A simple algorithm for sampling colourings of $G(n, d/n)$ up to Gibbs Uniqueness Threshold

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- show that the walk mixes in polynomial time
- for general G, we have polynomial mixing for any $k > \frac{11}{6}\Delta$ [Vigoda:'99]

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  - ... does not depend on the execution time

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MCMC sampling

There is a MCMC sampling k-colouring algorithm which has polynomial mixing for typical instances of $G(n, d/n)$, for any $k \geq \frac{11}{2}d$. [Efthymiou:'14]

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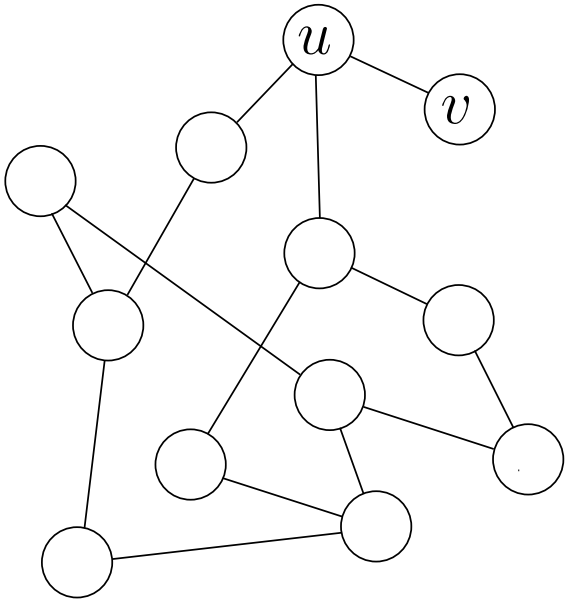
"Weitz-sampling"

There is a FPAUS sampling k-colouring algorithm for typical instances of $G(n, d/n)$, for any $k > 3d$. [Yin, Zhang:'15]

A simple observation



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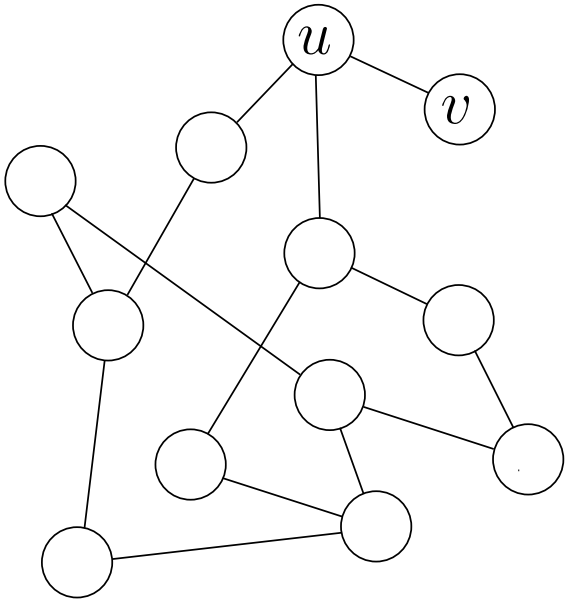

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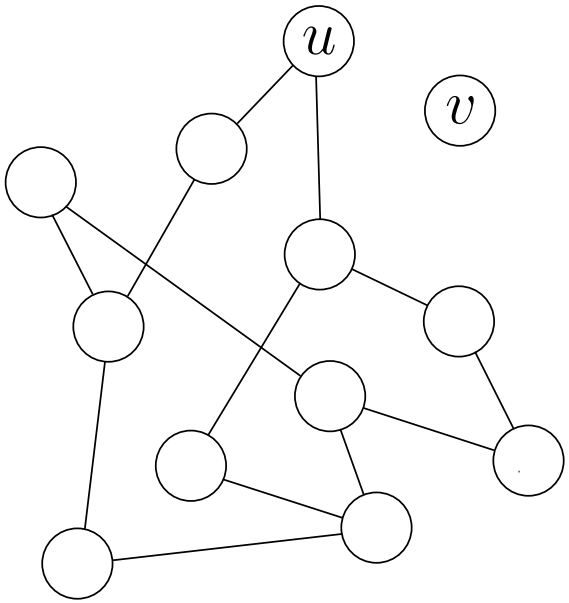

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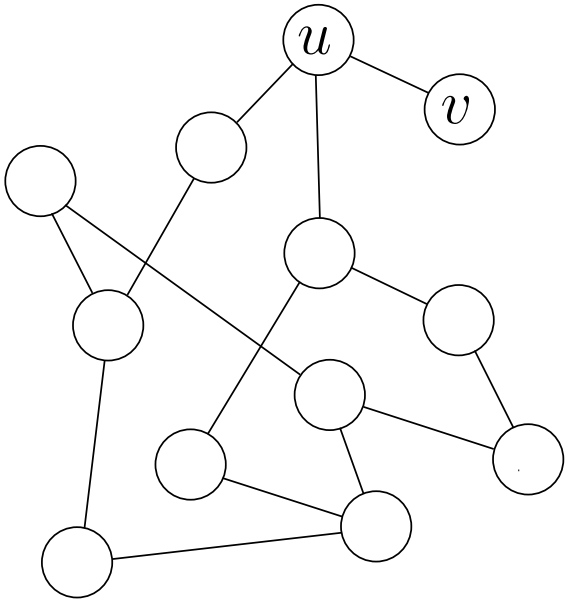

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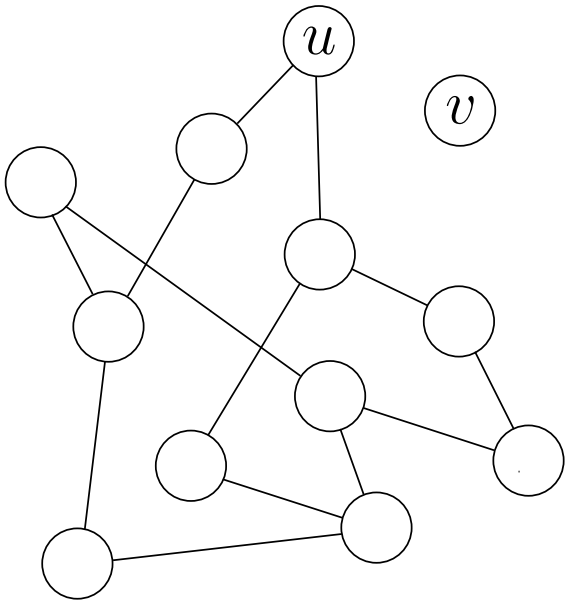

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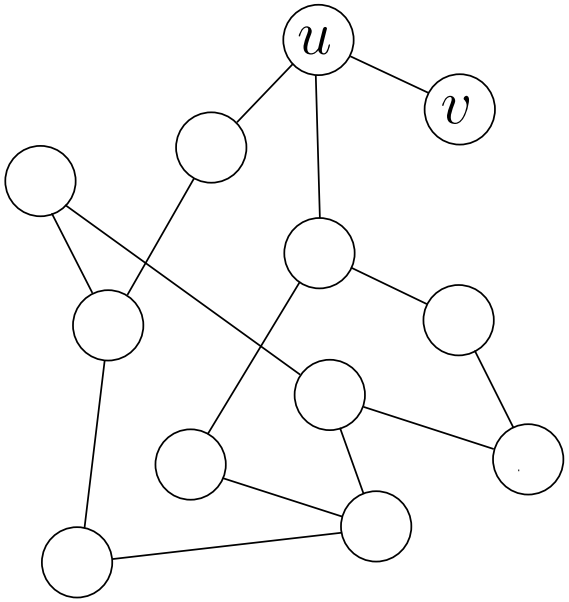

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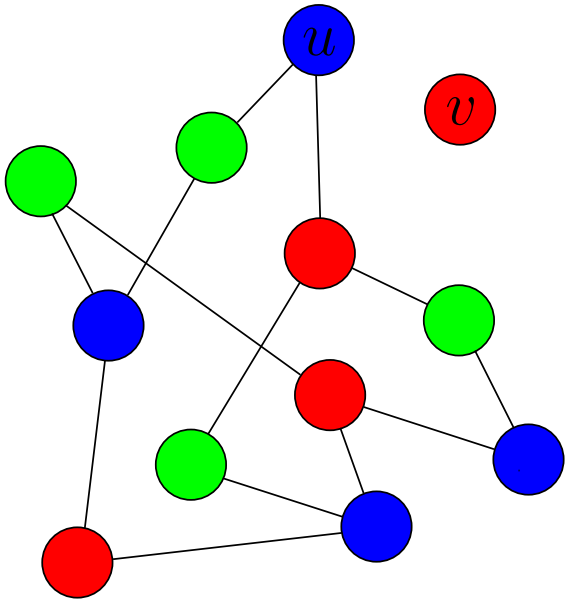

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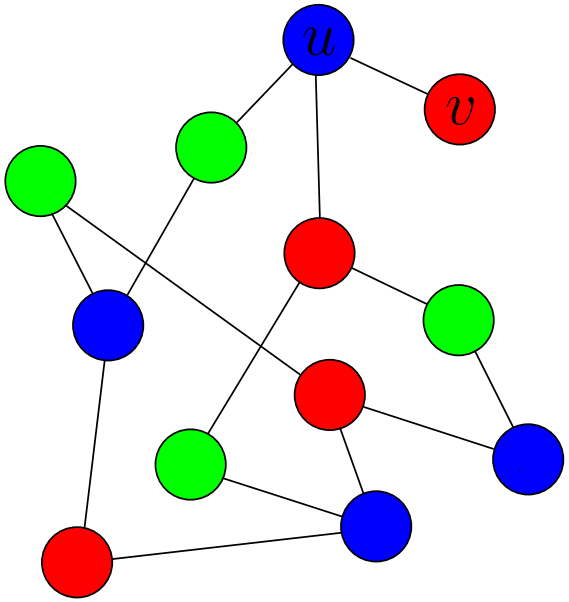

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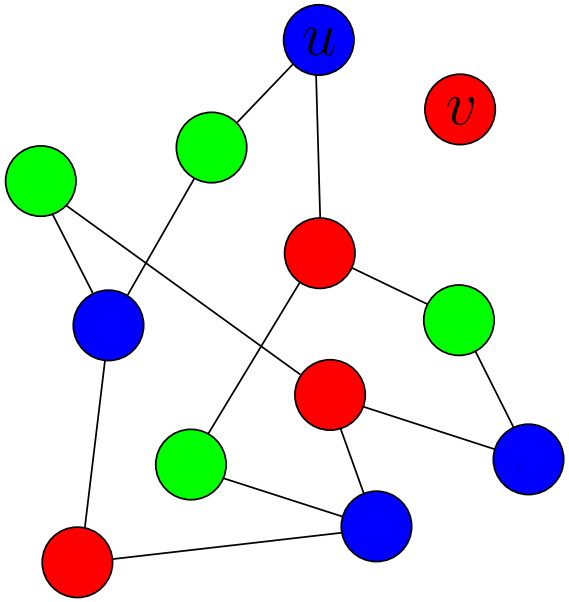

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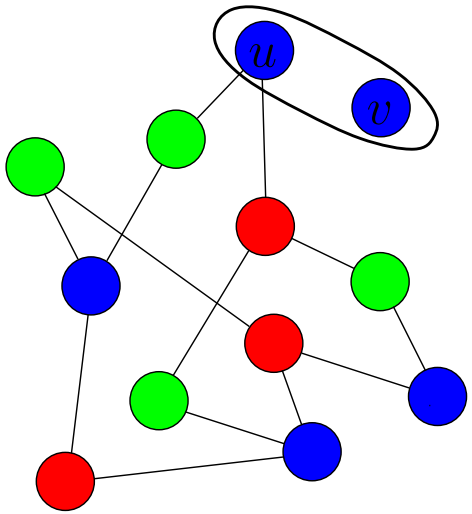

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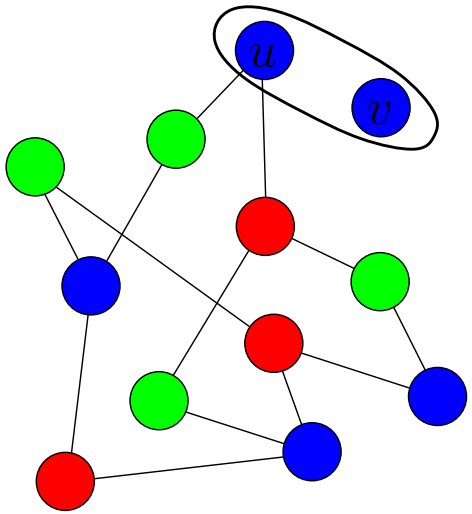

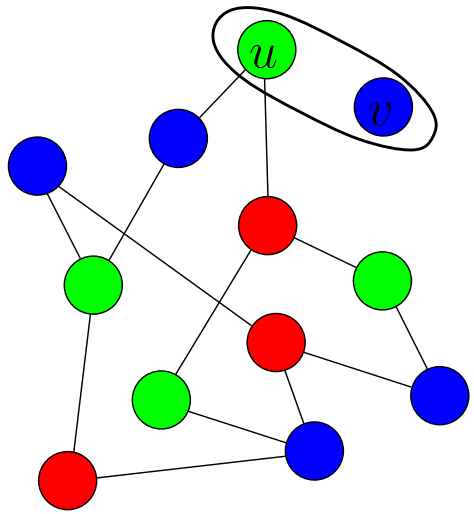

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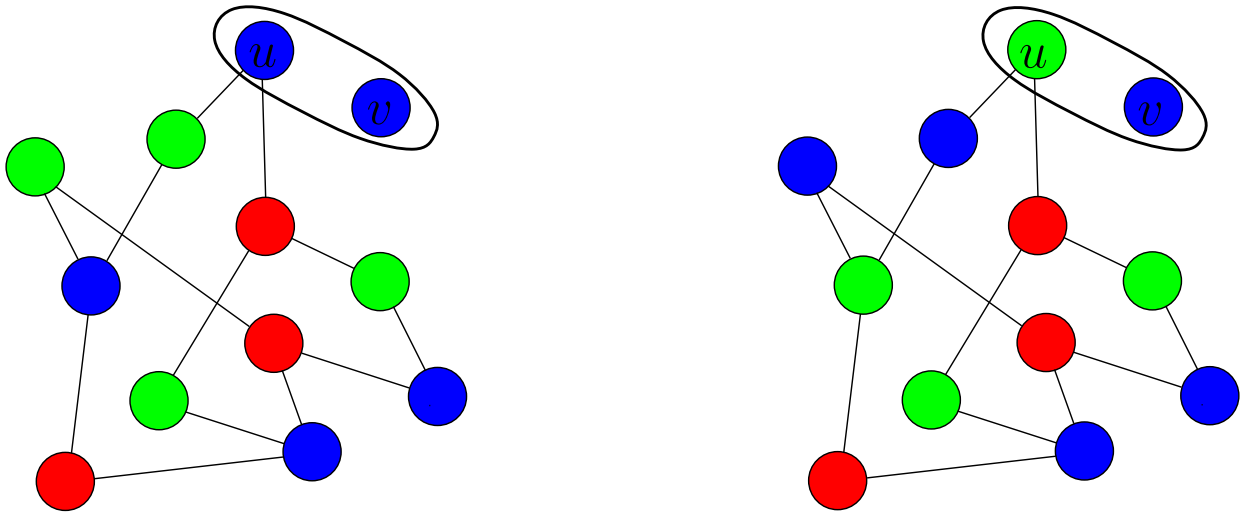

Be careful...

We can not change the colours of the vertices arbitrarily.

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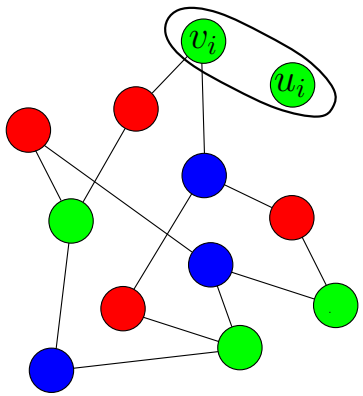

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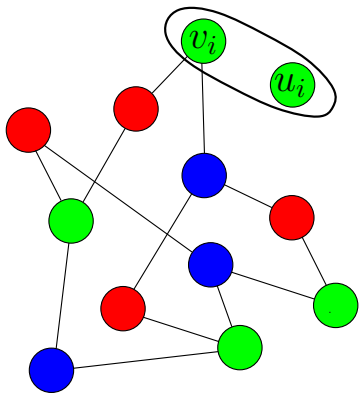

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Output: The colouring of G_r

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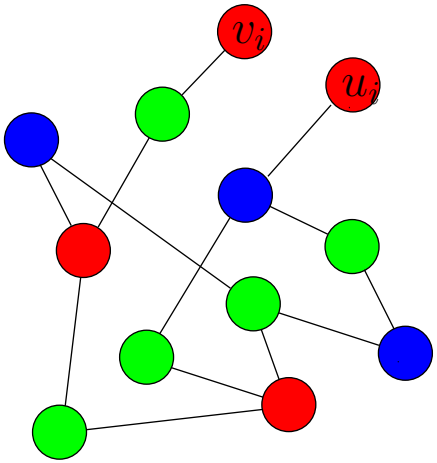

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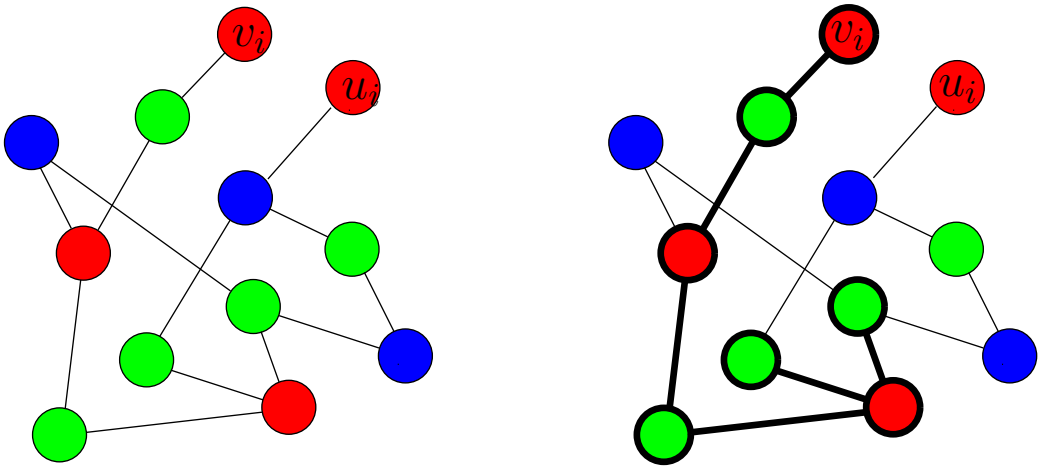

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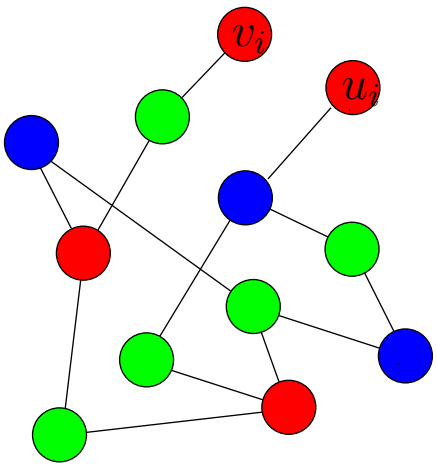

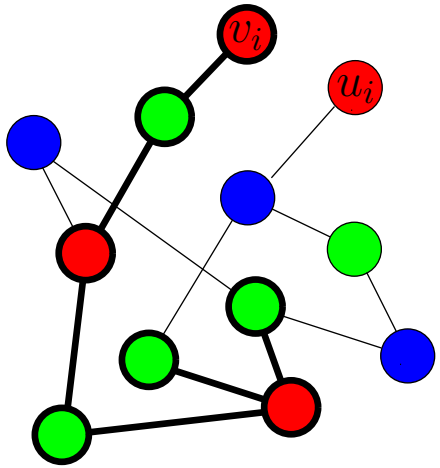

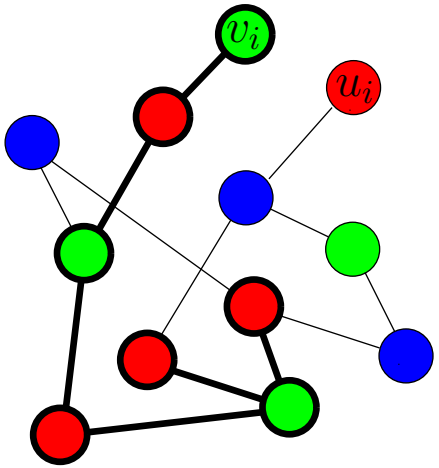

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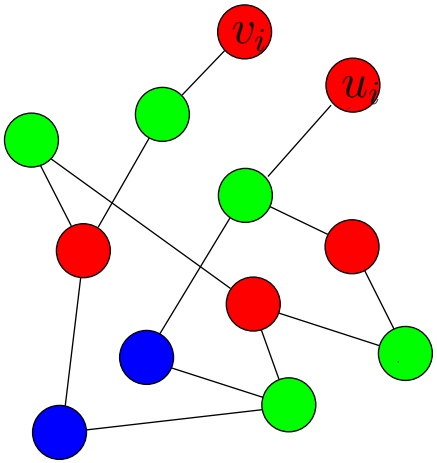

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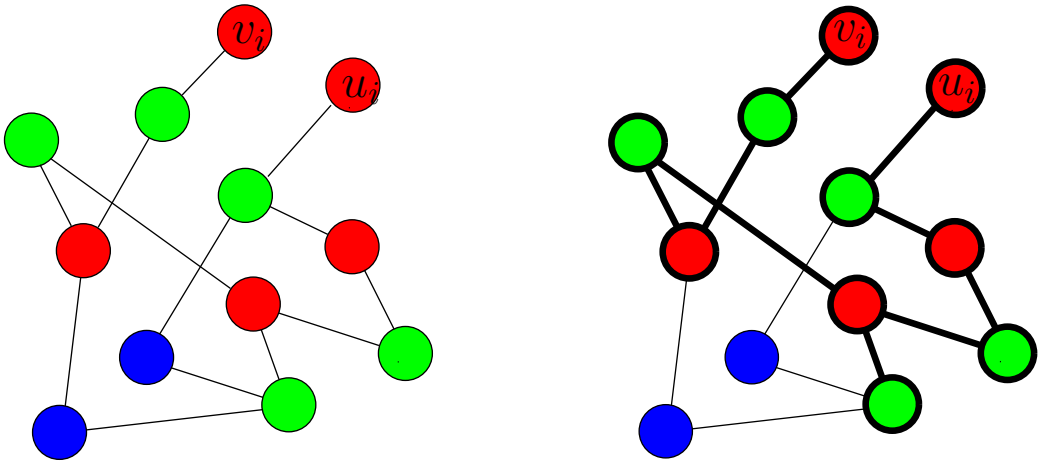

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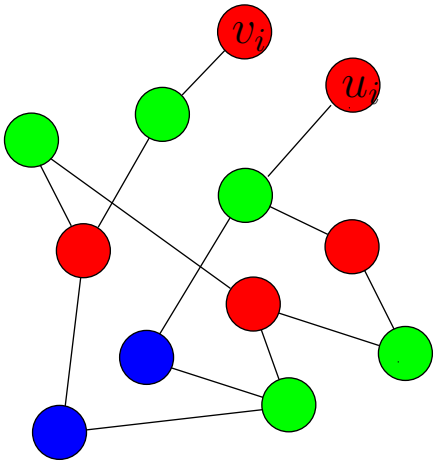

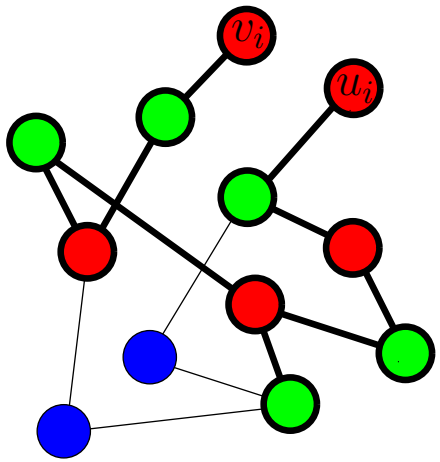

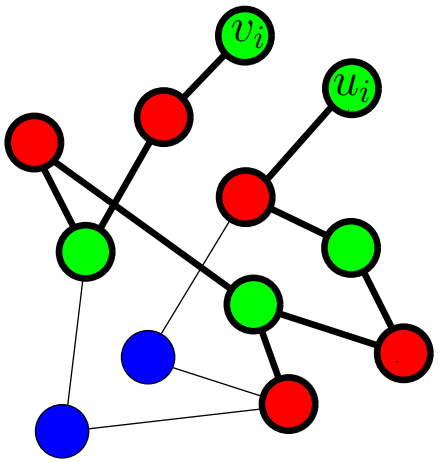

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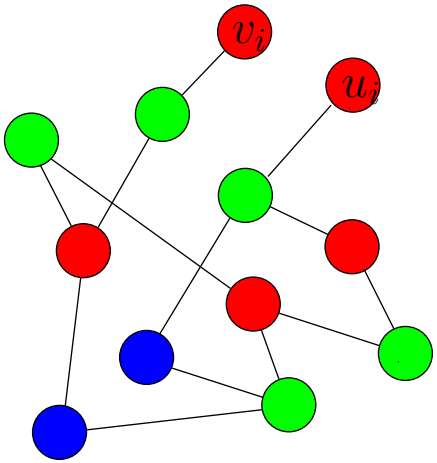

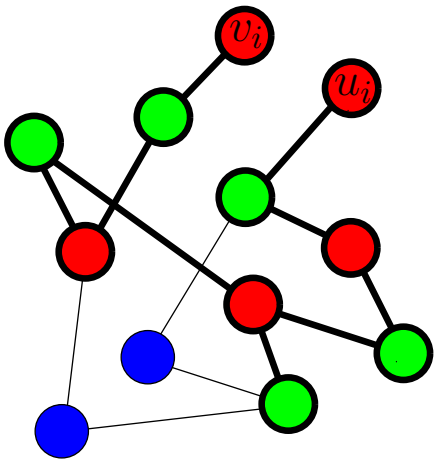

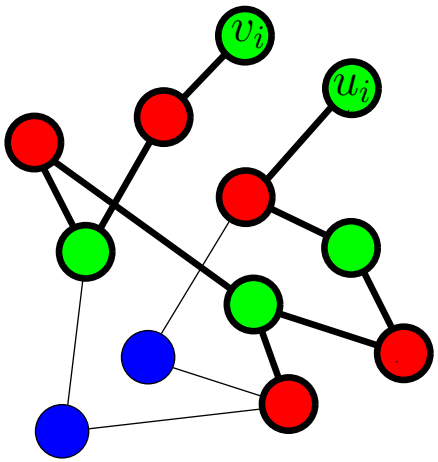

Pathological Colouring

Every k-colouring which specifies a 2-coloured path between v_i and u_i

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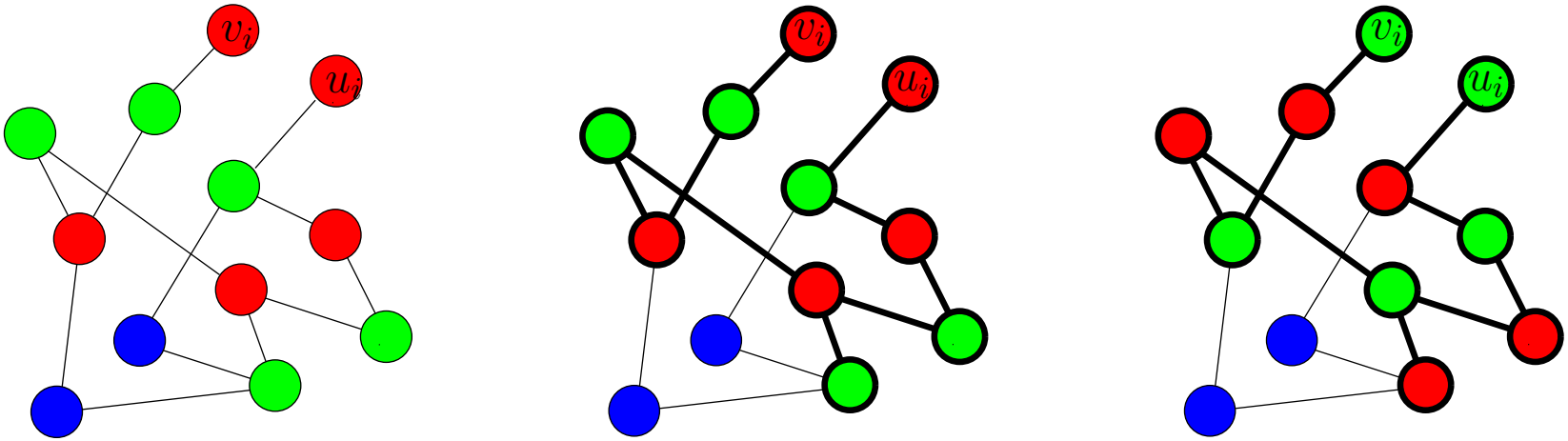

Pathological Colouring

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Remark

The existence of pathological colourings makes `UPDATE` an approximation algorithm

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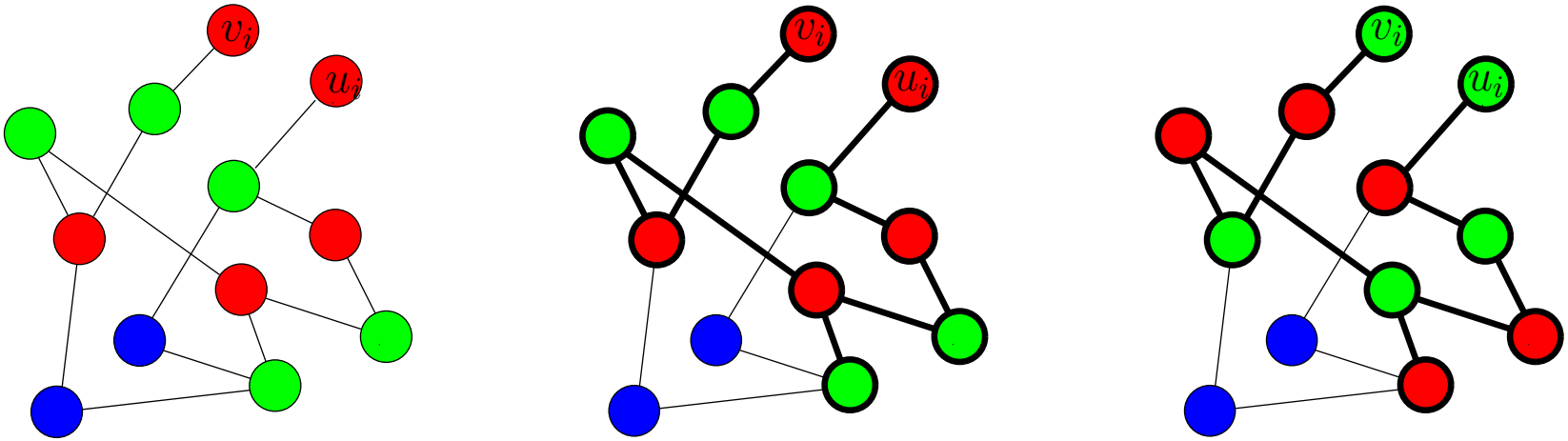

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Concequently...

The random colouring is an approximation one algorithm

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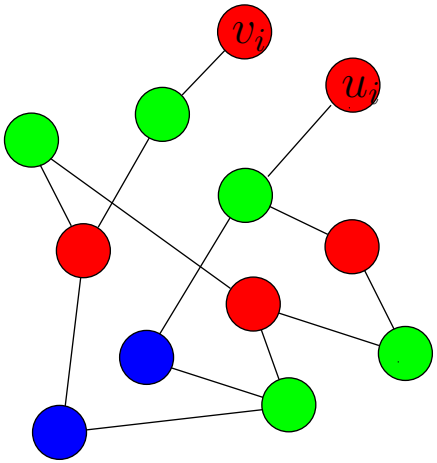

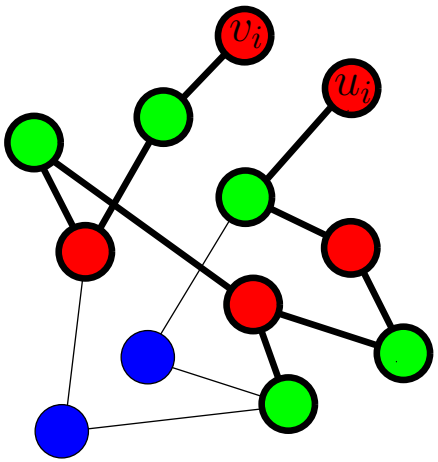

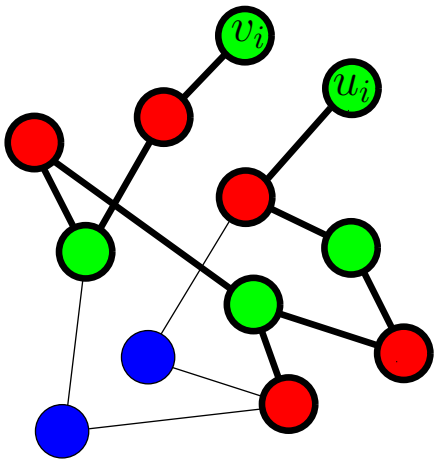

Pathological Colouring

Every k-colouring which specifies a 2-coloured path between v_i and u_i

Remark

The algorithm turns out to be accurate because the pathological colouring are relatively rare for the range of k we consider

Result -The algorithm

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input: $G(n, d/n)$, k

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Output: the colouring of G_r

Remark

Typically, each component of G_0 is either trivial or an isolated cycle.

Result

Theorem

Take $k = (1+\epsilon)d$ and assume the input graph is $G(n, d/n)$. Let μ, μ' be the Gibbs distribution of the k-colourings of the input graph and the distribution of the output of the algorithm, respectively. With probability $1 - O(n^{-\gamma})$ over $G(n, d/n)$ it holds that

$$||\mu - \mu'||_{TV} = O(n^{-\gamma})$$

Some remarks about the error

Definition

Given G_i, v_i, u_i and k, we let

- X_i a random colouring of G_i
- $Y_{i+1} = \texttt{Update}(X_i, v_i, u_i)$
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Theorem

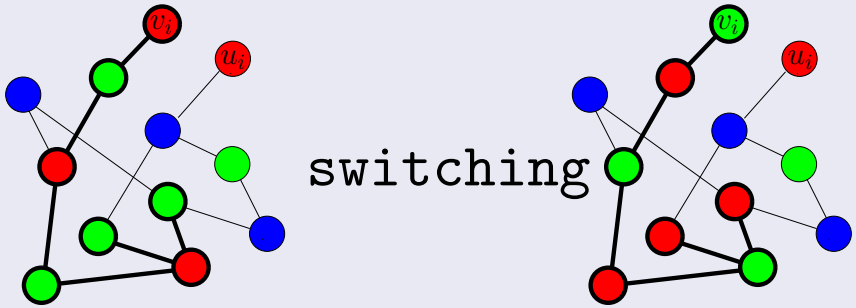
Let μ, μ' be the Gibbs distribution of the colourings of input graph and the distribution of the output of the algorithm, respectively. It holds that

$$||\mu - \mu'||_{TV} \leq \sum_{i=1}^{r} ||\mu_i - \nu_i||_{TV}$$

switching as a map

switching as a map

"switching from red to green"



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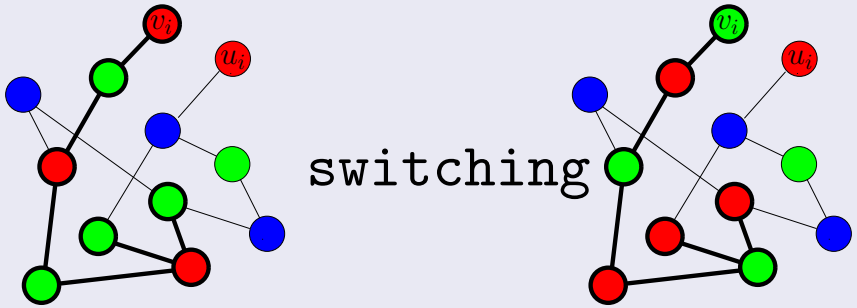

Remarks

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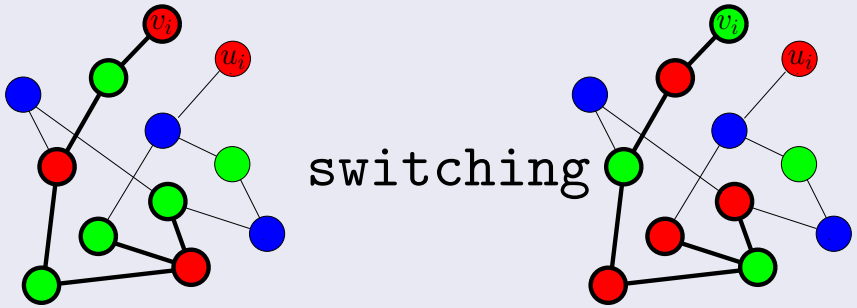

Remarks

- we want to implement a mapping from Ω_{rr} to Ω_{gr}

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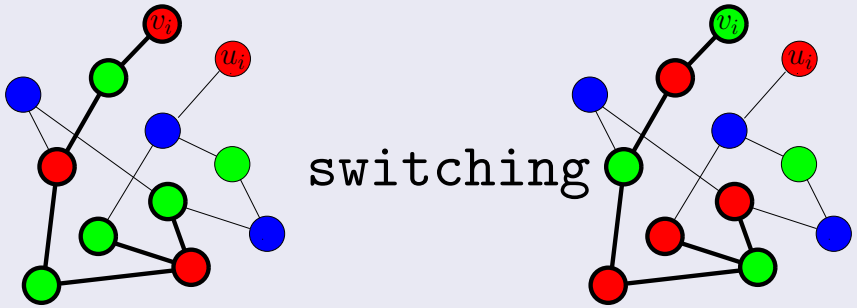

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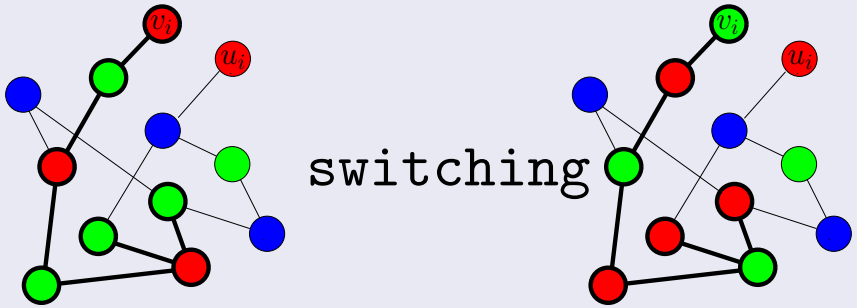

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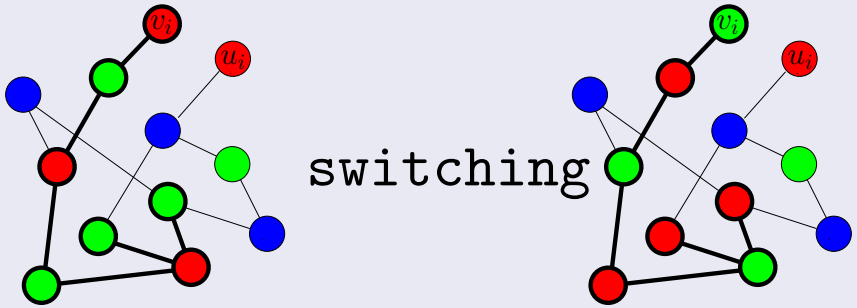

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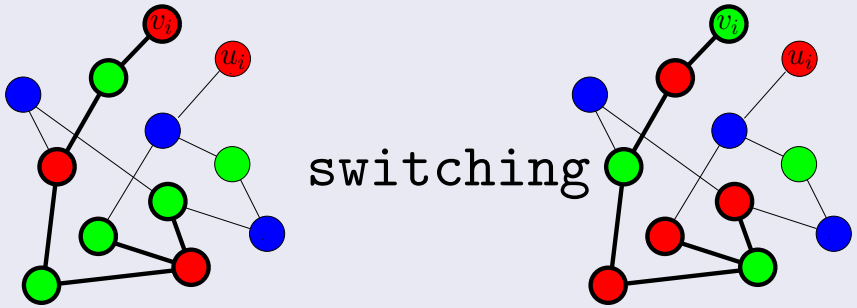

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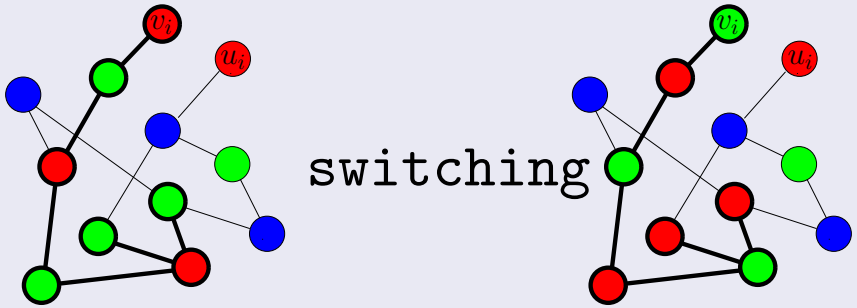

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The effect of pathological colouring

The effect of pathological colouring

... reminder

- G_i, v_i, u_i
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- $Y_{i+1} = \texttt{Update}(X_i, v_i, u_i)$

$$||\mu_i - \nu_i||_{TV} \leq ?$$

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Paths of disagreements in X_i

For $c, q \in [k]$ such that $c \neq q$, let $\varrho_i(c, q)$ be the expected number of paths from v_i to u_i coloured with (c, q), in X_i

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- show that $\mathbb{E}[\varrho_i(c, q)]$ is sufficiently small.

Upper bounding $\mathbb{E}[\varrho_i(c, q)]$

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Linearity of the expectation

- consider a permutation of, say, l vertices with v_i first and u_i last
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  - estimate the probability based on what we have revealed and the worst-case assumptions

The probability of a 2 coloured path I

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Graph first

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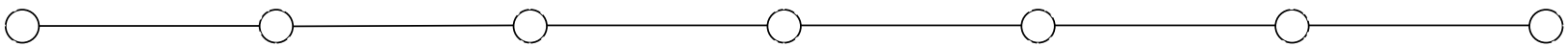

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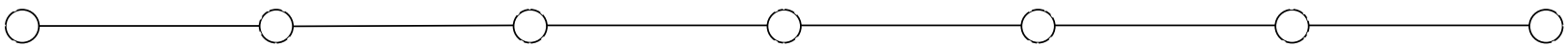

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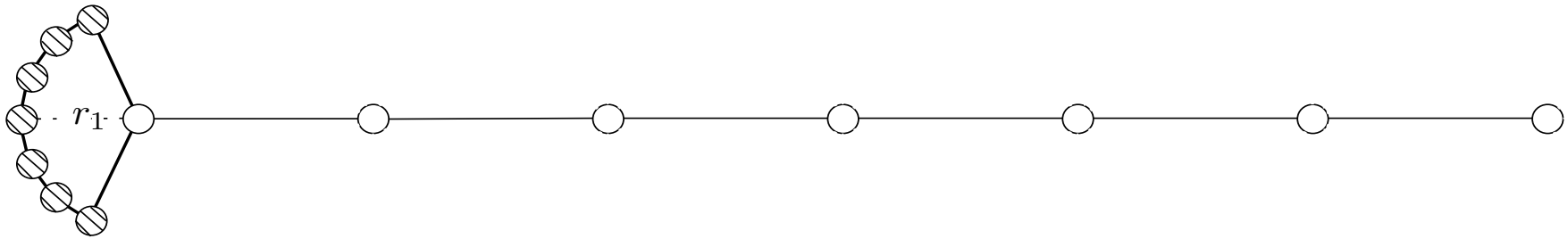

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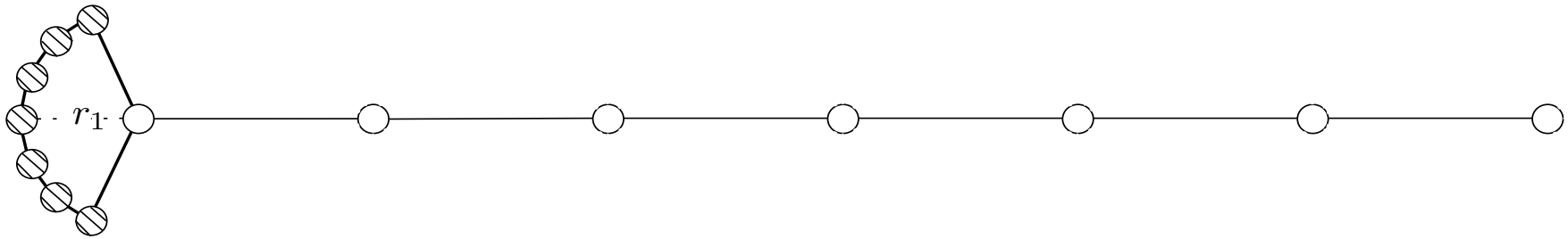

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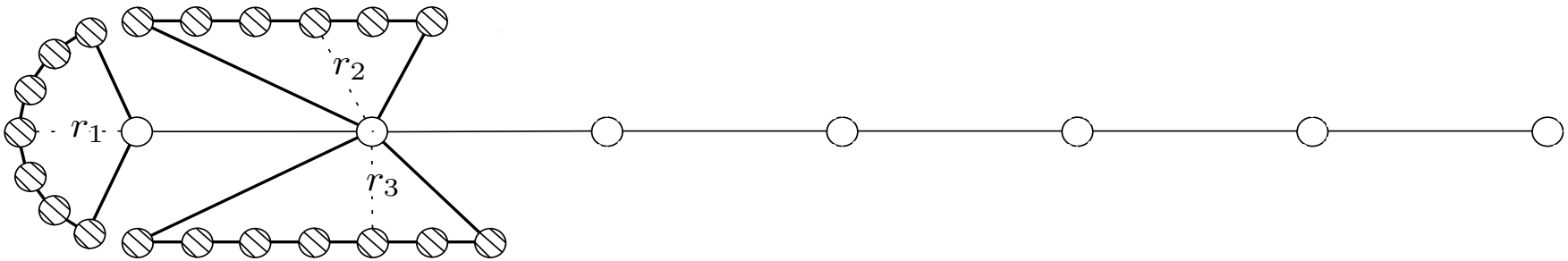

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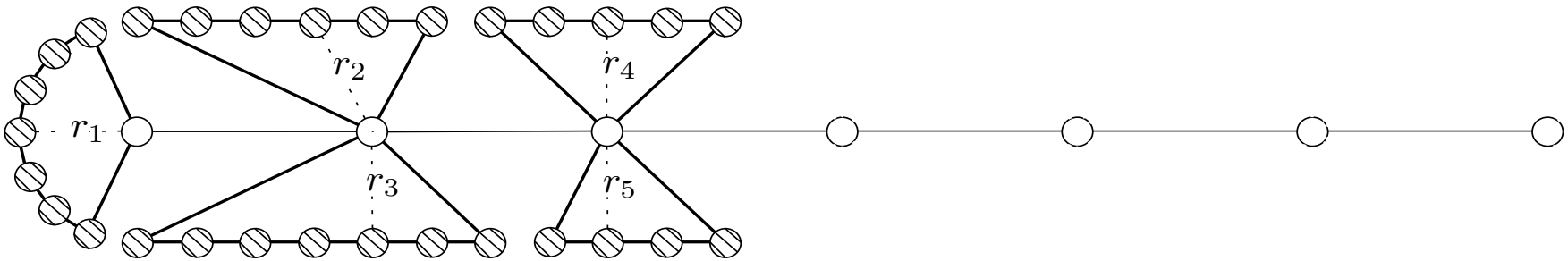

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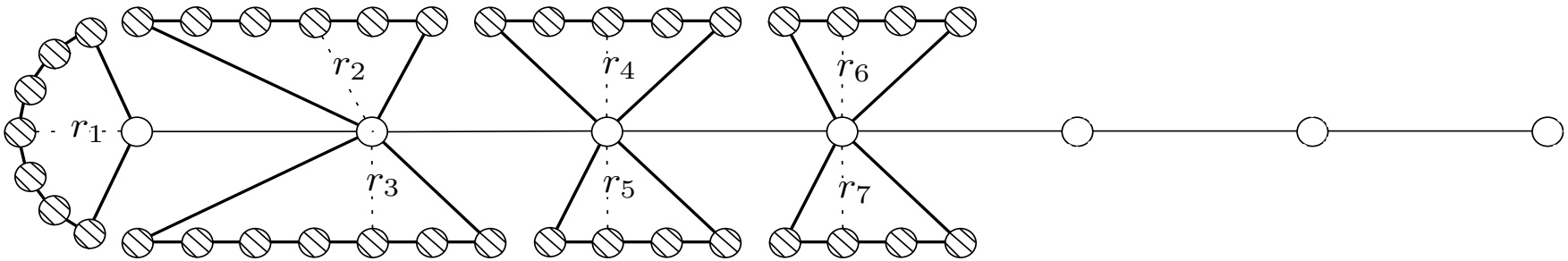

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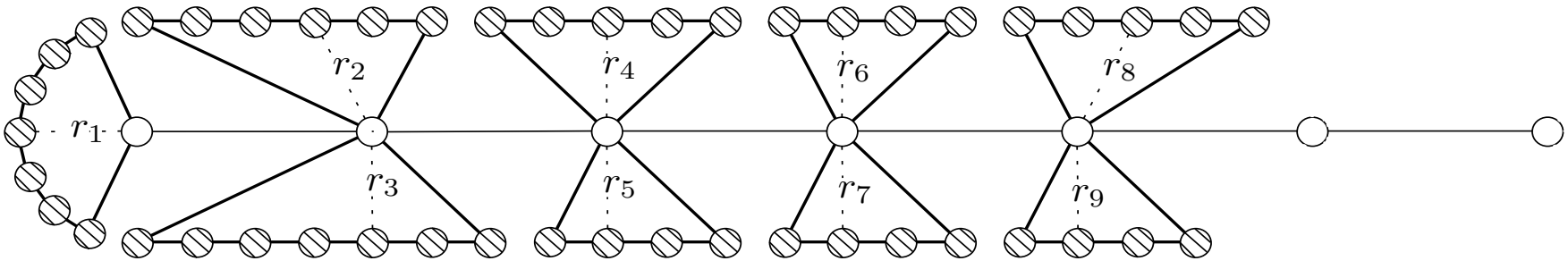

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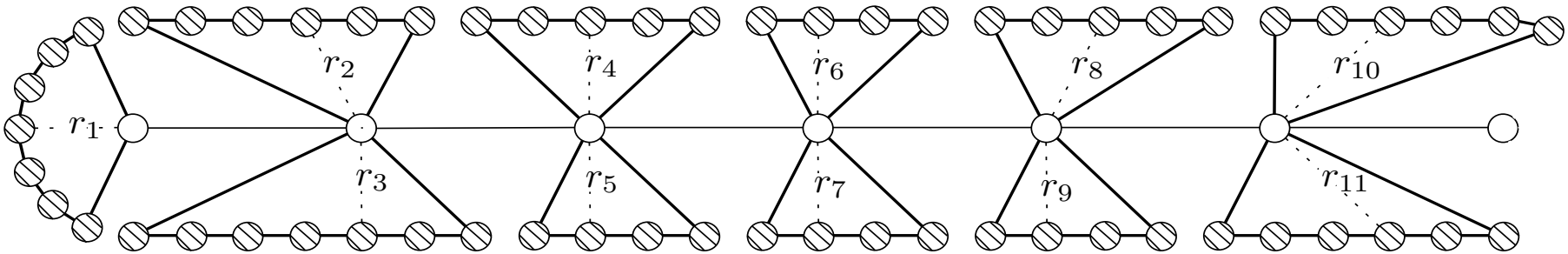

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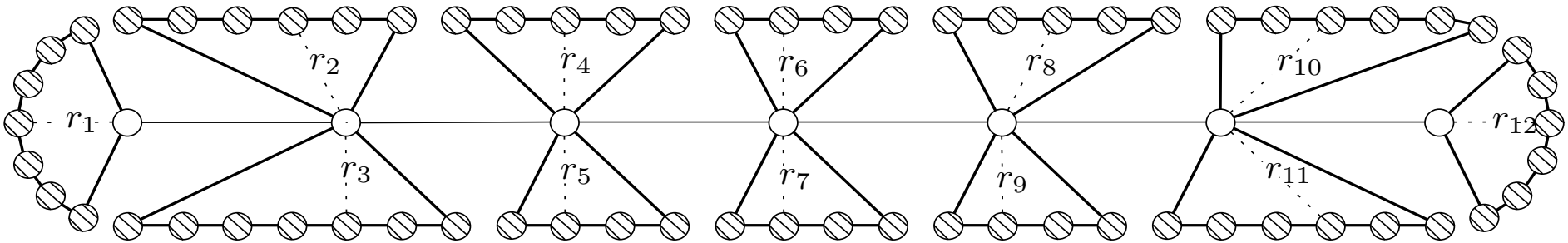

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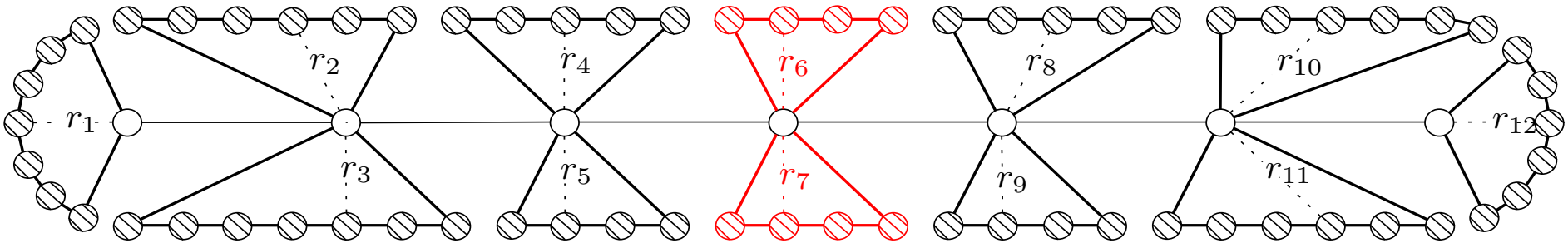
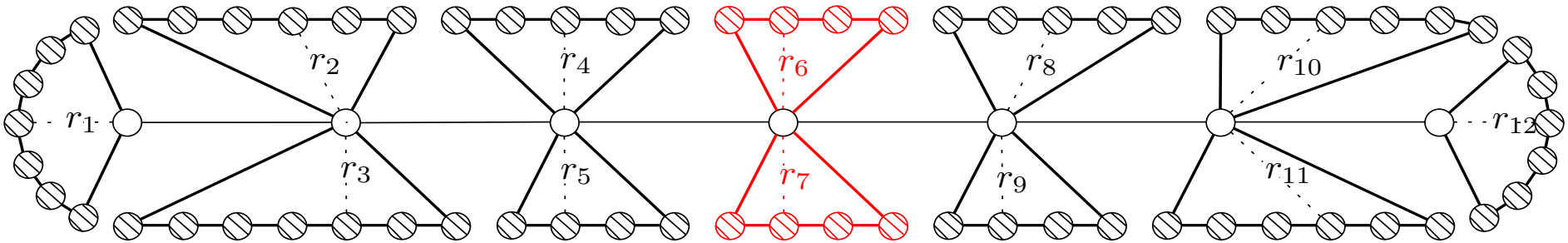


The probability of a 2 coloured path I

Graph first

- reveal a neighborhood around each vertex in a BFS manner
  - constant number of neighbours
  - everything is within radius r'
- "most of the times"
  - the neighborhood is a tree of height at most r'
  - maximum degree $< (1 + \epsilon/2)d$
  - the neighborhood does not intersect with others



The probability of a 2 coloured path II

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The random colouring part

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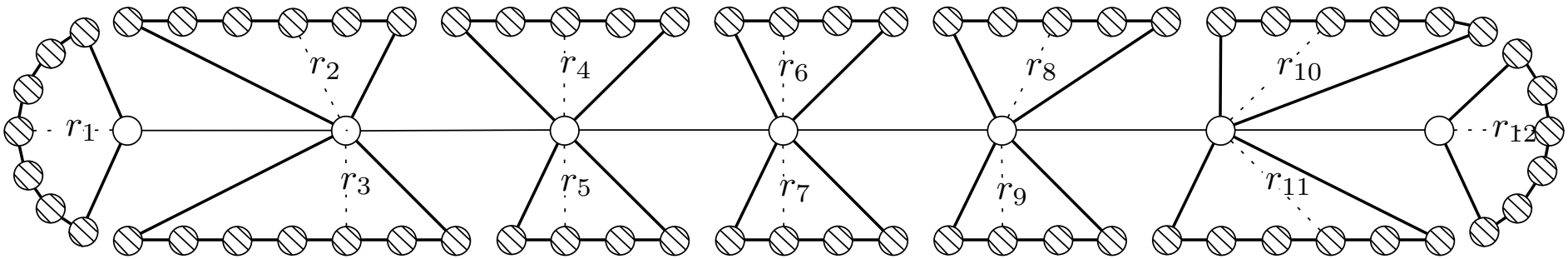

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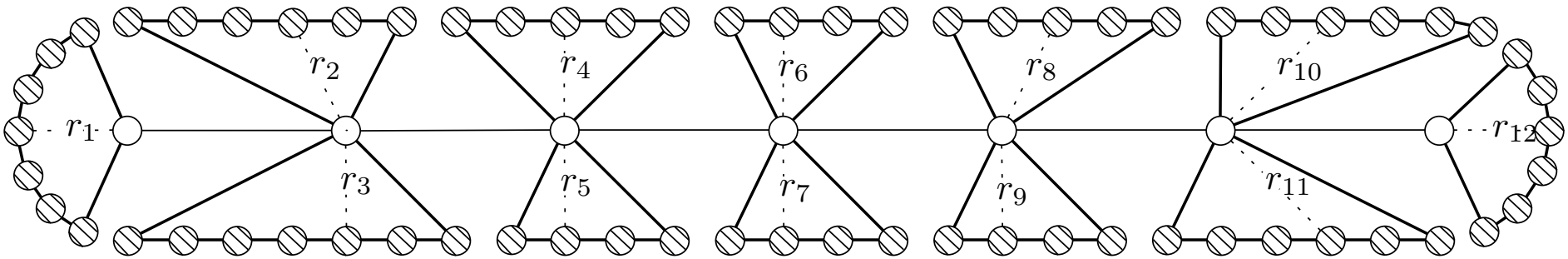
- ideally we consider a convex combination of boundary conditions



The probability of a 2 coloured path II

The random colouring part

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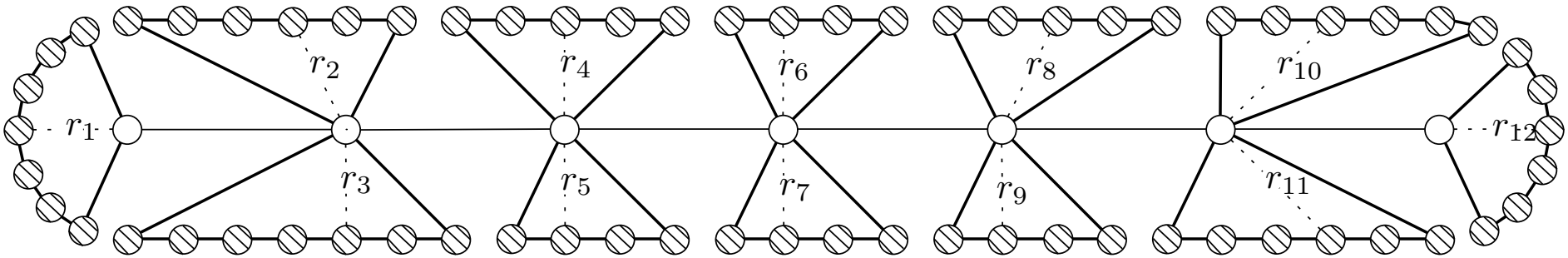
- instead we consider a worst case boundary condition



The probability of a 2 coloured path II

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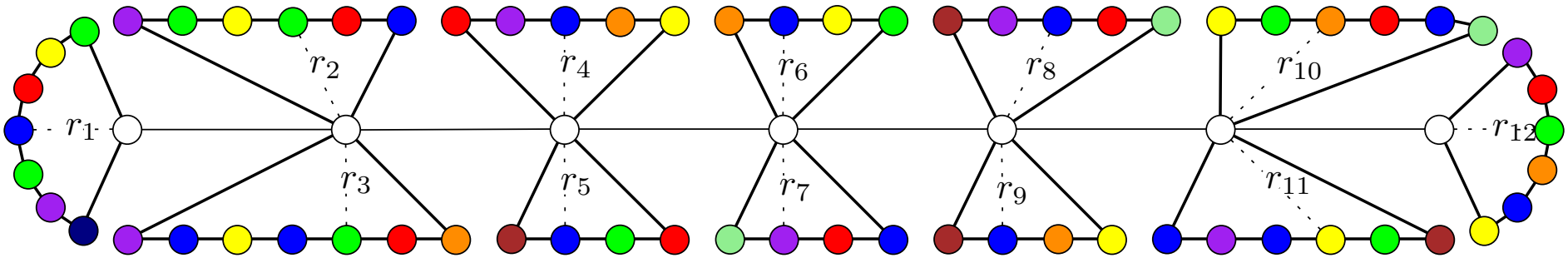

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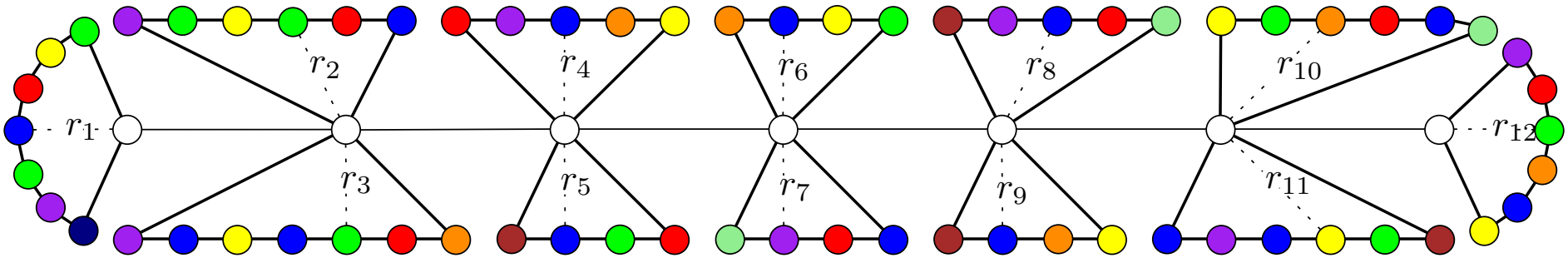

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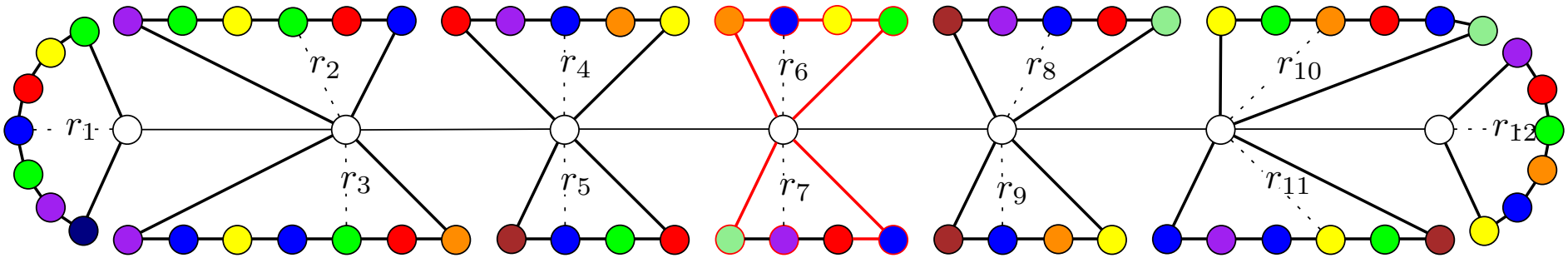

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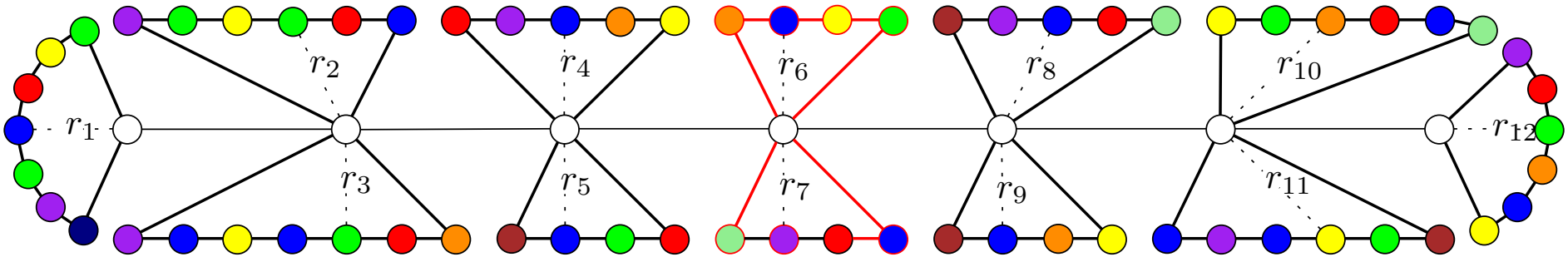

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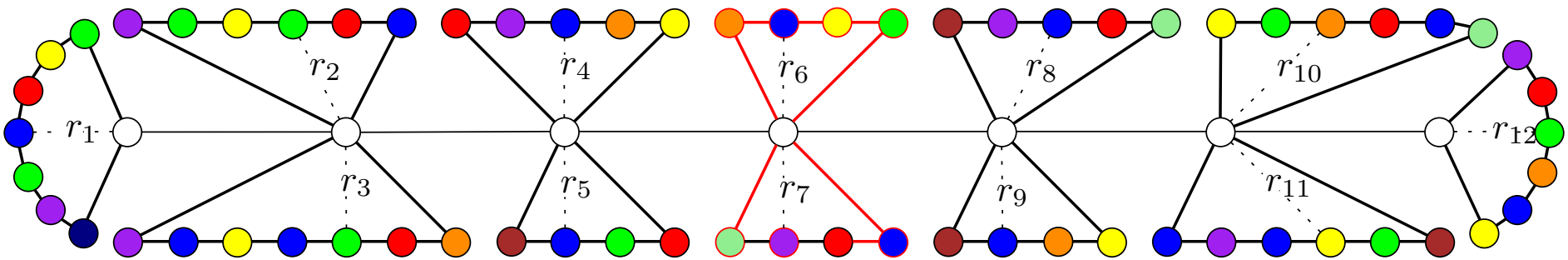

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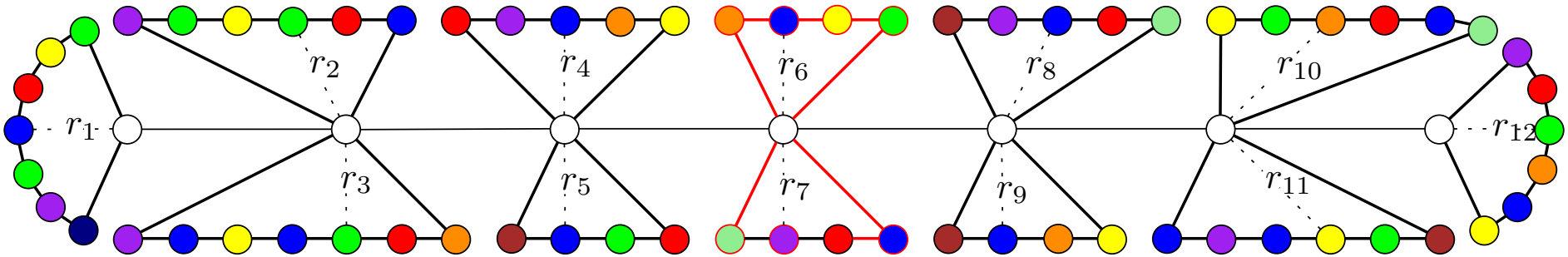

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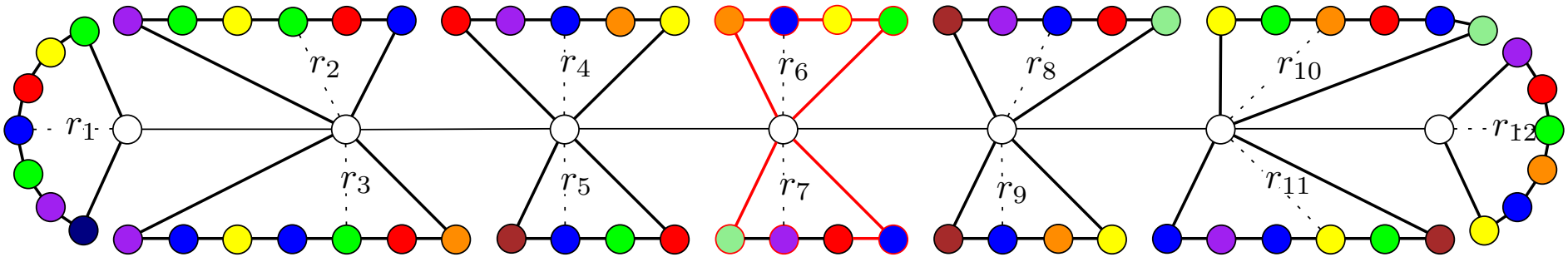

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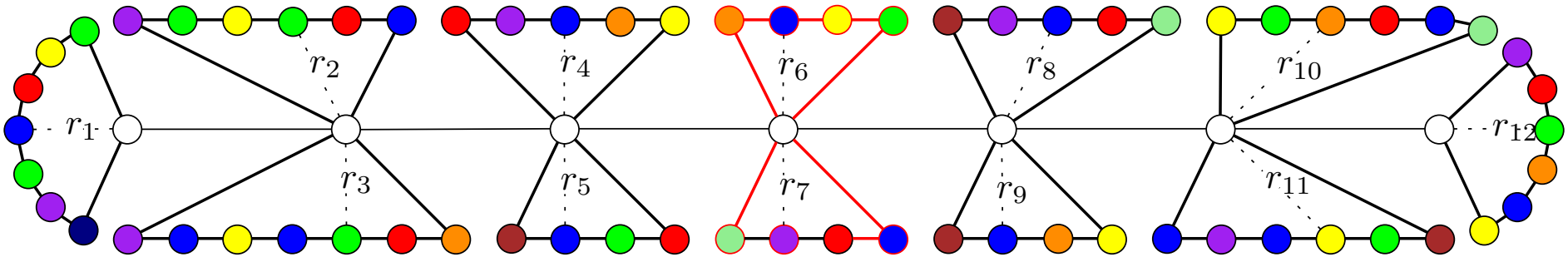
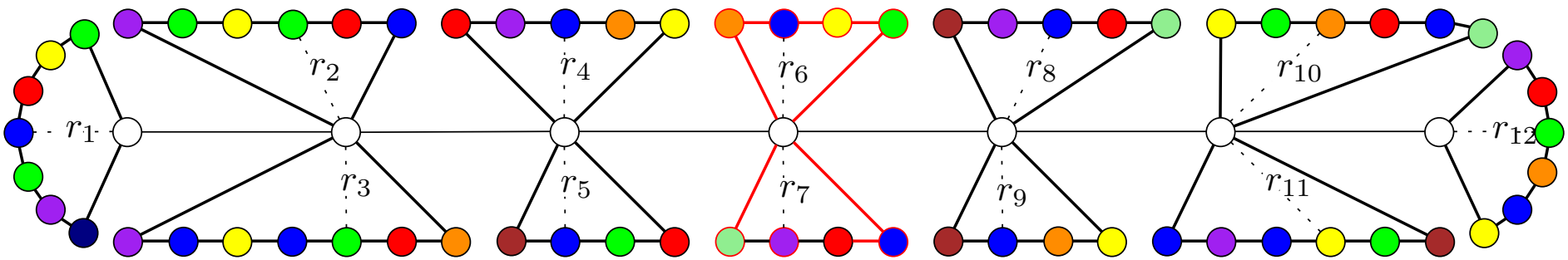


Effect of neighbour's structure



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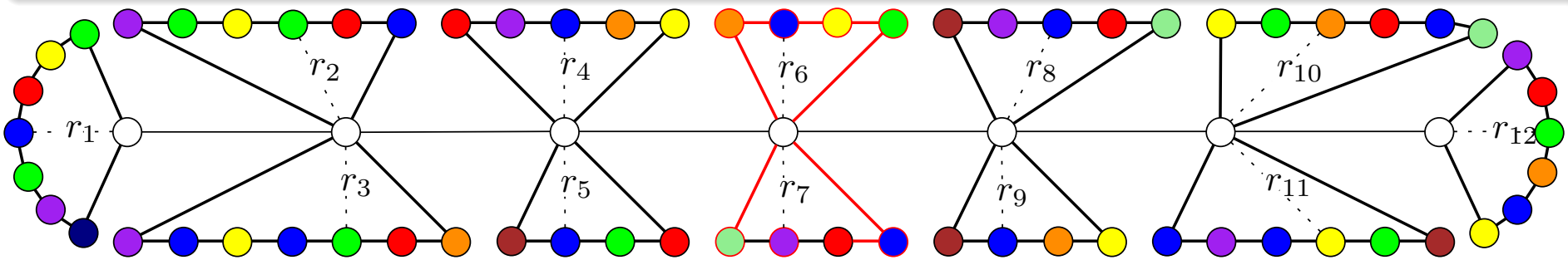

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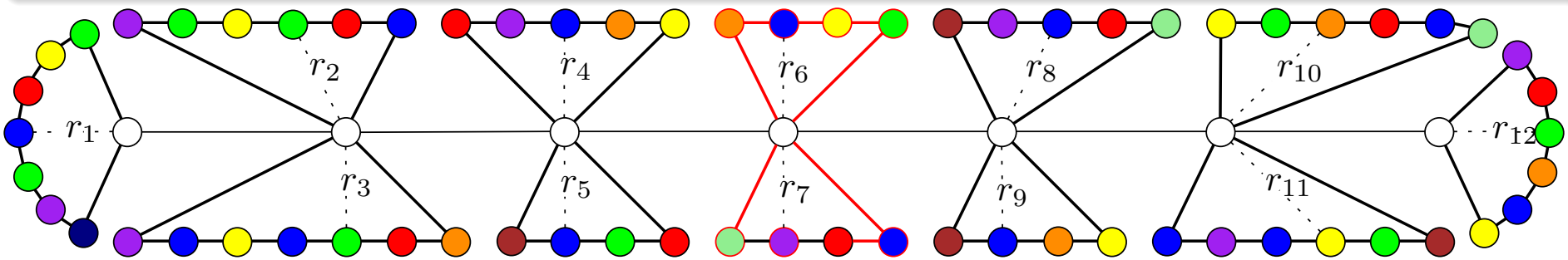

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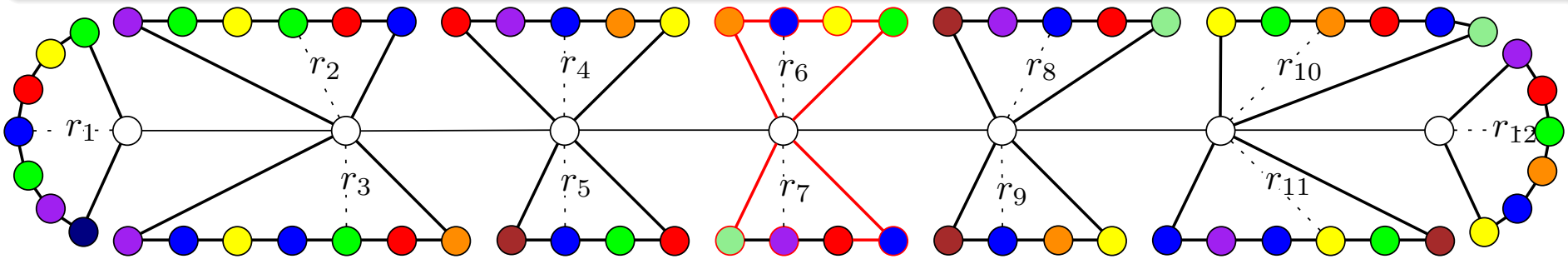

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For $k = (1+\epsilon)d$ and any $0 \leq i \leq r$, it holds that

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Corollary 2

For $k = (1+\epsilon)d$ and input graph $G(n, d/n)$ the following is true: Let μ, μ' be the Gibbs distribution of the k-colourings of $G(n, d/n)$ and the distribution of the output of the algorithm, respectively, then

$$\mathbb{E}||\mu - \mu'||_{TV} \leq O(n^{-\gamma}).$$

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- `UPDATE`, in its current form, is not expected to work for $k < d$

Thank You!!!