them to compute f(x,y) (worst x,y, best protocol).


0
1
1
⋮
0

Let M be the matrix of f, that is, $M_{x,y}=f(x,y)$

Viewing M as the adjacency matrix of a bipartite graph H with vertex classes X and Y the following holds:

$$bc(H) \le 2^{D(f)}$$

Therefore:

$$\log_2 (rank(M)) \le \log_2 (bc(H)) \le D(f)$$

Let M be the matrix of f, that is, $M_{x,y}=f(x,y)$

$$\log_2 (\mathrm{Rank}(M)) \leq D(f)$$

The log-rank conjecture [Lovász and Saks (87)]:

$$D(f) \leq (\log_2 (\mathrm{Rank}(M)))^{O(1)}$$

This is equivalent to the conjecture that for every graph with chromatic number χ and adjacency matrix of rank R

$$\log \chi \leq (\log R)^{O(1)}$$

Best known bounds:

Lovett (14): For every function f with matrix M of rank R: $D(f) \leq O(\sqrt{R})$

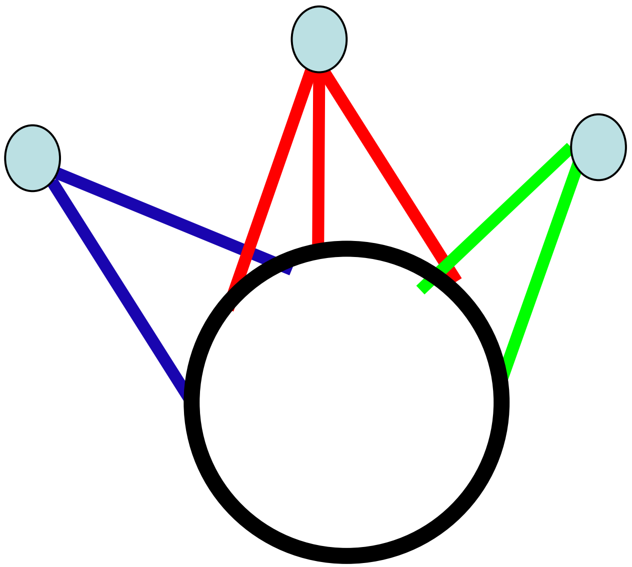
Kushilevitz, following Nisan Wigderson (95):
There are function f with

$$D(f) \geq \quad ((\log R)^{\log_3 6})$$

III Bipartite Decomposition of Random Graphs

Fact: **For every graph G with n vertices and independence number α :**

$$bc(G) \leq n - \alpha$$



Conjecture [Erdős, 88)]:
For almost every graph G with n vertices and independence number α:

$$bc(G) = n - \alpha$$

That is: for the random graph G=G(n,0.5):

$$bc(G) = n - \alpha(G)$$

with high probability (whp)

Conjecture [Chung and Peng (14)]: For the random graph G=G(n,p) with any Ω(1) ≤p< 0.5

$$bc(G) = n - (1 + o(1))\alpha(G)$$

whp

Thm [Chung and Peng (14)] : For any fixed p satisfying Ω(1) ≤ p ≤ 0.5

bc(G) ≥ n – O((log n)$^{3+ε}$)

$(= n - O((\alpha(G)^{3+\epsilon}))$ **)**

for any positive ε.

New Results

Thm 1: **For G=G(n,0.5) and "most" values of n**

$$bc(G) \leq n - \alpha(G) - 1$$

whp, whereas for infinitely many exceptional values of n

$$bc(G) \leq n - \alpha(G) - 2$$

with probability bounded away from zero. Thus, Erdős' conjecture is (slightly) false.

Thm 2: **There exists a constant c>0 so that for 2/n ≤ p ≤ c**

$$bc(G) = n - \Theta\big(\alpha(G)\big) = n - \Theta\left(\frac{log(np)}{p}\right)$$

whp.

A more precise statement for G=G(n,0.5):

Let $\beta(G)$ denote the largest number of vertices in an induced complete bipartite subgraph of G.
Clearly, for every G: $bc(G) \leq n - \beta(G) + 1$

Thm 1': **(i) If G=G(n,0.5) and $\alpha(G)$ is concentrated in a single point, then so is $\beta(G)$ and $\beta(G) = \alpha(G) + 2$ whp. In this case $bc(G) \leq n - \alpha(G) - 1$ whp.**

(ii) If $\alpha = \alpha(G)$ is concentrated in 2 points, then so is $\beta = \beta(G)$, and $\beta \in \{\alpha + 1, \alpha + 2, \alpha + 3\}$ whp, and each of the three possibilities holds with probability bounded away from 0 and 1.
In this case $bc(G) \leq n - \alpha - 2$ with probability bounded away from 0.

IV Something about the proof

(of Theorem 1')

Let $f(k) = \binom{n}{k} 2^{-\binom{k}{2}}$ **be the expected number of independent sets of size k in G=G(n,0.5). The independence number** $\alpha = \alpha(G)$ **of G is, whp, essentially the largest k for which f(k) ≥ 1, which is (2+o(1))** $\log_2 n$**.**

For this value of k, n=Θ($k2^{k/2}$**), and the expected number of induced complete bipartite subgraphs of G of size k+2 is g(k)=Θ(f(k))**

When for the largest k with f(k) ≥ 1, f(k) is far larger than 1, but f(k+1) is far smaller than 1, the second moment method implies that $\alpha(G) = k$ whp, and in this case g also satisfies g(k)>>1>>g(k+1) and $\beta(G) = k + 2$ whp.

When f(k)=Θ(1) then g(k)=Θ(1), and then by the (two-dimensional) Stein-Chen method, the two variables X=number of independent sets of size k and Y=number of induced complete bipartite graphs of size k+2 are approximately independent Poisson random variables with expectations f(k) and g(k).

Therefore, the events (X=0,Y=0), (X=0,Y>0), (X>0,Y=0) and (X>0,Y>0) all hold with probability bounded away from 0 and 1.

These correspond to $(\alpha = k-1, \beta = k+1)$, $(\alpha = k-1, \beta = k+2), (\alpha = k, \beta = k+1)$ **and** $(\alpha = k, \beta = k+2)$ **, respectively.**

Some computation can be saved by applying the one-dimensional Stein-Chen method, using the following:

Lemma: if X and Y are non-negative integers, and X+Y=0, then X=Y=0.

Thus, it suffices to show that X, Y and X+Y are approximately Poisson variables with the right expectations.

V Open Problems

Conjecture 1: **For G=G(n,0.5)**

$$\boldsymbol{bc(G) = n - O(\alpha(G))}$$

with high probability.

Conjecture 2 (Chung-Peng): **For G=G(n,p) with $\Omega(1) \leq p < 0.5$**

$$\boldsymbol{bc(G) = n - (1 + o(1))\alpha(G)}$$

with high probability.

Problem: Find a purely combinatorial proof of the Graham-Pollak Theorem:

$bc(K_n)=n-1$.