Global Optima from Local Algorithms

Dimitris Achlioptas [1] Fotis Iliopoulos [2]

[1]UC Santa Cruz

[2]UC Berkeley

The LLL Setting

- Probability space + Set of m "bad" events $B = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$.
- If $\{\mathcal{E}_i\}$ are independent, $\Pr[\text{Nothing bad happens}] = \prod_{i=1}^{m}(1 - p_i)$.
- But what if avoiding some bad events boosts some other bad events ?

Example: $\Omega = \{0,1\}^3$ with uniform measure, $F = (x_1 \vee x_2) \wedge (\overline{x_2} \vee x_3)$.

The LLL Setting

- Probability space + Set of m "bad" events $B = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$.
- If $\{\mathcal{E}_i\}$ are independent, $\Pr[\text{Nothing bad happens}] = \prod_{i=1}^{m}(1 - p_i)$.
- But what if avoiding some bad events boosts some other bad events ?

Example: $\Omega = \{0,1\}^3$ with uniform measure, $F = (x_1 \vee x_2) \wedge (\overline{x_2} \vee x_3)$.

General LLL (Erdős, Lovász '75)

If each $\mathcal{E}_i$ is mutually independent of all events in $B \setminus (\Gamma(i) \cup \mathcal{E}_i)$ and there exist $\{x_i\} \in [0,1)$ such that

$$x_i \geq \frac{\Pr[\mathcal{E}_i]}{\prod_{j \in \Gamma(i)}(1 - x_j)}$$

then $\Pr[\text{Nothing bad happens}] \geq \prod_{i=1}^{m}(1 - x_i)$.

The LLL Setting

- Probability space + Set of m "bad" events $B = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$.
- If $\{\mathcal{E}_i\}$ are independent, $\Pr[\text{Nothing bad happens}] = \prod_{i=1}^{m}(1 - p_i)$.
- But what if avoiding some bad events boosts some other bad events ?

Example: $\Omega = \{0,1\}^3$ with uniform measure, $F = (x_1 \vee x_2) \wedge (\overline{x_2} \vee x_3)$.

Asymmetric LLL (Very Handy Corrolary)

If each $\mathcal{E}_i$ is mutually independent of all events in $B \setminus (\Gamma(i) \cup \mathcal{E}_i)$
and

$$\sum_{j \in \Gamma(i)} \Pr[\mathcal{E}_j] \le \frac{1}{4}$$

then $\Pr[\text{Nothing bad happens}] > 0$.

The LLL Setting

- Probability space + Set of m "bad" events $B = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\}$.
- If $\{\mathcal{E}_i\}$ are independent, $\Pr[\text{Nothing bad happens}] = \prod_{i=1}^{m}(1 - p_i)$.
- But what if avoiding some bad events boosts some other bad events ?

Example: $\Omega = \{0,1\}^3$ with uniform measure, $F = (x_1 \vee x_2) \wedge (\overline{x_2} \vee x_3)$.

Lopsided LLL (Erdős, Spencer '91)

If each $\mathcal{E}_i$ is never boosted by any conditioning on events in $B \setminus (\Gamma(i) \cup \mathcal{E}_i)$ and there exist $\{x_i\} \in [0,1)$ such that

$$x_i \geq \frac{\Pr[\mathcal{E}_i]}{\prod_{j \in \Gamma(i)}(1 - x_j)}$$

then $\Pr[\text{Nothing bad happens}] \geq \prod_{i=1}^{m}(1 - x_i)$.

A Tight Example and a Breakthrough

Example

Every k-CNF formula where each clause shares variables with at most $\Delta \leq 2^k/\mathrm{e}$ other clauses is satisfiable.

A Tight Example and a Breakthrough

Example

Every k-CNF formula where each clause shares variables with at most $\Delta \leq 2^k/\mathrm{e}$ other clauses is satisfiable.

Theorem (Gebauer, Szabo, Tardos '11)

There exist unsatisfiable formulas with $\Delta = (1+\delta_k)2^k/\mathrm{e}$, where $\delta_k \to 0$.

A Tight Example and a Breakthrough

Example

Every k-CNF formula where each clause shares variables with at most $\Delta \leq 2^k/\mathrm{e}$ other clauses is satisfiable.

Theorem (Gebauer, Szabo, Tardos '11)

There exist unsatisfiable formulas with $\Delta = (1 + \delta_k)2^k/\mathrm{e}$, where $\delta_k \to 0$.

Algorithmic LLL: a s.t.a can be found efficiently if $\Delta \leq 2^{k/4}$

[Beck 91], [Alon 91], [Molloy, Reed 98], [Czumaj, Scheideler 00], [Srinivasan 08]

A Tight Example and a Breakthrough

Example

Every k-CNF formula where each clause shares variables with at most $\Delta \leq 2^k/\mathrm{e}$ other clauses is satisfiable.

Theorem (Gebauer, Szabo, Tardos '11)

There exist unsatisfiable formulas with $\Delta = (1+\delta_k)2^k/\mathrm{e}$, where $\delta_k \to 0$.

Algorithmic LLL: a s.t.a can be found efficiently if $\Delta \leq 2^{k/4}$

[Beck 91], [Alon 91], [Molloy, Reed 98], [Czumaj, Scheideler 00], [Srinivasan 08]

Theorem (Moser '09)

If $\Delta(F) < 2^{k-5}$ a sat assignment can be found in $O(|V| + |C| \log |C|)$.

Moser's ideas, with more care, yield $2^k/\mathrm{e}$. [Messner, Thierauf 11]

The Moser-Tardos Variable Setting

Assume that μ is a product probability space over n variables, i.e.,
$\mu(\omega) = \mu_1(\omega_1) \cdots \mu_n(\omega_n)$ for all $\omega \in \Omega$.

Theorem (Moser Tardos 09)

If $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m$ satisfy the conditions of the original LLL, an element $\omega \in \Omega$ such that none of the $\{\mathcal{E}_i\}$ holds can be found in time $O(m)$.

Resample

1: Sample $\omega \in \Omega$ according to μ — Easy since μ is product
2: **while** some bad event occurs **do**
3: Select any occurring bad event $\mathcal{E}_i$
4: Resample the variables of $\mathcal{E}_i$ according to μ
5: **return** ω

The Moser-Tardos Variable Setting

Assume that μ is a product probability space over n variables, i.e.,
$\mu(\omega) = \mu_1(\omega_1) \cdots \mu_n(\omega_n)$ for all $\omega \in \Omega$.

Theorem (Moser Tardos 09)

If $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m$ satisfy the conditions of the original LLL, an element $\omega \in \Omega$ such that none of the $\{\mathcal{E}_i\}$ holds can be found in time $O(m)$.

- Covers majority of existing LLL applications
- It can be parallelized [Moser, Tardos 09]
- It can be derandomized [Chandrasekaran, Goyal, Haeupler 09]
- May work even if m is exponential [Haeupler, Saha, Srinivasan 10]
- Shearer's condition suffices (weaker than LLL) [Kolipaka, Szegedy 11]

Limitations of the Moser Tardos Proof

The proof depends heavily on μ being a product measure

- Does not extend to Lopsided LLL
- Hard to extend even to simple non-product spaces

  e.g., uniform measure on permutations [Harris, Srinivasan 13]

Limitations of the Moser Tardos Proof

The proof depends heavily on μ being a product measure

- Does not extend to Lopsided LLL
- Hard to extend even to simple non-product spaces

  e.g., uniform measure on permutations [Harris, Srinivasan 13]

- Product measures require a variable decomposition.
- Product measures are algorithmically limiting:

Resamplings are state-independent

Example

LLL requires $e\Delta$ colors on graphs with max degree Δ instead of just $\Delta + 1$ colors

What to do?

Get Rid of the Measure

No Measure

Let Ω be an arbitrary finite set, such as:

- $\{0, 1\}^n$
- P_n — Permutations of $[n]$
- $\mathrm{Ham}(G) :=$ Hamiltonian cycles of graph G.
- $\mathrm{Col}_q(G) :=$ Valid edge-q-colorings of graph G.

No Measure

Let Ω be an arbitrary finite set, such as:

- $\{0,1\}^n$
- P_n Permutations of $[n]$
- $\text{Ham}(G) :=$ Hamiltonian cycles of graph G.
- $\text{Col}_q(G) :=$ Valid edge-q-colorings of graph G.

Let $F = \{f_1, f_2, \ldots, f_m\}$ be arbitrary subsets of Ω called flaws.

Examples (Flaw f_i is the subset of....)

- $\{0,1\}^n$ that violates clause c_i. SAT
- P_n in which $\pi(i) = i$. Derangements
- $\text{Col}_q(G)$ in which cycle C_i is bichromatic. Acyclic Edge Coloring

The LLL as a Random Walk

How to Find Flawless Objects

- Specify a directed graph D on Ω such that:
  - Every flawed object has outdegree at least 1.
  - Every flawless object has outdegree 0.
- Start at an arbitrary $\sigma_1 \in \Omega$
- Take a random walk on D until you reach a sink.

The LLL as a Random Walk

How to Find Flawless Objects

- Specify a directed graph D on Ω such that:
  - Every flawed object has outdegree at least 1.
  - Every flawless object has outdegree 0.
- Start at an arbitrary $\sigma_1 \in \Omega$
- Take a random walk on D until you reach a sink.

Definition (writing $\sigma \in \Omega$ instead of $\omega \in \Omega$ to emphasize "state" view)

- For $\sigma \in \Omega$, let $U(\sigma) = \{f_i : \sigma \in f_i\}$. σ is flawless iff $U(\sigma) = \emptyset$.
- $\forall \sigma \in \Omega, \forall f \ni \sigma$ define a non-empty set $A(f, \sigma) \subseteq \Omega$ of actions.
- Each $\tau \in A(f, \sigma)$ becomes an arc $\sigma \xrightarrow{f} \tau$ in D. D is a multidigraph

The Classics

k-**CNF** formula $F = c_1 \wedge \cdots \wedge c_m$ with n variables.

- $\Omega = \{0,1\}^n$.
- $f_i = \{\sigma \in \Omega : \sigma$ violates clause $c_i\}$.
- $A(f_i, \sigma) = \{$The 2^k mutations of σ through $\mathrm{var}(c_i)\}$.

$$\boxed{\Delta(F) < 2^k/\mathrm{e}}$$

The Classics

k-**CNF** formula $F = c_1 \wedge \cdots \wedge c_m$ with n variables.

- $\Omega = \{0,1\}^n$.
- $f_i = \{\sigma \in \Omega : \sigma$ violates clause $c_i\}$.
- $A(f_i, \sigma) = \{$The 2^k mutations of σ through $\mathrm{var}(c_i)\}$.

$$\boxed{\Delta(F) < 2^k/\mathrm{e}}$$

q-**coloring** graph $G(V,E)$ with n vertices.

- $\Omega = [q]^n$.
- $f_{u,v} = \{\sigma \in \Omega : \mathrm{col}(u) = \mathrm{col}(v)\}$
- $A(f_{u,v}, \sigma) = \{$All q mutations of σ through $v\}$

$$\boxed{q > \mathrm{e}\Delta}$$

The Classics

k-**CNF** formula $F = c_1 \wedge \cdots \wedge c_m$ with n variables.

- $\Omega = \{0,1\}^n$.
- $f_i = \{\sigma \in \Omega : \sigma$ violates clause $c_i\}$.
- $A(f_i, \sigma) = \{$The 2^k mutations of σ through $\mathrm{var}(c_i)\}$.

$\Delta(F) < 2^k/\mathrm{e}$

q-**coloring** graph $G(V,E)$ with n vertices.

- $\Omega = [q]^n$.
- $f_{u,v} = \{\sigma \in \Omega : \mathrm{col}(u) = \mathrm{col}(v)\}$
- $A(f_{u,v}, \sigma) = \{$All q mutations of σ through $v\}$ — $q > \mathrm{e}\Delta$
- $A(f_{u,v}, \sigma) = \{$Only conflict-free mutations$\}$ — $q \geq \Delta + 1$

Measuring Flaws

Persistence

Let D_i be the subgraph of arcs labeled f_i. The persistence of f_i is

$$A_i = \max_{\sigma \in f_i} \frac{\mathrm{In}(\sigma)}{\mathrm{Out}(\sigma)}$$

Potential Causality

Write $i \to j$ if there is any $\sigma \xrightarrow{f_i} \tau$ such that $f_j \in U(\tau) \setminus (U(\sigma) \setminus f_i)$

$i \to i$ if f_i is still present in τ and $i \to j$ if f_j is a "new" flaw in τ

Transience

The digraph D is transient if for every $i \in [m]$,

$$\sum_{j \leftarrow i} A_j < \frac{1}{\mathrm{e}} \ .$$

Main Result

- Let $\sigma_1 \in \Omega$ be arbitrary.
- For $t = 1, 2, \ldots$
  - Let f_i be a random flaw present in σ_t.
  - Address f_i by taking a uniformly random action in $A(f_i, \sigma)$.

Theorem

Let $T_0 = \log|\Omega| + |U(\sigma_1)|$. If D is transient the probability that the walk does not each a sink within $t = O(T_0 + s)$ steps is less than 2^{-s}.

Features

- No need for a uniform sample to start off.
- Running time depends on $|U(\sigma_1)|$, not $|F|$. Ω-sampling $\implies T_0 = O(|F|)$.

Flaw-choice: we can do better than random

Left-handed version

General LLL

... If there exist positive real numbers $\{x_i\}$ such that for all $i \in [m]$,

$$\Pr(A_i) \prod_{j \in \{i\} \cup \Gamma(i)} (1 + x_j) \leq x_i \ \ldots$$

Theorem (Main result)

.... If there exist positive real numbers $\{x_f\}$ such that for every $f \in F$,

$$A_f \prod_{g \in \Gamma(f)} (1 + x_g) < x_f \ \ldots$$

In fact, it suffices to have

$$A_f \sum_{S \in \mathrm{Ind}(\Gamma(f))} \prod_{g \in S} x_g < x_f \ \ldots$$

Rainbow Perfect Matchings in a Complete Graph

Input: an edge-colored K_{2n}. Ouput: a rainbow perfect matching

- Let Ω be the set of all perfect matchings of K_{2n}.
- For each pair of vertex-disjoint edges $\{e_i, e_j\}$ with the same color let

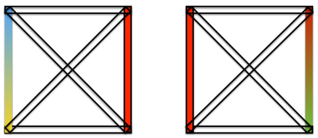
$$f_{i,j} = \{M \in \Omega : \{e_i, e_j\} \subset M\} \ .$$

Result: If no color is used more than $n/(2\mathrm{e})$ times... $O(n^2 \log n)$ time.

Algorithm Design via Atomicity

Algorithm: swap both same-color edges out.

Atomicity: Consider $M \xrightarrow{f_{i,j}} M'$. Add to M' edges $\{e_i, e_j\}$ and then close the two cycles.

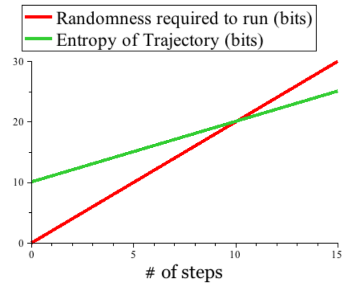


Remark: Swapping out only one of the two edges does not work.

Proof

Entropic Method

- Atomicity $\Rightarrow$ if each state transition emits the flaw addressed, the trajectory up to moment t can be reconstructed from: $I(t) = \langle f^1, f^2, \ldots, f^t, \sigma_t \rangle$
- Amenability $\Rightarrow$ Every t-step trajectory must consume at least $L(t)$ bits
- Transcience $\Rightarrow$ $H(I_t) < L(t)$ for $t > T_0$
- $\Longrightarrow$ $\exists$ likely trajectories of length $< t$



Summary + Future Work

- No probability function — Just Ω and subsets
- No variables — Arbitrary Ω and subsets
- State-dependent actions — Non-trivial Algorithms

Summary + Future Work

- No probability function — Just Ω and subsets
- No variables — Arbitrary Ω and subsets
- State-dependent actions — Non-trivial Algorithms

Future Work

- Quantum version
- Backtracking algorithms
- Sampling

Thanks!