Power concave functions and Borell-Brascamp-Lieb inequalities

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Notations: $p$-means of non-negative numbers

Let $p \in [-\infty, +\infty]$ and $\mu \in (0, 1)$. Given two real numbers $a > 0$ and $b > 0$, the quantity

\[
M_p(a, b; \mu) = \begin{cases} 
\max\{a, b\} & p = +\infty \\
((1 - \mu)a^p + \mu b^p)^{\frac{1}{p}} & \text{for } p \neq -\infty, 0, +\infty \\
a^{1-\mu}b^\mu & p = 0 \\
\min\{a, b\} & p = -\infty.
\end{cases}
\]

is the ($\mu$-weighted) $p$-mean of $a$ and $b$.

For $a, b \geq 0$, we set $M_p(a, b; \mu) = 0$ if $ab = 0$ and $p \in \mathbb{R}$. 
Let \( \Omega \) be a convex set in \( \mathbb{R}^n \) and \( p \in [\infty, \infty] \). A nonnegative function \( u \) defined in \( \Omega \) is said \( p \)-concave if

\[
u((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y); \lambda)\]

for all \( x, y \in \Omega \) and \( \lambda \in (0, 1) \). In the cases \( p = 0 \) and \( p = -\infty \), \( u \) is also said log-concave and quasi-concave in \( \Omega \), respectively.
Let $\Omega$ be a convex set in $\mathbb{R}^n$ and $p \in [-\infty, \infty]$. A nonnegative function $u$ defined in $\Omega$ is said $p$-concave if

$$u((1 - \lambda)x + \lambda y) \geq M_p(u(x), u(y); \lambda)$$

for all $x, y \in \Omega$ and $\lambda \in (0, 1)$. In the cases $p = 0$ and $p = -\infty$, $u$ is also said log-concave and quasi-concave in $\Omega$, respectively.

In other words, a non-negative function $u$, with convex support $\Omega$, is $p$-concave if:
- it is a non-negative constant in $\Omega$, for $p = +\infty$;
- $u^p$ is concave in $\Omega$, for $p > 0$;
- $\log u$ is concave in $\Omega$, for $p = 0$;
- $u^p$ is convex in $\Omega$, for $p < 0$;
- it is quasi-concave, i.e. all of its superlevel sets are convex, for $p = -\infty$.

For $p = 1$ corresponds to usual concavity. From Jensen’s inequality it follows that if $u$ is $p$-concave, then $u$ is $q$-concave for every $q \leq p$. 
Let $0 < \lambda < 1$, $-\frac{1}{n} \leq p \leq \infty$. Let $u_0, u_1, h$ be nonnegative integrable functions defined on $\mathbb{R}^n$, satisfying

$$h((1 - \lambda)x + \lambda y) \geq M_p(u_0(x), u_1(y), \lambda)$$

for all $x \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x) \, dx \geq M_q \left( \int_{\mathbb{R}^n} u_0(x) \, dx, \int_{\mathbb{R}^n} u_1(x) \, dx, \lambda \right)$$

where

$$q = \begin{cases} 
\frac{1}{n} & p = +\infty \\
\frac{p}{pn + 1} & p \in (-1/n, +\infty) \\
-\infty & p = -1/n.
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Henstock-Macbeath (1953), Dinghas (1957)
Borell (1975), Brascamp-Lieb (1976)
The case $p = 0$

Prékopa-Leindler inequality

Let $0 < \lambda < 1$ and let $u_0$, $u_1$ and $h$ be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h((1 - \lambda)x + \lambda y) \geq u_0(x)^{1-\lambda}u_1(y)^\lambda,$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x) \, dx \geq \left( \int_{\mathbb{R}^n} u_0(x) \, dx \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} u_1(x) \, dx \right)^\lambda.$$

PL is a functional version of the Brunn-Minkowski inequality (in fact, the same could be said for BBL for any $p$).

**The Brunn-Minkowski inequality**

For measurable sets $K_0, K_1$ and $\lambda \in [0, 1]$, $K_\lambda = (1 - \lambda)K_0 + \lambda K_1$ and $+$ is the Minkowski addition, then

$$|K_\lambda| \geq M_{1/n}(|K_0|, |K_1|; \lambda) = \left[(1 - \lambda)|K_0|^{\frac{1}{n}} + \lambda|K_1|^{\frac{1}{n}}\right]^n \quad (0.1)$$

provided $K_\lambda$ is measurable as well.
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**Multiplicative form of BM**

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$$|K_\lambda| \geq |K_0|^{1-\lambda}|K_1|^\lambda$$

The BM inequality has strong and unexpected relations with many other fundamental analytic and geometric inequalities.
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$k_0, k_1$ measurable sets, $\lambda \in [0, 1]$, $k_\lambda = (1 - \lambda)k_0 + \lambda k_1$ and $+$ is the Minkowski addition, then

$$|k_\lambda| \geq M_{1/n}(|k_0|, |k_1|; \lambda) = \left((1 - \lambda)|k_0|^{\frac{1}{n}} + \lambda |k_1|^{\frac{1}{n}}\right)^n$$

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**Multiplicative form of BM**

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The BM inequality has strong and unexpected relations with many other fundamental analytic and geometric inequalities.

For references and a nice presentation, see R. J. Gardner (2002)
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Equality holds in BM if and only if $K_0$ and $K_1$ are convex and homothetic.
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**Equality conditions in BBL - Dubuc (1977)**

Equality holds in BBL for some $p \in [-1/n, \infty)$ if and only if

$h$ is $p$-concave

and there exist suitable $A, B, m, n > 0$ and $x_1, x_\lambda \in \mathbb{R}^n$ such that

$$u_0(x) = A h(mx + x_1), \quad u_1(x) = B h(nx + x_\lambda).$$
What happens if we are close to equality in one of the above mentioned inequalities?


(All for log-concave functions)
Stability/Improvements

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There are stability/quantitative results for the Brunn-Minkowski inequality for convex sets.


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Paolo Salani (DiMal - Università di Firenze)
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The Ball-Böröczky [1] result is for log-concave functions in dimension 1 and it is written as a stability result: if \( \int_{\mathbb{R}} h \, dx \leq (1 + \epsilon) \sqrt{\int_{\mathbb{R}} u_0 \, dx \int_{\mathbb{R}} u_1 \, dx} \), then there exist \( a > 0 \) and \( b \in \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}^n} |a(-1)^i u_i(x + (-1)^i b) - h(x)| \, dx \leq \gamma \epsilon^{1/6} |\log \epsilon|^{2/3} \int_{\mathbb{R}^n} h \, dx.
\]

This is extended to dimension \( n > 1 \) in [2], but only for log-concave even functions.
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Bucur-Fragalà [3] use the 1-dimensional result by Ball-Böröczky to write a quantitative version of the PL for log-concave functions in terms of some (a little bit involved) distance between $u_0$ and $u_1$, that is

$$\int_{\mathbb{R}^n} h \, dx \geq [1 + \psi_{\lambda,n}(d_n(u_0, u_1))] \left( \int_{\mathbb{R}^n} u_0 \, dx \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} u_1 \, dx \right)^{\lambda}$$

where $d_n$ measure the distance of $u_0$ and $u_1$ from coinciding up to an homothety.
Joint work with D. Ghilli (Università di Padova).
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Let $H$ denotes the Hausdorff distance between sets in $\mathbb{R}^n$, we set

$$H_0(K, L) = H(\tau_0 K, \tau_1 L),$$

(0.2)

where $\tau_1, \tau_0$ are two homotheties (i.e. translation plus dilation) such that $|\tau_0 K| = |\tau_1 L| = 1$ and such that the centroids of $\tau_0 K$ and $\tau_1 L$ coincide.

Theorem 1 (Ghilli-S. 2015)

Let $p > 0$ and assume that $u_0$ and $u_1$ are $L_1^p$-concave functions, with convex compact supports $\Omega_0$ and $\Omega_1$ respectively. Then, if $H_0(\Omega_0, \Omega_1)$ is small enough, it holds

$$\int_{\Omega_0} \lambda h(x) \, dx \geq M_{np+1}(I_0, I_1, \lambda) \left[ 1 + \beta H_0(\Omega_0, \Omega_1)(n+1)(p+1) \right]$$

(0.3)

where $\beta$ is a constant depending on $n$, $\lambda$, $M_{np+1}(I_0, I_1, \lambda)$ and the diameters and the measures of $\Omega_0$ and $\Omega_1$. 
Quantitative BBL for \( p \)-concave functions with \( p > 0 \)

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\[ A(K, L) := \inf_{x \in \mathbb{R}^n} \left\{ \frac{|K \Delta (x + \lambda F)|}{|K|}, \lambda = \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \right\}, \quad (0.4) \]

where \( \Delta \) denotes the operation of symmetric difference, i.e. \( \Omega \Delta B = (\Omega \setminus B) \cup (B \setminus \Omega) \).

**Theorem 2** (Ghilli-S. 2015)

In the same assumptions and notation of Theorem 1, if \( A(\Omega_0, \Omega_1) \) is small enough it holds

\[ \int_{\Omega} \lambda h(x) \, dx \geq M_{np+1} \left[ 1 + \delta A(\Omega_0, \Omega_1)^2 (p+1)^p \right], \quad (0.5) \]

where \( \delta \) is a constant depending only on \( n, \lambda, p, M_{np+1} \) and on the measures of \( \Omega_0 \) and \( \Omega_1 \).
Quantitative BBL for $p$-concave functions with $p > 0$  

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Let $A$ denote the *relative asymmetry* of two sets, that is

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In the same assumptions and notation of Theorem 1, if $A(\Omega_0, \Omega_1)$ is small enough it holds

$$\int_{\Omega_0} h(x) \, dx \geq M^{\frac{p}{np+1}} (l_0, l_1, \lambda) \left[ 1 + \delta A(\Omega_0, \Omega_1)^{\frac{2(p+1)}{p}} \right], \quad (0.5)$$

where $\delta$ is a constant depending only on $n, \lambda, p, M^{\frac{p}{np+1}} (l_0, l_1, \lambda)$ and on the measures of $\Omega_0$ and $\Omega_1$. 
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**Remarks.** 0. Case $p = 1$ is easy! (In fact the same can be said for $p = 1/k$, $k \in \mathbb{N}$)
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2. We in fact prove more than what stated above and the support sets $\Omega_0$ and $\Omega_1$ could be replaced by any couple of level sets of $u_0$ and $u_1$, suitably related.
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Main theorem

If for some (small enough) $\epsilon > 0$ it holds

$$\int_{\Omega} \lambda h(x) \, dx \leq M p^n p + 1 \left( \int_{\Omega_0} u_0(x) \, dx, \int_{\Omega_1} u_1(x) \, dx; \lambda \right) + \epsilon,$$

then

$$|\Omega_\lambda| \leq M 1 n \left( |\Omega_0|, |\Omega_1|, \lambda \right) \left[ 1 + \eta \epsilon p p + 1 \right].$$
Quantitative BBL for \( p \)-concave functions with \( p > 0 \)

**Remarks.**

0. Case \( p = 1 \) is easy! (In fact the same can be said for \( p = 1/k, \ k \in \mathbb{N} \))

1. In both theorems it is not necessary that all the involved functions are \( p \)-concave, it is just sufficient that \( h \) only is \( p \)-concave.

2. We in fact prove more than what stated above and the support sets \( \Omega_0 \) and \( \Omega_1 \) could be replaced by any couple of level sets of \( u_0 \) and \( u_1 \), suitably related. However, for the application we have in mind (quantitative BM inequalities for variational functionals), we are mainly interested in the support sets.

3. The proof of both theorems essentially amounts to proving the following and then applying existing quantitative results for the classical BM inequality.

**Main theorem**

If for some (small enough) \( \epsilon > 0 \) it holds

\[
\int_{\Omega_\lambda} h(x) \, dx \leq M \frac{p}{np+1} \left( \int_{\Omega_0} u_0(x) \, dx, \int_{\Omega_1} u_1(x) \, dx ; \lambda \right) + \epsilon, \tag{0.6}
\]

then

\[
|\Omega_\lambda| \leq M_1 (|\Omega_0|, |\Omega_1|, \lambda) \left[ 1 + \eta \epsilon \frac{p}{p+1} \right]. \tag{0.7}
\]
Sketch of the proof

Let

\[ I_i = \int_{\Omega_i} u_i \, dx \quad i = 0, 1, \]

\[ I_{\lambda} = \int_{\Omega_{\lambda}} h \, dx \]

and

\[ L_i = \max_{\Omega_i} u_i \quad i = 0, 1, \quad L_{\lambda} = \max_{\Omega_{\lambda}} h \]

Consider the distribution functions

\[ \mu_i(s) = |\{ u_i \geq s \}| \quad i = 0, 1, \quad \mu_{\lambda}(s) = |\{ u_{p,\lambda} \geq s \}| \]

Then

\[ I_i = \int_0^{L_i} \mu_i(s) \, ds \quad i = 0, 1, \lambda. \]
Sketch of the proof

Notice that the assumption of BBL is equivalent to

\[ \{ h \geq \mathcal{M}_p(s_0, s_1; \lambda) \} \supseteq (1 - \lambda) \{ u_0 \geq s_0 \} + \lambda \{ u_1 \geq s_1 \} \]  

(0.8)

for \( s_0 \in [0, L_0], \; s_1 \in [0, L_1] \).
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for \( s_0 \in [0, L_0], \ s_1 \in [0, L_1] \). Then, using the Brunn-Minkowski inequality, we get

\[ \mu_\lambda(\mathcal{M}_p(s_0, s_1; \lambda)) \geq \mathcal{M}_{\frac{1}{\lambda}}(\mu_0(s_0), \mu_1(s_1), \lambda). \quad (0.9) \]

Now define the functions \( s_i : [0, 1] \rightarrow [0, L_i] \) for \( i = 0, 1 \) such that

\[ s_i(t) : \frac{1}{l_i} \int_0^{s_i(t)} \mu_i(s) \, ds = t \quad \text{for } t \in [0, 1], \quad (0.10) \]
Notice that the assumption of BBL is equivalent to

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for $s_0 \in [0, L_0]$, $s_1 \in [0, L_1]$. Then, using the Brunn-Minkowski inequality, we get

$$\mu_\lambda(M_p(s_0, s_1; \lambda)) \geq M_{\frac{1}{n}}(\mu_0(s_0), \mu_1(s_1), \lambda).$$

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Now define the functions $s_i : [0, 1] \to [0, L_i]$ for $i = 0, 1$ such that

$$s_i(t) : \frac{1}{l_i} \int_0^{s_i(t)} \mu_i(s) \, ds = t \quad \text{for } t \in [0, 1],$$

(0.10)

and set

$$s_\lambda(t) = M_p(s_0(t), s_1(t), \lambda) \quad t \in [0, 1].$$
Sketch of the proof

Thanks to (0.9), we get

\[ \mu_\lambda(s_\lambda(t)) \geq M_{1/n}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) \quad t \in [0, 1] \]

(0.11)
Thanks to (0.9), we get
\[ \mu_\lambda (s_\lambda(t)) \geq \mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) \quad t \in [0, 1] \] (0.11)

Now, given any \( \alpha \in (0, 1) \), set
\[ F_{\epsilon} = \{ t \in [0, 1] : \mu_\lambda(s_\lambda(t)) > \mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) + \epsilon^{1-\alpha} \} \] (0.12)

and
\[ \Gamma_{\epsilon} = \{ s_\lambda(t) : t \in F_{\epsilon} \} . \] (0.13)

We want to find a bound of \( |\Gamma_{\epsilon}| \) in terms of \( \epsilon \) and, playing with the integrals and using the assumption, it is actually possible and we find
\[ |\Gamma_{\epsilon}| \leq \epsilon^\alpha . \] (0.14)
Sketch of the proof

Thanks to (0.9), we get

$$\mu_\lambda(s_\lambda(t)) \geq M_1 \left( \mu_0(s_0(t)), \mu_1(s_1(t)), \lambda \right), \quad t \in [0, 1]$$  \hspace{1cm} (0.11)

Now, given any $\alpha \in (0, 1)$, set

$$F_\epsilon = \{ t \in [0, 1] : \mu_\lambda(s_\lambda(t)) > M_1 \left( \mu_0(s_0(t)), \mu_1(s_1(t)), \lambda \right) + \epsilon^{1-\alpha} \}$$  \hspace{1cm} (0.12)

and

$$\Gamma_\epsilon = \{ s_\lambda(t) : t \in F_\epsilon \}.$$  \hspace{1cm} (0.13)

We want to find a bound of $|\Gamma_\epsilon|$ in terms of $\epsilon$ and, playing with the integrals and using the assumption, it is actually possible and we find

$$|\Gamma_\epsilon| \leq \epsilon^\alpha.$$  \hspace{1cm} (0.14)

Now choosing the right power $\alpha = \frac{p}{p+1}$ and using the $p$-concavity of $h$ and the Brunn-Minkowski inequality we get the conclusion.
Examples of applications


2. Quantitative Urysohn’s inequaities for the same functionals.

I will show in some detail the case of torsional rigidity.
The leading idea is to find a way to compare solutions of different equations in different domains, I mean.... whiteboard —>
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More explicitly, consider two sets $\Omega_0$ and $\Omega_1$ and a real number $\mu \in (0, 1)$, and denote by $\Omega_\mu$ the \textit{Minkowski convex combination} (with coefficient $\mu$) of $\Omega_0$ and $\Omega_1$, that is

$$\Omega_\mu = (1 - \mu)\Omega_0 + \mu \Omega_1 = \{(1 - \mu)x_0 + \mu x_1 : x_0 \in \Omega_0, x_1 \in \Omega_1 \}.$$
The leading idea is to find a way to compare solutions of different equations in different domains, I mean.... whiteboard —> 

More explicitly, consider two sets \( \Omega_0 \) and \( \Omega_1 \) and a real number \( \mu \in (0, 1) \), and denote by \( \Omega_\mu \) the \textit{Minkowski convex combination} (with coefficient \( \mu \)) of \( \Omega_0 \) and \( \Omega_1 \), that is 

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\]

Correspondingly, let \( u_0 \), \( u_1 \) and \( u_\mu \) be the solutions of 

\[
(P_i) \quad \begin{cases}
F_i(x, u_i, Du_i, D^2 u_i) = 0 & \text{in } \Omega_i, \\
u_i = 0 & \text{on } \partial \Omega_i, \\
u_i > 0 & \text{in } \Omega_i,
\end{cases}
\]

where \( F_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S_n \rightarrow \mathbb{R} \) is a continuous (proper) degenerate elliptic operator, i.e. decreasing with respect to \( u \) and increasing w.r.t. to the (Hessian) matrix variable \( A \).
Combinations of solutions in different sets

\[ u_0 \text{ sol. of } (P_0) \text{ in } \Omega_0 \quad u_1 \text{ sol. of } (P_1) \text{ in } \Omega_1 \]
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\[ x_0 \in \Omega_0 \quad \rightarrow \quad u_0(x_0) \quad \quad x_1 \in \Omega_1 \quad \rightarrow \quad u_1(x_1) \]
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**Question:** are there suitable assumptions on the operators \( F_0, F_1 \) and \( F_\mu \) which permit to compare \( u_\mu \) with (a suitable convolution of) \( u_0 \) and \( u_1 \) ?
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\[ u_0 \text{ sol. of } (P_0) \text{ in } \Omega_0 \quad u_1 \text{ sol. of } (P_1) \text{ in } \Omega_1 \n\]
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**Question:** are there suitable assumptions on the operators \( F_0, F_1 \) and \( F_\mu \) which permit to compare \( u_\mu \) with (a suitable convolution of) \( u_0 \) and \( u_1 \)? Precisely, we want to find suitable conditions on the operators \( F_0, F_1, F_\mu \) which guarantee

\[ u_\mu((1 - \mu)x_0 + \mu x_1) \geq M_p(u_0(x_0), u_1(x_1); \mu) \]

for every \( x_0 \in \Omega_0, x_1 \in \Omega_1, \) for some \( p \in \mathbb{R} \).
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\[ u_\mu((1 - \mu)x_0 + \mu x_1) \geq M_p(u_0(x_0), u_1(x_1); \mu) \]

for every \( x_0 \in \Omega_0, x_1 \in \Omega_1 \), for some \( p \in \mathbb{R} \).

Equivalently

\[ u_\mu(x) \geq \sup\{M_p(u_0(x_0), u_1(x_1); \mu) : x_0 \in \Omega_0, x_1 \in \Omega_1, x = (1 - \mu)x_0 + \mu x_1\} \]

for every \( x \in \Omega_\mu = (1 - \mu)\Omega_0 + \mu \Omega_1 \).
The $p$-concave convolution

Let us define the function $u_{p,\mu}^* : \Omega_\mu \rightarrow [0, +\infty)$ as follows:

$$u_{p,\mu}^*(x) = \sup\{M_p(u_0(x_0), u_1(x_1); \mu) : x_0 \in \Omega_0, x_1 \in \Omega_1, x = (1 - \mu)x_0 + \mu x_1\}$$

and call it $p$-concave convolution of $u_0$ and $u_1$ (with weight $\mu$).
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\]

and call it \( p\)-concave convolution of \( u_0 \) and \( u_1 \) (with weight \( \mu \)).

When \( p = 1 \) it is the usual supremal convolution (from convex analysis) and, geometrically, it simply corresponds to the function whose graph is the Minkowski linear combination (in \( \mathbb{R}^{n+1} \)) of the graphs of \( u_0 \) and \( u_1 \).
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For $p > 0$ it corresponds to make the sup-conv (that is the Minkowski combination of the graphs) of $u_0^p$ and $u_1^p$ and then to raise to power $1/p$. 
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For \( p = 0 \) it corresponds to \( \exp(\text{sup-conv of } \log u_0 \text{ and } \log u_1) \).
Let us define the function \( u^*_p,\mu : \Omega_\mu \to [0, +\infty) \) as follows:

\[
    u^*_p,\mu(x) = \sup\{ M_p(u_0(x_0), u_1(x_1); \mu) : x_0 \in \Omega_0, x_1 \in \Omega_1, x = (1 - \mu)x_0 + \mu x_1 \}
\]

and call it \( p\)-concave convolution of \( u_0 \) and \( u_1 \) (with weight \( \mu \)).

When \( p = 1 \) it is the usual \textit{supremal convolution} (from convex analysis) and, geometrically, it simply corresponds to the function whose graph is the \textit{Minkowski linear combination (in} \( \mathbb{R}^{n+1} \) \textit{) of the graphs of} \( u_0 \) and \( u_1 \).

For \( p > 0 \) it corresponds to make the sup-conv (that is the Minkowski combination of the graphs) of \( u^p_0 \) and \( u^p_1 \) and then to raise to power \( 1/p \).

For \( p = 0 \) it corresponds to \( \exp(\sup\text{-conv of } \log u_0 \text{ and } \log u_1) \).

\textbf{STRATEGY:} if \( F_\mu \) satisfies the \textit{comparison principle}, to get the goal we have just to prove that \( u^*_p,\mu \) is a \textit{viscosity subsolution} of \( (P_\mu) \).
For $i = 0, 1, \mu$ and for a given $p \geq 0$, for every fixed $\theta \in \mathbb{R}^n$ we define $G^{(\theta)}_{i,p} : \Omega_i \times (0, +\infty) \times S_n \to \mathbb{R}$ as follows:

\[
G^{(\theta)}_{i,p}(x, t, A) = F_i(x, t^{\frac{1}{p}} - 1 \theta, t^{\frac{1}{p} - 3} A) \quad \text{for } p > 0, \tag{0.15}
\]

\[
G^{(\theta)}_{i,0}(x, t, A) = F_i(x, e^t, e^t \theta, e^t A). \tag{0.16}
\]

**Assumption** $(A_{\mu, p})$. Let $\mu \in (0, 1)$ and $p \geq 0$. We say that $F_0, F_1, F_\mu$ satisfy the assumption $(A_{\mu, p})$ if, for every fixed $\theta \in \mathbb{R}^n$, the following holds:

\[
G^{(\theta)}_{\mu,p}((1 - \mu)x_0 + \mu x_1, (1 - \mu)t_0 + \mu t_1, (1 - \mu)A_0 + \mu A_1) \geq \min\{G^{(\theta)}_{0,p}(x_0, t_0, A_0); \ G^{(\theta)}_{1,p}(x_1, t_1, A_1)\}
\]

for every $x_0 \in \Omega_0$, $x_1 \in \Omega_1$, $t_0, t_1 > 0$ and $A_0, A_1 \in S_n$. 
Assumptions on the operators

For $i = 0, 1, \mu$ and for a given $p \geq 0$, for every fixed $\theta \in \mathbb{R}^n$ we define $G_{i,p}^{(\theta)} : \Omega_i \times (0, +\infty) \times S_n \to \mathbb{R}$ as follows:

$G_{i,p}^{(\theta)}(x, t, A) = F_i(x, t^{1/p}, t^{1/p-1}\theta, t^{1/p-3}A)$ for $p > 0$, \hspace{1cm} (0.15)

$G_{i,0}^{(\theta)}(x, t, A) = F_i(x, e^t, e^t\theta, e^tA)$. \hspace{1cm} (0.16)

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$G_{\mu,p}^{(\theta)}( (1 - \mu)x_0 + \mu x_1, (1 - \mu)t_0 + \mu t_1, (1 - \mu)A_0 + \mu A_1 ) \geq \min\{ G_{0,p}^{(\theta)}(x_0, t_0, A_0); G_{1,p}^{(\theta)}(x_1, t_1, A_1)\}$

for every $x_0 \in \Omega_0, x_1 \in \Omega_1, t_0, t_1 > 0$ and $A_0, A_1 \in S_n$.

If $F_0 = F_1 = F_\mu$, we are simply requiring the operator $G_p^{\theta}$ to be quasi-concave, i.e. with convex superlevel sets.

Let $\mu \in (0, 1)$, $\Omega_i$ an open bounded convex set and $u_i$ a classical solution of $(P_i)$ for $i = 0, 1$. Assume that $F_0, F_1, F_\mu$ satisfy the assumption $(A_{\mu,p})$ for some $p \in [0, 1)$. If $p > 0$, assume furthermore that for $i = 0, 1$ it holds

$$\liminf_{y \to x} \frac{\partial u_i(y)}{\partial \nu} > 0$$

(0.17)

for every $x \in \partial \Omega_i$, where $\nu$ is any inward direction of $\Omega_i$ at $x$. Then $u_{p,\mu}^*$ is a viscosity subsolution of $(P_\mu)$.
Let $\mu \in (0, 1)$, $\Omega_i$ an open bounded convex set and $u_i$ a classical solution of $(P_i)$ for $i = 0, 1$. Assume that $F_0, F_1, F_{\mu}$ satisfy the assumption $(A_{\mu,p})$ for some $p \in [0, 1)$. If $p > 0$, assume furthermore that for $i = 0, 1$ it holds

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**Corollary (1)**

*In the same assumption of the previous theorem, if $F_{\mu}$ satisfies a Comparison Principle and $u_\mu$ is a viscosity solution of $(P_{\mu})$, then*

$$u_\mu((1 - \mu) x_0 + \mu x_1) \geq M_p(u_0(x_0), u_1(x_1); \mu)$$

(0.18)

*for every $x_0 \in \Omega_0$, $x_1 \in \Omega_1$.***
Comparison

By a combination of the previous result with the BBL inequality, we can compare the $L^r$ norms of the involved functions:

**Corollary (2)**

*With the same assumptions and notation of the previous corollary, for every $r \in (0, +\infty]$ we have*

\[
\| u_\mu \|_{L^r(\Omega_\mu)} \geq M_q(\| u_0 \|_{L^r(\Omega_0)}, \| u_1 \|_{L^r(\Omega_1)}; \mu),
\]

*where*

\[
q = \begin{cases} 
\frac{pr}{np+r} & \text{for } r \in (0, +\infty) \\
p & \text{for } r = +\infty.
\end{cases}
\]
A simple Example

For instance, let $u_0$ and $u_1$ be the solutions of the following problems

\[
\begin{cases}
\Delta u_0 + f_0(x) = 0 & \text{in } \Omega_0 \\
u_0 = 0 & \text{on } \partial \Omega_0
\end{cases}
\]

and

\[
\begin{cases}
\Delta u_1 + f_1(x) = 0 & \text{in } \Omega_1 \\
u_1 = 0 & \text{on } \partial \Omega_1.
\end{cases}
\]

Then take $\mu \in (0, 1)$ and set

$$
\Omega = (1 - \mu)\Omega_0 + \mu \Omega_1.
$$

Now let $u_\mu$ be the solution of

\[
\begin{cases}
\Delta u_\mu + f_\mu(x) = 0 & \text{in } \Omega \\
u_\mu = 0 & \text{on } \partial \Omega.
\end{cases}
\]
A simple example

Then assumption \((A_{\mu,p})\) for \(p=1/3\) reads

\[
f_{\mu}\left((1-\mu)x_0 + \mu x_1\right) \geq (1-\mu)f_0(x_0) + \mu f_1(x_1)
\]  

(0.20)

The main theorem tells that we can estimate \(u_{\mu}\) in terms of \(u_0\) and \(u_1\); precisely it holds

\[
u_{\mu}\left((1-\mu)x_0 + \mu x_1\right) \geq \left[(1-\mu)^{3/2}u_0(x_0) + \mu^{3/2}u_1(x_1)\right]^{3}
\]

for every \(x_0 \in \Omega_0, x_1 \in \Omega_1\). Moreover, by using the Borell-Brascamp-Lieb inequality we get

\[
\|u_{\mu}\|_{L^r(\Omega_{\mu})} \geq M_q(\|u_0\|_{L^r(Q)}, \|u_1\|_{L^r(B(0,1))}; \mu)
\]

for every \(r \in (0, +\infty]\), where

\[
q = \begin{cases} 
\frac{r}{n+3r}, & r \in (0, +\infty) \\
1/3, & r = +\infty
\end{cases}
\]
Examples

Notice in particular that, if \( f_0 = f_1 = f_\mu = f : \mathbb{R}^n \to [0, +\infty) \), condition (0.20) simply means \( f \) is concave. In this particular case, we can write the following result.

**Corollary**

Let \( f \) be a smooth nonnegative function defined in \( \mathbb{R}^n \). Let \( \mu \in (0, 1) \) and \( \Omega_0 \) and \( \Omega_1 \) be convex subsets of \( \mathbb{R}^n \) and denote by \( u_0 \), \( u_1 \) and \( u_\mu \) the solutions of

\[
\begin{cases}
\Delta u_i + f(x) = 0 & \text{in } \Omega_i \\
u_i = 0 & \text{on } \partial \Omega_i
\end{cases}
\]

for \( i = 0, 1, \mu \) respectively, where \( \Omega_\mu = (1 - \mu)\Omega_0 + \mu\Omega_1 \), as usual. Assume \( f \) is \( \beta \)-concave for some \( \beta \geq 1 \), that is \( f^\beta \) is concave. Then (0.18) holds with

\[
p = \frac{\beta}{1 + 2\beta}.
\]

In case \( f \) is a positive constant (\( \beta = +\infty \)), the same conclusions follow with \( p = 1/2 \).
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1. it can be used to prove concavity properties of solutions to elliptic and parabolic problems in convex domains;

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Consequences

This technique has several applications:

1. it can be used to prove concavity properties of solutions to elliptic and parabolic problems in convex domains;

2. it is possible to define a new kind of rearrangement (the mean-width rearrangements) which apply to operators not in divergence form and permits to obtain Talenti-like results for the associated equations;

3. it permits to prove Brunn-Minkowski type (and also Urysohn’s type) inequalities for many variational functionals.
When the involved equation is the Euler equation of some variational functional, we can obtain a BM inequality for such a functional.
When the involved equation is the Euler equation of some variational functional, we can obtain a BM inequality for such a functional. Probably the simplest case is that one of *Torsional rigidity*:

\[
\frac{1}{\tau(\Omega)} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\left( \int_{\Omega} |u| \right)^2} : u \in W^{1,2}_0(\Omega), \int_{\Omega} |u| \, dx > 0 \right\}
\]
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\]

**BM inequality for \( \tau \)** [Borell, 1985]

\[
\tau(\Omega_\mu) \geq M_{1/(n+2)}(\tau(\Omega_0), \tau(\Omega_1); \mu) = \left[ (1 - \mu) \tau(\Omega_0)^{1/(n+2)} + \mu \tau(\Omega_1)^{1/(n+2)} \right]^{n+2}
\]
When the involved equation is the Euler equation of some variational functional, we can obtain a BM inequality for such a functional. Probably the simplest case is that one of Torsional rigidity:

$$\frac{1}{\tau(\Omega)} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\left( \int_{\Omega} |u| \right)^2} : u \in W^{1,2}_0(\Omega), \int_{\Omega} |u| dx > 0 \right\}$$

BM inequality for $\tau$ [Borell, 1985]

$$\tau(\Omega^\mu) \geq M_{1/(n+2)}(\tau(\Omega_0), \tau(\Omega_1); \mu) = \left[ (1 - \mu)\tau(\Omega_0)^{1/(n+2)} + \mu \tau(\Omega_1)^{1/(n+2)} \right]^{n+2}$$

Equality holds if and only if $\Omega_0$ and $\Omega_1$ are homothetic [Colesanti 2005].
A proof of BM for $\tau$ with the above technique is quite simple.
A proof of BM for \( \tau \) with the above technique is quite simple. Notice that

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\tau(\Omega_i) = \int_{\Omega_i} u_i \, dx \quad i = 0, 1, \mu
\]

where \( u_i \) is the solution of the torsion problem

\[
(P_{\mu}) \begin{cases}
\Delta u_i + 1 = 0 & \text{in } \Omega_i, \\
u_i = 0 & \text{on } \partial\Omega_i.
\end{cases} \quad i = 0, 1, \mu
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\Delta u_i + 1 = 0 & \text{in } \Omega_i, \\
u_i = 0 & \text{on } \partial\Omega_i.
\end{cases}$$

Set

$$u^*_1(\mu)(x) = \sup\{(1 - \mu)\sqrt{u_0(x_0)} + \mu\sqrt{u_1(x_1)} : (1 - \mu)x_0 + \mu x_1 = x\}$$

By Theorem 3, $u^*_1(\mu)$ is a subsolution to the torsion problem in $\Omega_\mu$. 
A proof of BM for $\tau$ with the above technique is quite simple. Notice that

$$\tau(\Omega_i) = \int_{\Omega_i} u_i \, dx \quad i = 0, 1, \mu$$

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\Delta u_i + 1 = 0 & \text{in } \Omega_i, \\
 u_i = 0 & \text{on } \partial\Omega_i.
\end{cases} \quad (P_\mu)$$

Set

$$u^{*}_{1/2, \mu}(x) = \sup\{(1 - \mu)\sqrt{u_0(x_0)} + \mu\sqrt{u_1(x_1)} : (1 - \mu)x_0 + \mu x_1 = x\}$$

By Theorem 3, $u^{*}_{1/2, \mu}$ is a subsolution to the torsion problem in $\Omega_\mu$. Then

$$u_\mu \geq u^{*}_{1/2, \mu}$$

and we can use the BBL inequality to get the desired result, see Corollary 2.
Quantitative BM inequalities for $\tau$

Now it's easy to understand that it is possible to use the quantitative versions of BBI to get corresponding quantitative versions of the BM inequality for $\tau$. 

Let $\Omega_0$ and $\Omega_1$ be convex bodies in $\mathbb{R}^n$, then the following hold:

$$\tau(\Omega_\lambda) \geq M_1 n + 2 \left( \tau(\Omega_0), \tau(\Omega_1), \lambda \right) + \beta H_0(\Omega_0, \Omega_1)$$

$$\tau(\Omega_\lambda) \geq M_1 n + 2 \left( \tau(\Omega_0), \tau(\Omega_1), \lambda \right) + \delta A(\Omega_0, \Omega_1)$$

where $\beta$ and $\delta$ are constants depending on $n$, $\lambda$, $M_p$, $n p + 1$, $\tau(\Omega_0, \tau(\Omega_1))$, and the diameters and the measures of $\Omega_0$ and $\Omega_1$. 

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Quantitative BM inequalities for $\tau$

Now it’s easy to understand that it is possible to use the quantitative versions of BBl to get corresponding quantitative versions of the BM inequality for $\tau$.

Quantitative BM for $\tau$ [Ghilli-S. (2014)]

Let $\Omega_0$ and $\Omega_1$ be convex bodies in $\mathbb{R}^n$, then the following hold:

$$\tau(\Omega_\lambda) \geq M \frac{1}{n+2} (\tau(\Omega_0), \tau(\Omega_1), \lambda) + \beta H_0(\Omega_0, \Omega_1)^{3(n+1)},$$  \hfill (0.21)

$$\tau(\Omega_\lambda) \geq M \frac{1}{n+2} (\tau(\Omega_0), \tau(\Omega_1), \lambda) + \delta A(\Omega_0, \Omega_1)^6,$$  \hfill (0.22)

where $\beta$ and $\delta$ are constants depending on $n$, $\lambda$, $M$, $\frac{\rho}{np+1}$, $(\tau(\Omega_0), \tau(\Omega_1), \lambda)$ and the diameters and the measures of $\Omega_0$ and $\Omega_1$. 

Paolo Salani (DiMaI - Università di Firenze)
Quantitative BM inequalities for $\tau$

Now it's easy to understand that it is possible to use the quantitative versions of BBl to get corresponding quantitative versions of the BM inequality for $\tau$.

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where $\beta$ and $\delta$ are constants depending on $n$, $\lambda$, $M \frac{p}{np+1} (\tau(\Omega_0, \tau(\Omega_1), \lambda)$ and the diameters and the measures of $\Omega_0$ and $\Omega_1$. 
Given a convex set $\Omega$, we say that $\Omega^\sharp_m$ is a rotation mean of $\Omega$ if there exist a number $m \in \mathbb{N}$ and $\rho_1, \ldots, \rho_m \in SO(n)$ such that

$$
\Omega^\sharp_m = \frac{1}{m} \left( \rho_1 \Omega + \cdots + \rho_m \Omega \right).
$$

The following theorem is due to Hadwiger.

**Theorem (Hadwiger)**

Given an open bounded convex set $\Omega$, there exists a sequence of rotation means of $\Omega$ converging in Hausdorff metric to a ball $\Omega^\sharp$ with diameter equal to the mean width $w(\Omega)$ of $\Omega$.

Notice that in the plane the mean width of a convex set coincides essentially with its perimeter. Precisely:

$$
w(\Omega) = \frac{|\partial \Omega|}{\pi}.
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Then $\Omega^\sharp$ is a circle with the same perimeter as $\Omega$. 
Given a convex set $\Omega$, we say that $\Omega_m^\#$ is a rotation mean of $\Omega$ if there exist a number $m \in \mathbb{N}$ and $\rho_1, \ldots, \rho_m \in SO(n)$ such that

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Urysohn’s inequality for $\tau$

Given a convex set $\Omega$, we say that $\Omega^\#_m$ is a \textit{rotation mean} of $\Omega$ if there exist a number $m \in \mathbb{N}$ and $\rho_1, \ldots, \rho_m \in SO(n)$ such that

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**Urysohn’s ineq. for $\tau$**

$$\tau(\Omega) \leq \tau(\Omega^\#)$$

and $=\$ holds if and only if $\Omega = \Omega^\#$.

In other words: among sets with given mean width, the torsional rigidity is maximized by balls.
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In other words: among sets with given mean width, the torsional rigidity is maximized by balls.
Let $\Omega$ be an open bounded convex set of $\mathbb{R}^n$, $n \geq 2$ with centroid in the origin. Let $\Omega^\#$ be the ball with the same mean-width of $\Omega$ with center in the origin. Then the following hold

$$\tau(\Omega^\#) \geq \tau(\Omega) \left(1 + \mu H^{3(n+1)}\right),$$  

(0.23)

$$\tau(\Omega^\#) \geq \tau(\Omega) \left(1 + \nu A^6\right),$$  

(0.24)

where $H = H(\Omega, \Omega^\#)$ and $A = \max\{A(\Omega, \Omega_\rho) : \rho \text{ rotation in } \mathbb{R}^n\}$ are small enough, $\mu$ and $\nu$ are constants, the former depending on $n$, $\tau(\Omega)$ and the diameter of $\Omega$, the latter depending only on $n$ and $\tau(\Omega)$.
Brunn-Minkowski type inequalities have been proved for several variational functionals:

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THANKS!