Unbalanced random matching markets: the stark effect of competition

Itai Ashlagi, Yash Kanoria, Jacob D. Leshno

Columbia Business School
Matching markets

Markets characterized by:
- Indivisibilities
- Capacity constraints

Often two-sided
Matching markets

Dating/marriage market
Matching markets

Applicants

Programs

School/college admissions
Matching markets

Workers

Firms

Labor market
Matching markets

Housing market
Our goal

We study the effect of competition in matching markets
We study matching markets where

- Agents have heterogeneous preferences
  - Each agent has an ordered preference list over the other side
- There are no transfers.

[Gale & Shapley 1962]

Successful solution concept of stable matchings:

- Market designs
  - e.g. school admissions, residency match, engineering college admissions in India
- Verified in empirical studies of decentralized markets
Stable Matchings = Core Allocations

- A matching \( \mu \) is **stable** if there is no **blocking pair**: man and woman \((m, w)\) who both prefers each other over their current match.

- The set of stable matchings is a non-empty lattice, whose extreme points are the **Men Optimal Stable Match** (MOSM) and the **Women Optimal Stable Match** (WOSM)
  - Men are matched to their most preferred stable woman under the MOSM and their least preferred stable woman under the WOSM
Example

- The preferences of 3 men and 4 women:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>
Example

- A non stable matching (A and 2 would block)

\[
\begin{array}{ccc}
A & B & C \\
1 & 2 & 4 \\
2 & 4 & 1 \\
3 & 3 & 2 \\
4 & 1 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
A & C & A & B \\
B & A & C & C \\
C & B & B & A \\
\end{array}
\]
Example

- The MOSM:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

- (3 is unmatched)
Example

- The WOSM:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

(3 is unmatched)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>C</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td></td>
</tr>
</tbody>
</table>
Our model: random matching markets

Matching markets with heterogeneous preferences and no transfers.

Random Matching Market:
A set of $n$ men $\mathcal{M}$ and a set of $n + k$ women $\mathcal{W}$

- Man $m$ has complete preferences over women, drawn i.i.d. uniformly at random.
- Woman $w$ has complete preferences over men, drawn i.i.d. uniformly at random.
Questions

- **Average rank, or who gets to choose?**
  - Define men’s average rank of their wives under $\mu$

$$R_{\text{Men}}(\mu) := \frac{1}{|\mu^{-1}(\mathcal{W})|} \sum_{m \in \mu^{-1}(\mathcal{W})} \text{Rank}_m(\mu(m))$$

  excluding unmatched men from the average

- Average rank is 1 if all men got their most preferred wife, higher rank is worse.

- **How many agents have multiple stable assignments?**
  - Agents can manipulate iff they have multiple stable partners
Previous Literature

- Pittel (1989), Knuth, Motwani, Pittel (1990), Roth, Peranson (1999) – when there are equal number of men and women
  - Under MOSM men’s average rank of wives is $\log n$, but it is $n/\log n$ under the WOSM
  - The core is large – most agents have multiple stable partners
- Roth, Peranson (1999) document that in the NRMP the MOSM and WOSM are almost the same: suggest short lists as reason
- Immorlica, Mahdian (2005) and Kojima, Pathak (2009) show that if one side has short random preference lists the core is small
- Holzman and Samet (2013) show if preferences are highly correlated the core is small.

Is the small observed core driven by features like short lists & correlation in preferences?
We consider random matching markets with unequal number of men and women.
Men’s average rank of wives, $|\mathcal{W}| = 40$

\[
\mathbb{E}[R_{Men}] = \frac{10}{\log(40)} \approx 4.0 \quad \text{and} \quad \frac{40}{\log(40)} \approx 10.0
\]
Men’s average rank of wives, $|\mathcal{W}| = 40$
Men’s average rank of wives, $|\mathcal{W}| = 40$
Men’s average rank of wives, $|\mathcal{W}| = 40$
Percent of matched men with multiple stable partners $|\mathcal{W}| = 40$
Main Theorem

**Theorem:** Consider a random market with $n$ men and $n + k$ women for $k = k(n) \geq 1$. With high probability in any stable matching,

\[ R_{\text{Men}} \leq 1.01 \left( \frac{n+k}{n} \right) \log \left( \frac{n+k}{k} \right) \]

and

\[ R_{\text{Women}} \geq \frac{n}{\left[ 1 + 1.01 \left( \frac{n+k}{n} \right) \log \left( \frac{n+k}{k} \right) \right]} \]

Moreover,

\[ R_{\text{Men}}(\text{WOSM}) \leq (1 + o(1))R_{\text{Men}}(\text{MOSM}) \]

\[ R_{\text{Women}}(\text{WOSM}) \geq (1 - o(1))R_{\text{Women}}(\text{MOSM}) \]

And $o(n)$ of men and $o(n)$ of women have multiple stable matches.
Main Theorem

- That is, under all stable matchings
  - men do almost as well as they would if they chose, ignoring women’s preferences.
  - Women are either unmatched or roughly getting a randomly assigned man.

- The core is small
  - Limited choice for centralized mechanisms
  - Implies limited scope for manipulation of stable mechanisms
  - Facilitates comparative statics etc.
Corollary 1: One women makes a difference

**Corollary:** In a random market with $n$ men and $n + 1$ women, with high probability

$$R_{\text{Men}} \leq 1.01 \log n$$

and

$$R_{\text{Women}} \geq \frac{n}{1.01 \log n}$$

in all stable matches, and a vanishing fraction of agents have multiple stable partners.
Corollary 2: Large Unbalance

**Corollary:** Consider a random market with $n$ men and $(1 + \lambda)n$ women for $\lambda > 0$. Let $\kappa = 1.01 (1 + \lambda) \log(1 + 1/\lambda)$. With high probability, in all stable matchings

$$R_{\text{Men}} \leq \kappa$$

and

$$R_{\text{Women}} \geq \frac{n}{1 + \kappa}$$

Ashlagi, Braverman, Hassidim (2011) showed that the core is small in this setting.
Intuition

- In a competitive assignment market with \( n \) homogenous buyers and \( n \) homogenous sellers the core is large, but the core shrinks when there is one extra seller.
- In a matching market the addition of an extra woman makes all the men better off
  - Every man has the option of matching with the single woman
  - But only some men like the single woman
  - Changing the allocation of some men requires changing the allocation of many men: If some men are made better, and some women are made worse off, creating more options for men. All men benefit, and the core is small.
Proof overview

Calculate the WOSM using:

- **Algorithm 1**: Men-proposing Deferred Acceptance gives MOSM
- **Algorithm 2**: MOSM $\rightarrow$ WOSM

Both algorithms use a sequence of proposals by men

Stochastic analysis by sequential revelation of preferences
Algorithm 1: Men-proposing DA (Gale & Shapley)

Everyone starts unmatched. We add the men one at a time, running a ‘chain’ for each man.

Chain for adding $m_k$:

- Set $m_k$ to be the proposer.
- The proposer proposes to his (next) most preferred woman $w$.
  - If $w$ is unmatched, end chain and continue to add $m_{k+1}$
  - Otherwise, $w$ rejects her less preferred man between $m_k$ and her current partner. Repeat, with rejected man proposing.
Algorithm 2: MOSM $\rightarrow$ WOSM

We look for stable improvement cycles for women.

We iterate:

Phase: For candidate woman $\hat{w}$, reject her match $m$, starting a chain.

Two possibilities for how the chain ends:

- (Improvement phase) Chain reaches $\hat{w} \Rightarrow$ new stable match.
- (Terminal phase) Chain ends with unmatched woman $\Rightarrow m$ is $\hat{w}$’s best stable match. $\hat{w}$ is no longer a candidate.
Illustration of Algorithm 2: MOSM → WOSM

$w_1$ is matched to $m_1$ under MOSM

$w_2$ prefers $m_1$ to $m_2$

rejects $m_1$ to start a chain
Illustration of Algorithm 2: MOSM → WOSM

\[ w_2 \text{ prefers } m_1 \text{ to } m_2 \]
Illustration of Algorithm 2: MOSM → WOSM
Illustration of Algorithm 2: MOSM → WOSM

\[ w_1 \text{ prefers } m_3 \text{ to } m_1 \]
Illustration of Algorithm 2: MOSM $\rightarrow$ WOSM

New stable match found. Update match and continue.
Illustration of Algorithm 2: MOSM → WOSM
Illustration of Algorithm 2: MOSM → WOSM
Illustration of Algorithm 2: MOSM → WOSM

\( w_1 \) \( m_3 \)

\( w_4 \) \( m_4 \)

\( w_4 \) prefers \( m_3 \) to start a chain
Illustration of Algorithm 2: MOSM → WOSM

$w_5$ prefers $m_4$ to $m_5$
Chain ends with a proposal to unmatched woman $\bar{w}$

$\Rightarrow m_3$ is $w_1$’s best stable partner

and similarly $w_4$ and $w_5$ already had their best stable partner
Illustration of Algorithm 2: MOSM → WOSM

Chain ends with a proposal to unmatched woman \( \bar{w} \)

\( \Rightarrow m_3 \) is \( w_1 \)'s best stable partner

and similarly \( w_4 \) and \( w_5 \) already had their best stable partner
Illustration of Algorithm 2: MOSM → WOSM

Chain ends with a proposal to unmatched woman $\bar{w}$

$\Rightarrow m_3$ is $w_1$’s best stable partner

and similarly $w_4$ and $w_5$ already had their best stable partner
Illustration of Algorithm 2: MOSM → WOSM
Illustration of Algorithm 2: MOSM → WOSM

$w_1$ $w_4$ $w_5$ $\tilde{w}$

$m_3$ $m_4$ $m_5$ $m_8$

$w_8$ rejects $m_8$ to start a new chain
Illustration of Algorithm 2: MOSM $\rightarrow$ WOSM

$w_4$ prefers $m_8$ to $m_4$

But $w_4$ is already matched to her best stable partner
Illustration of Algorithm 2: MOSM $\rightarrow$ WOSM

$w_1$ prefers $m_3$ to $m_4$

But $w_4$ is already matched to her best stable partner

$\Rightarrow m_8$ is $w_8$'s best stable partner
Illustration of Algorithm 2: MOSM → WOSM

$w_4$ prefers $m_8$ to $m_4$

But $w_4$ is already matched to her best stable partner

$\Rightarrow m_8$ is $w_8$’s best stable partner
Algorithm 2: MOSM → WOSM

Initialize $S = \mathcal{W}_{\text{unmatched}}$

1. Choose $\hat{w}$ in $\mathcal{W}\setminus S$ if non-empty.

2. **Phase:** Record the current match as $\hat{\mu}$. Woman $\hat{w}$ rejects her partner, man $m$, starting a chain where $\hat{w}$ accepts a proposal only if the proposal is preferred to $m$.

3. **Two possibilities for how the chain ends:**
   - (Improvement phase) If the chain ends with acceptance by $\hat{w}$, we have found a new stable match. Return to Step 2.
   - (Terminal phase) Else the chain ends with acceptance by $\mathcal{W}_{\text{unmatched}}$. Woman $\hat{w}$ has found her best stable partner. Roll the match back to $\hat{\mu}$. Add $\hat{w}$ to $S$ and return to Step 1.
Overview of stochastic analysis

- Analysis of MPDA is similar to that of Pittel (1989)

- Analysis of Algorithm 2: MOSM → WOSM more involved.

  Key finding: In a typical market, very few agents participate in improvement cycles.
Proof idea:

- Analysis of MPDA is similar to that of Pittel (1989)
  - Coupon collectors problem

- Analysis of Algorithm 2: MOSM → WOSM more involved.
  - $S$ grows quickly
  - Once $S$ is large improvement phases are rare
  - Together, in a typical market, very few agents participate in improvement cycles.
Strategic implications

- Men proposing DA (MPDA) is strategyproof for men, but no stable mechanism is strategyproof for all agents.
- A woman can manipulate MPDA only if she has multiple stable husbands
  - Misreport truncated preferences.
- In unbalanced matching market a diminishing number of women have multiple stable husbands
  - Mechanism is approximately strategyproof
Further questions

- Can we allow correlation in preferences?
  - Perfect correlation leads to a unique core.
  - But “short side” depends on more than $N, K$
    - Tiered market: 30 men, 40 women: 20 top, 20 mid
  - What is a general balance condition? Who chooses in more general settings?

- Are there any real/natural matching markets with large cores?
  - Extensive simulations suggest the answer is “no”
Men’s average rank of wives

<table>
<thead>
<tr>
<th>n</th>
<th>diff</th>
<th>-10</th>
<th>-1</th>
<th>0</th>
<th>+1</th>
<th>+5</th>
<th>+10</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>MOSM</td>
<td>29.5</td>
<td>20.3</td>
<td>5.0</td>
<td>4.1</td>
<td>3.0</td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td>WOSM</td>
<td>30.1</td>
<td>23.6</td>
<td>20.3</td>
<td>4.9</td>
<td>3.2</td>
<td>2.6</td>
</tr>
<tr>
<td>200</td>
<td>MOSM</td>
<td>53.6</td>
<td>35.3</td>
<td>5.7</td>
<td>4.8</td>
<td>3.7</td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td>WOSM</td>
<td>54.7</td>
<td>41.0</td>
<td>35.5</td>
<td>5.7</td>
<td>3.8</td>
<td>3.2</td>
</tr>
<tr>
<td>500</td>
<td>MOSM</td>
<td>115.8</td>
<td>75.9</td>
<td>6.7</td>
<td>5.7</td>
<td>4.5</td>
<td>3.9</td>
</tr>
<tr>
<td></td>
<td>WOSM</td>
<td>118.0</td>
<td>86.6</td>
<td>76.2</td>
<td>6.7</td>
<td>4.7</td>
<td>4.0</td>
</tr>
<tr>
<td>1000</td>
<td>MOSM</td>
<td>203.8</td>
<td>136.2</td>
<td>7.4</td>
<td>6.4</td>
<td>5.2</td>
<td>4.6</td>
</tr>
<tr>
<td></td>
<td>WOSM</td>
<td>207.5</td>
<td>155.1</td>
<td>137.3</td>
<td>7.4</td>
<td>5.4</td>
<td>4.7</td>
</tr>
<tr>
<td>2000</td>
<td>MOSM</td>
<td>364.5</td>
<td>249.6</td>
<td>8.1</td>
<td>7.1</td>
<td>5.9</td>
<td>5.3</td>
</tr>
<tr>
<td></td>
<td>WOSM</td>
<td>370.8</td>
<td>280.7</td>
<td>249.1</td>
<td>8.1</td>
<td>6.1</td>
<td>5.4</td>
</tr>
<tr>
<td>5000</td>
<td>MOSM</td>
<td>793.1</td>
<td>560.0</td>
<td>9.1</td>
<td>8.1</td>
<td>6.8</td>
<td>6.2</td>
</tr>
<tr>
<td></td>
<td>WOSM</td>
<td>804.7</td>
<td>622.5</td>
<td>560.2</td>
<td>9.1</td>
<td>7.0</td>
<td>6.3</td>
</tr>
</tbody>
</table>
Percent of men with multiple stable matches

<table>
<thead>
<tr>
<th>$n$</th>
<th>Diff:</th>
<th>-10</th>
<th>-1</th>
<th>0</th>
<th>+1</th>
<th>+5</th>
<th>+10</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td></td>
<td>2.1</td>
<td>15.1</td>
<td>75.3</td>
<td>15.4</td>
<td>4.5</td>
<td>2.3</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td>2.2</td>
<td>14.6</td>
<td>83.6</td>
<td>14.6</td>
<td>4.1</td>
<td>2.1</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>2.0</td>
<td>12.6</td>
<td>91.0</td>
<td>13.1</td>
<td>3.6</td>
<td>2.0</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>1.9</td>
<td>12.3</td>
<td>94.5</td>
<td>12.2</td>
<td>3.4</td>
<td>2.0</td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td>1.8</td>
<td>11.1</td>
<td>96.7</td>
<td>11.1</td>
<td>2.9</td>
<td>1.7</td>
</tr>
<tr>
<td>5000</td>
<td></td>
<td>1.5</td>
<td>10.1</td>
<td>98.4</td>
<td>10.2</td>
<td>2.8</td>
<td>1.5</td>
</tr>
</tbody>
</table>
Many-to-one simulations

- Similar features observed when capacities are small
- See the paper for details
Conclusion

Random unbalanced matching markets are surprisingly competitive:

- The short side chooses in all stable matchings
- The core is small – most agents have a single stable partner.

Our results suggest that matching markets generically have small cores

(See also Kanoria, Saban & Sethuraman (2015) for a similar result on matching markets with transfers)
Thank you!