Solution Clusters for Locked CSP's

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Locked Constraint Satisfaction Problems

Introduced by Zdeborová and Mézard 2008

- no constraint has two solutions that differ on a single variable
- each variable lies in at least two constraints

(1-in-k)-SAT
(2-or-5-or-9-in-10)-SAT
XOR-SAT - a.k.a. linear equations mod 2

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Roughly 1000 variables can't be solved with Belief Propogation (=SP); BP with reinforcement; Stochastic Local Search.

Note: Changing a variable in a solution will force a cascade of changes, as we must change at least two variables in each affected constraint.

Clusters in Locked Problems

ZM08 studied random locked CSP's with truncated Poisson degree sequences; minimum degree 2.

They found a threshold μ^* such that

- constraint-density below μ^*: all solutions are in a single cluster
- constraint-density above μ^*: every cluster has size $O(1)$.

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We provide a rigorous proof that:

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We provide a rigorous proof that:

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EXCEPT: for general degree sequences, we don't have a proof that XORSAT is satisfiable in the clustered phase.

Satisfiability threshold for XORSAT

Random k-XORSAT on a degree sequence with min degree two.

Choose a random k-uniform hypergraph on that degree sequence. Then treat each hyperedge as a clause by signing it randomly.

Random k-XORSAT on a degree sequence with min degree two.

n: number of variables

m: number of edges

Hypothesis

- $m < (1-\epsilon)n$: w.h.p. satisfiable
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Completed for truncated Poisson degree sequences:

- $k = 3$ Dubois, Mandler 2002
- $k > 3$ Dietzfelbinger et al 2010, Pittel and Sorkin 2012.

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FALSE for some degree sequences. Lelarge 2013

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The subgraph remaining after repeatedly deleting any vertices of degree < 2, and any edges containing those vertices.

Analyzed for random graphs on a fixed degree sequence by Fernholz and Ramachandran 2003

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Rigorous proofs:

Ibrahimi, Kanoria, Kranning and Montanari (2011)

Achlioptas and M (2011)

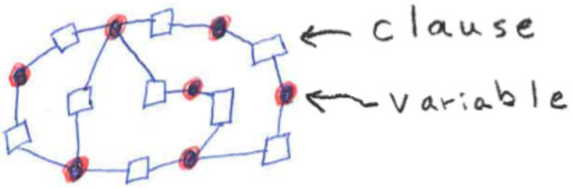
Inside the 2-core threshold: Gao and M (2013)

Why is the 2-core frozen?

Flippable set: A subset S of the variables such that every clause contains an even number of members of S.

Note: Every pair of solutions differs on a flippable set. So

$$\text{“every variable is frozen”} = \text{“every flippable set has linear size”}$$



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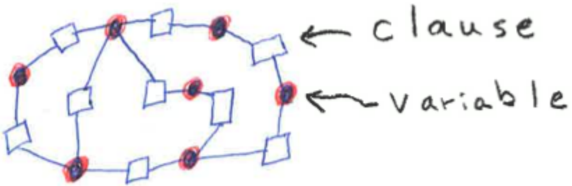

Note: Flippable sets depend only on the underlying hypergraph, not the signs on the clauses, or the actual solution.
So the cluster structure is determined by the hypergraph.

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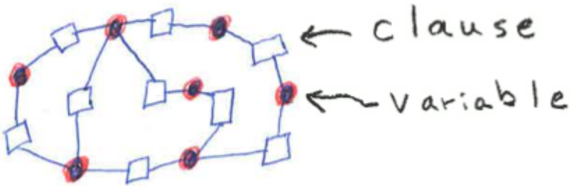

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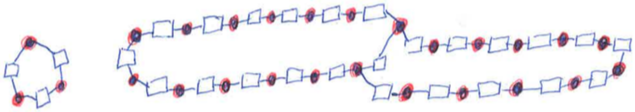
So small sets must look like:



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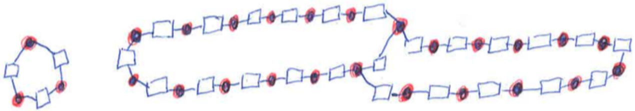
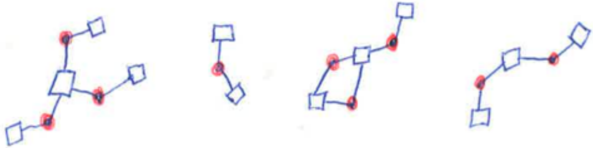


But the subgraph induced by the degree two variables only has small components.



Components in the 2-graph

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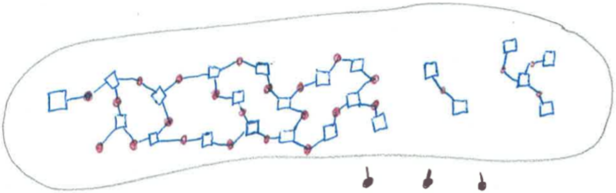

W.h.p. one of the vertices removed near the end of the stripping process would be adjacent to the giant component.
The deletion of that vertex would lead to the deletion of the entire giant component.

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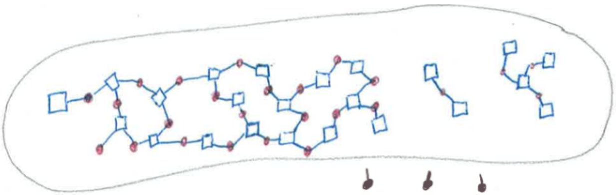

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Therefore, if a 2-core is reached by a stripping process, then (nearly) every variable in the 2-core is frozen

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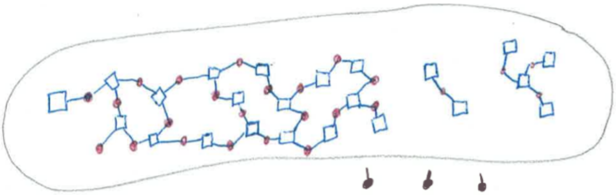


However: If we fix an initial degree sequence with minimum degree 2, then we can choose one in which the 2-graph has a giant component.

Degree sequences with minimum degree 2

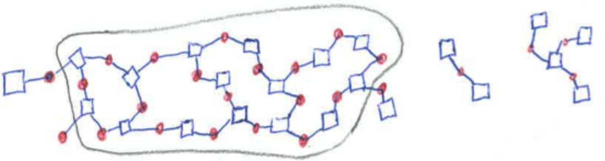
Type 1: The 2-graph has only small components.

Every flippable set has linear size, other than $O(1)$ short cycles.

So every cluster has size $O(1)$.

Type 2: The 2-graph has a giant component

Every vertex in the 2-core of the giant component lies in a flippable cycle of size $O(\log n)$, and so is not frozen.



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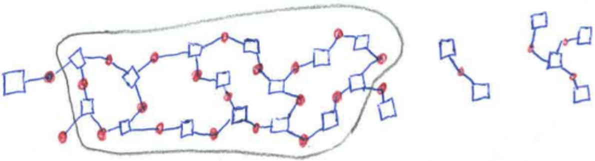
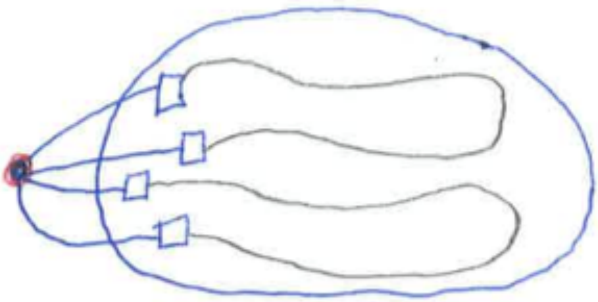


This yields exponentially large clusters, formed by flipping these cycles.

Other flippable sets



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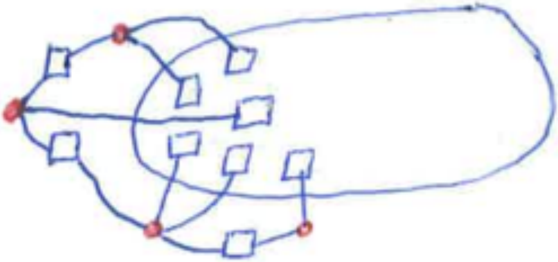

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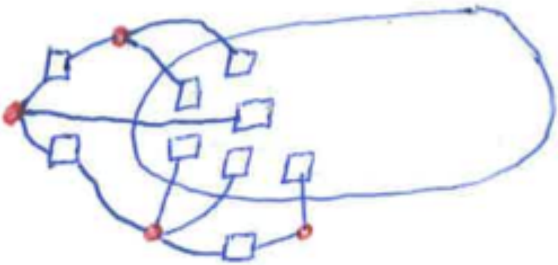

Lemma: *Every vertex lies in one of these flippable sets unless the graph obtained by deleting the giant component of the 2-graph has a 2-core.*

Degree sequences with minimum degree 2

Type 2a: The 2-graph has a giant component. Removing that giant component leaves a graph with no 2-core.

Every vertex in the graph lies in a flippable set of size poly(log n), and so is not frozen.

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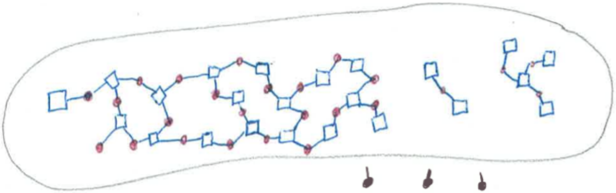

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Lemma: *Truncated Poisson sequences are Type 1 or Type 2a.*

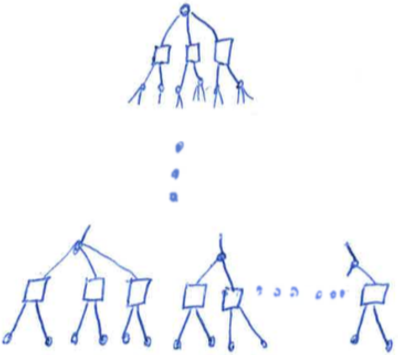
This confirms the cluster description from Zdeborová and Mézard.

Noisy Reconstruction

Choose a Galton-Watson hypertree of height h. Sign the hyperedges to be XORSAT constraints.

Take a random solution, and fix the values of the leaves.

What is the probability that the root is determined (as $h \to \infty$)?



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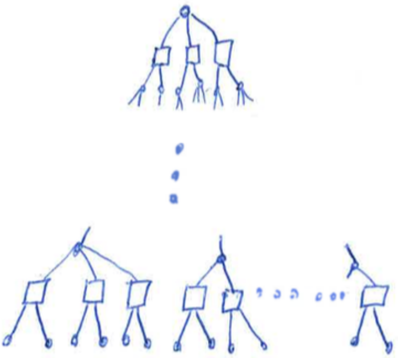

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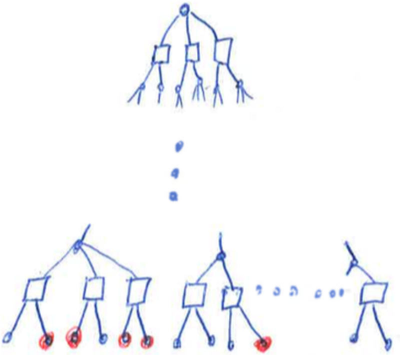

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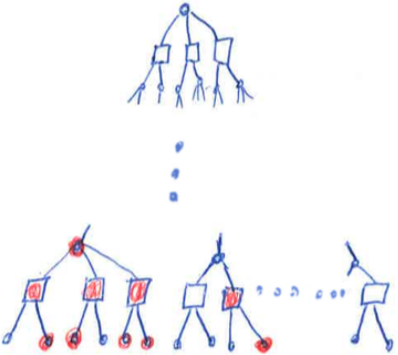

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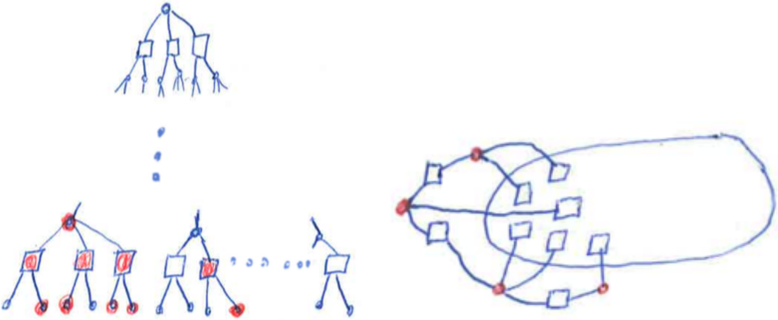

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Type 1: 1
Type 2a: 0
Type 2b: $0 < p < 1$
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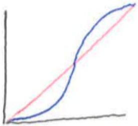
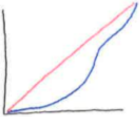


Type 1

Type 2b



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For many balanced problems, eg (2-or-4-in-6)-SAT, the planted model should be contiguous to the standard model, but that requires extensive second moment analysis.

This will imply, eg. that there are a large number of clusters. But we used the linear algebra structure of XORSAT to prove that there is a single cluster.