Bifractional Brownian Motion: Existence and Border Cases

M. Lifshits (St. Petersburg and Linköping)

Minneapolis, April, 2015
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This is a joint work with Ksenia Volkova from St. Petersburg State University (Russia).
fractional Brownian motion (fBm)

Classical fractional Brownian motion (fBm) $W^{(H)}(t)$, $t \in \mathbb{R}$, with parameter $H \in (0, 1]$, is a centered Gaussian process on $\mathbb{R}$ with covariance

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fBm appears in many limit theorems related to long range dependence.

It is so widely known and used that it needs no further recommendations.
fractional Brownian motion $W^H$ (continued)

Characteristic properties of fBm:

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is $H$-self-similar:

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Bifractional Brownian motion (bfBm) $\mathcal{B}^{(H,K)}$ is a centered Gaussian process on $\mathbb{R}$ with covariance

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There is one more special case of bfBm directly related to the usual fBm.
Another relation between bfBm $B$ and fBm $W$.

Consider an anti-symmetrized version of fBm, 

$$V^{(H)}(t) := W^{(H)}(t) - W^{(H)}(-t), \quad t \geq 0.$$
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with $H = \frac{1}{2}$, we see that bfBm $B^{(1/2,K)}$, $0 < K \leq 2$, consists, up to a scaling factor, of the two independent versions of $V^{(K/2)}$, – one for positive, another for negative times.
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Further studies and extensions of bfBm appear in the works by Alpay and Levanoy, El-Nouty, Es-Sebaiy and Tudor, Kruk et al, Lei and Nualart, J.Liu, C.Ma, Russo and Tudor, Tudor and Xiao, W.Wang, etc.
We study the existence of bfBm for a given pair of parameters \((H, K)\) and encounter some related limiting processes. Initially, Houdré and Villa proved the existence of bfBm on \(\mathbb{R}\) for

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(Their result may be extended !)

Later on, Bardina and Es-Sebaiy enlarged the zone of existence. Using an idea of Lei and Nualart, they proved that bfBm exists on \(\mathbb{R}\) for

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To the moment when we started this work, it was still unknown whether bfBm exists for any other pairs \((H, K)\).

We show that conditions \(K \leq 2\) and \(HK \leq 1\) are necessary for the existence of bfBm on \(\mathbb{R}_+\).
General picture

\[ W^{(H)} \]

\[ HK = 1 \quad K \leq 2, \quad HK \leq 1 \quad \text{necessary} \]

Lifshits–Volкова

New zone

Bifractional BM

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General picture

$W^{(H)}$

$V^{(H)}$

$HK = 1$ and $HK \leq 1$ necessary

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Houdré–Villa

Bardina–Es-Sebaiy

$HK = 1$
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Lifshits–Volkova
An extension of Houdré–Villa result

The arguments of Houdré and Villa actually have nothing to do with fBm or bfBm.

Recall that Bernstein function is a function $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ which admits the following Lévy-Khintchine representation

$$f(\lambda) = a + b\lambda + \int_{0}^{\infty} \left(1 - e^{-x\lambda}\right)\mu(dx),$$

where $a, b \geq 0$ are constants and $\mu$ is a measure on $(0, \infty)$ satisfying the integrability condition

$$\int_{0}^{\infty} \min\{x, 1\}\mu(dx) < \infty.$$

Typical examples are $\lambda \rightarrow \log(1 + \lambda)$ and $\lambda \rightarrow \lambda K$ for $0 < K \leq 1$.

Theorem

Let $Y(t), t \in \mathbb{R}$ be a centered process with stationary increments and finite second moments $\sigma(t)^2 := \mathbb{E}Y(t)^2$. Then for any Bernstein function $f(\cdot)$ there exists a process with covariance

$$R_f,\sigma(s, t) := f(\sigma(s)^2 + \sigma(t)^2) - f(\sigma(s-t)^2),$$

$s, t \in \mathbb{R}$. Another extension was given by Chunsheng Ma. (TPA, 2013)
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Lei–Nualart decomposition

Following Lei and Nualart, consider the real Gaussian process \( (K \in (0, 2)) \)

\[
X^{(K)}(t) := \sqrt{\frac{K}{\Gamma(1 - K)}} \int_0^\infty (1 - e^{-rt})r^{-(1+K)/2}\mathcal{W}(dr), \quad t \geq 0,
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where \( \mathcal{W} \) is a Gaussian white noise.
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\[
R_X^{(H,K)}(s, t) = \left(|s|^{2HK} + |t|^{2HK} - (|s|^{2H} + |t|^{2H})^K\right).
\]
Lei–Nualart decomposition (continued)

By comparing covariances

\[ R_X^{(H,K)}(s, t) = \left( |s|^{2HK} + |t|^{2HK} - (|s|^{2H} + |t|^{2H})^K \right) \]

and

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we have the Lei–Nualart decomposition for fBm (in the zone of parameters $K < 1$, $HK \leq 1$). If bfBm $B^{(H,K)}$ exists, then

$$\sqrt{2} W^{(HK)}(t) = X^{(H,K)}(t) + \sqrt{2^K} B^{(H,K)}(t), \quad t \in \mathbb{R},$$

with independence of processes on the right hand side.
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Since \( X^{(K)}(\cdot) \) is a smooth process, it becomes obvious that the local properties of \( B^{(H,K)}(\cdot) \) are the same as those of fBm \( W^{(HK)}(\cdot) \).
Lamperti transformation

If a process $Y(t), t \geq 0$ is $H$-self-similar, then its transformation

$$U_Y(\tau) := e^{-H\tau} Y(e^{\tau})$$

is a stationary process.
Stationary versions

In addition to the self-similar processes $W^{(H)}$, $X^{(K)}$, $X^{(H,K)}$, $B^{(H,K)}$, let us introduce their stationary versions

$$U_W^{(H)}(\tau) := e^{-H \tau} W^{(H)}(e^\tau);$$
$$U_X^{(K)}(\tau) := e^{-K \tau/2} X^{(K)}(e^\tau);$$
$$U_X^{(H,K)}(\tau) := e^{-HK \tau} X^{(H,K)}(e^\tau);$$
$$U_B^{(H,K)}(\tau) := e^{-HK \tau} B^{(H,K)}(e^\tau).$$

Notice that $U_W^{(H)}$ is one of the well known versions of the fractional Ornstein–Uhlenbeck process. We also have $U_X^{(H,K)}(\tau) = U_X^{(K)}(2H \tau)$, which means for spectral densities $f^{(H,K)}(\tau) = \frac{1}{2H} f^{(K)}(2H \tau)$. 
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In addition to the self-similar processes $W^H$, $X^K$, $X^{(H,K)}$, $B^{(H,K)}$, let us introduce their stationary versions

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U^{(H)}_W(\tau) := e^{-H\tau} W^H(e^\tau);
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Notice that $U^{(H)}_{W}$ is one of the well known versions of the fractional Ornstein–Uhlenbeck process.

We also have

\[
U^{(H,K)}_{X}(\tau) = U^{(K)}_{X}(2H\tau).
\]

which means for spectral densities

\[
f^{(H,K)}_{X}(u) = \frac{1}{2H} f^{(K)}_{X} \left( \frac{u}{2H} \right).
\]
Spectral criterion of existence

Lei–Nualart decomposition

\[
\sqrt{2} W^{(H,K)}(t) = X^{(H,K)}(t) + \sqrt{2^K} B^{(H,K)}(t), \quad t \geq 0,
\]

becomes in the stationary setting

\[
\sqrt{2} U^{(H,K)}_W(t) = U^{(H,K)}_X(t) + \sqrt{2^K} U^{(H,K)}_B(t), \quad t \in \mathbb{R}.
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It follows

**Theorem**

*Let* \( K \in (0, 1) \), \( H > 0 \). *Then* \( \text{bfBm} \, B^{(H,K)} \) *exists on* \( \mathbb{R}_+ \) *iff*

\[ f^{(H,K)}_X(u) \leq 2 f^{(HK)}_W(u), \quad u \in \mathbb{R}. \]
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**Theorem**

*Let $K \in (0, 1)$, $H > 0$. Then BfBm $B^{(H,K)}$ exists on $\mathbb{R}^+_+$ iff*

$$f^{(H,K)}_X(u) \leq 2 f^{(H,K)}_W(u), \quad u \in \mathbb{R}.$$

It remains to find the spectral densities $f^{(K)}_X$ and $f^{(H,K)}_W$ and to compare them.
After some simple but not totally trivial computations, we find

$$f_W^{(H)}(u) = \frac{A_H \cosh(\pi u)}{\sin^2(\pi H) \cosh^2(\pi u) + \cos^2(\pi H) \sinh^2(\pi u)}$$

with

$$A_H = \frac{\Gamma(2H + 1) \sin(\pi H)}{2(u^2 + H^2)|\Gamma(H + iu)|^2}.$$
Spectral density for fractional OU process

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with

\[ A_H = \frac{\Gamma(2H + 1) \sin(\pi H)}{2(u^2 + H^2)|\Gamma(H + iu)|^2}. \]

In the classical case \( H = 1/2 \) this reduces to the familiar formula

\[ f_W^{(1/2)}(u) = \frac{2}{\pi(4u^2 + 1)}. \]
Spectral density for fractional OU process

After some simple but not totally trivial computations, we find

\[ f_W^{(H)}(u) = \frac{A_H \cosh(\pi u)}{\sin^2(\pi H) \cosh^2(\pi u) + \cos^2(\pi H) \sinh^2(\pi u)} \]

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Alternative series formula for \( f_W^{(H)}(u) \) due to Barndorff-Nielsen and Perez-Abreu is also known.
Spectral density for $X$

After similar calculations we get

$$f_X^{(k)}(u) = \frac{k}{\Gamma(1 - k)} \frac{|\Gamma(-iu - k/2)|^2}{2\pi},$$

whereas

$$f_X^{(H,k)}(u) = \frac{1}{2H} \frac{k}{\Gamma(1 - k)} \frac{|\Gamma(-iu/2H - k/2)|^2}{2\pi},$$
Spectral density for $X$

After similar calculations we get

$$f_X^{(\kappa)}(u) = \frac{\kappa}{\Gamma(1 - \kappa)} \frac{|\Gamma(-iu - \kappa/2)|^2}{2\pi},$$

whereas

$$f_X^{(\nu, \kappa)}(u) = \frac{1}{2\nu} \frac{\kappa}{\Gamma(1 - \kappa)} \frac{|\Gamma(-iu/2\nu - \kappa/2)|^2}{2\pi},$$

... and numerical comparison of the two spectral densities, $f_W^{(\nu, \kappa)}(\cdot)$ and $f_X^{(\nu, \kappa)}(\cdot)$, yields the numerical boundary between the existence and non existence zones of parameters $(\nu, \kappa)$. 
Open problem

If bfBm $B^{(H,K)}(\cdot)$ exists on the half-line $\mathbb{R}_+$, does it admit an extension to entire real line $\mathbb{R}$?
Again, in addition to the self-similar process $B^{(H,K)}$, consider its stationary (Lamperti) transformation

$$U_B^{(H,K)}(\tau) := e^{-HK\tau}B^{(H,K)}(e^{\tau}).$$
Back to simple necessary conditions: \( \kappa \leq 2 \) and \( HK \leq 1 \)

Again, in addition to the self-similar process \( B^{(H,K)} \), consider its stationary (Lamperti) transformation

\[
U_B^{(H,K)}(\tau) := e^{-HK\tau} B^{(H,K)}(e^\tau).
\]

It has the covariance function

\[
R_B^{(H,K)}(\tau) = (\cosh(H\tau))^\kappa - 2^{(2H-1)\kappa} |\sinh(\tau/2)|^{2HK}.
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When $\tau \to \infty$, we get asymptotics

$$R_B^{(H,K)}(\tau) = 2^{-\kappa} \kappa e^{(K-2)H\tau} (1 + o(1)) + \kappa H 2^{1-K} e^{(K-1)\tau} (1 + o(1)).$$

Therefore, the boundedness of $R_B^{(H,K)}$ implies that both conditions $\kappa \leq 2$ and $\kappa H \leq 1$ are necessary for the existence of $B^{(H,K)}$ on $\mathbb{R}_+$. 
Back to simple necessary conditions: \( k \leq 2 \) and \(HK \leq 1\)

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R_{B}^{(H,K)}(\tau) = 2^{-k}Ke^{(K-2)H\tau}(1 + o(1)) + HK 2^{1-K} e^{(HK-1)\tau}(1 + o(1)).
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Therefore, the boundedness of \( R_{B}^{(H,K)} \) implies that both conditions \( k \leq 2 \) and \( HK \leq 1 \) are necessary for the existence of \( B^{(H,K)} \) on \( \mathbb{R}_+ \).
Back to the general picture
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\[ HK = 1 \]
A limiting process for $K \to 0$

Consider a limiting behavior of the bfBm when $H > 0$ is fixed and $K \to 0$. 

Consider the stationary version (Lamperti transformation) 

$$U(H, K) B(\tau) := e^{-HK\tau} B(H, K)(e^{\tau}).$$

and its covariance

$$R(H, K) B(\cdot).$$

For any $\tau > 0$ we have, as $K \to 0$,

$$K^{-1} R(H, K) B(\tau) \to \left[ \ln(2 \cosh(H\tau)) - H\tau \right] + 2H \left[ \tau/2 - \ln(2 \sinh(\tau/2)) \right] := R_1(\tau) + R_2(\tau).$$

We want to find the spectrum corresponding to this limiting covariance.

Notice a logarithmic explosion of the term $R_2$ at zero. This means that the limiting process is not a usual process defined pointwise but a generalized one. This feature may not be repaired by time scaling.
A limiting process for $K \to 0$

Consider a limiting behavior of the bfBm when $H > 0$ is fixed and $K \to 0$. Consider the stationary version (Lamperti transformation)

$$U_{B_{(H,K)}}^{(H,K)}(\tau) := e^{-HK\tau}B^{(H,K)}(e^{\tau}).$$

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A limiting process for $K \to 0$ (continued)

For respective spectral densities we get

$$f_1(u) = H\pi^{-1}u^{-2}\left[1 - \frac{\pi u/H}{2 \sinh(\pi u/2H)}\right]$$

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A limiting process for $k \to 0$ (continued)

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$$f_1(u) = H\pi^{-1} u^{-2} \left[ 1 - \frac{\pi u / H}{2 \sinh(\pi u / 2H)} \right]$$

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The limiting process is related to stationary random fields on the hyperbolic space studied earlier by S. Cohen and Lifshits.
THANK YOU FOR COMING!