Academic wages, Singularities, Phase Transitions and Pyramid Schemes

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Outline

1. Mathematical challenges in economic theory
   - Steady-state matching coupling the education and labor markets
2. A mathematical model
   - A variational approach to competitive equilibria
3. Results
   - Existence of equilibrium wages and matchings
   - Specialization, uniqueness, and structural properties
   - Description of singularities
4. Conclusions
5. References

Background, challenge, universality

• despite some celebrated successes, economic theory presents a largely untapped source of interesting mathematical problems

• e.g. in a heterogeneous population of N collaborator/competitors, is

$$\lim_{N \to \infty} \frac{top\ wage}{average\ wage} < +\infty?$$

i.e.

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$$\lim_{\textit{firm size} \to \infty} \frac{\textit{CEO salary}}{\textit{average salary}} = +\infty?$$

i.e. does $(\textit{total economy}) \in L^1$ imply $(\textit{individual payoffs}) \in L^\infty$?

• some flavor of questions in statistical physics;

• do parallels exist that can be developed?

Matching in the education and labor markets

EDUCATION MARKET

- different students willing to pay teachers to enhance their skills
- different teachers seek students to pay their salaries

LABOR MARKET

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- adults choose a profession (worker, manager, teacher) based on earnings potential given their skills (innate or acquired)
- workers seek managers to produce output (commensurate with skills)
- managers seek workers...
- fruits of output divided competitively (according to what each will bear)
- teachers seek students to educate (depending on the skills of each...)

Interrelation between these markets has unexpected potential for feedback!

Steady-state competitive equilibrium

PROFIT MOTIVE: individuals driven to maximize share of wealth (generated by labor production b_L plus any external value b_E of education)

LARGE MARKET HYPOTHESIS: no individual or small group has market power (i.e. can affect outcomes for a positive fraction of population)

EQUILIBRIA are STABLE: no individual or small group should prefer to abandon their partners in favor of collaboration with each other

STEADY-STATE: educational matching should reproduce the same endogenous distribution of adult skills α at each generation, given an exogenously specified distribution κ of student skills at each generation

A mathematical model

Student skills: $k \in K = [0, \bar{k}[$ distributed according to $d\kappa \geq 0$ on $\bar{K} \subset \mathbf{R}$

Adult skill level $a \in \bar{A}$ has value $cb_E(a)$ outside the labor market, where $0 < b_E \in C^1(\bar{A})$ is *strongly* convex increasing, $c \geq 0$, and w.l.o.g. $A = K$

EDUCATION MARKET: parameterized by $0 < \theta < 1 \leq N$ and $b_E(\cdot)$

- a teacher can teach N students, each inheriting a fraction θ of their skill

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EDUCATION MARKET: parameterized by $0 < \theta < 1 \leq N$ and $b_E(\cdot)$

• a teacher can teach N students, each inheriting a fraction θ of their skill i.e., if $k \in K$ studies with $a \in A$ they acquire skill $z^\theta(k, a) = (1 - \theta)k + \theta a$.

LABOR MARKET: parameterized by $0 < \theta' < 1 \leq N'$ and $b_L(\cdot)$ like $b_E(\cdot)$

• worker $a \in A$ and manager $a' \in A$ produce output $b_L((1 - \theta')a + \theta' a')$

• each manager can manage up to N' workers

Payoffs and matchings

Recall: a map $z : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ *pushes* a measure $\mu \geq 0$ on $\mathbf{R}^m$ *forward* to a measure $z_{\#}\mu$ on $\mathbf{R}^n$ assigning mass $\mu[z^{-1}(V)]$ to each $V \subset \mathbf{R}^n$ (all Borel)

Seek real functions u, v on $K = A$ and measures $\epsilon, \lambda \geq 0$ on $\bar{K} \times \bar{A}$ where

$u(k) =$ lifetime net income of student of skill k (minus tuition invested)
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$d\epsilon(k, a) =$ fraction of skill k students who study with skill a teachers
$d\lambda(a, a') =$ number of skill a workers who match with skill a' managers

whose marginals $\epsilon^i = \pi^i_{\#}\epsilon$ under $\pi^1(k, a) = k$ and $\pi^2(k, a) = a$
and push-forward $z^\theta_{\#}\epsilon$ through $z^\theta(k, a) := (1 - \theta)k + \theta a$ satisfy...

MNEMONIC TABLE

Generation	Skill range	Skill distribution	Distribution type
Kids Adults	$K = [0, \bar{k}[$ $A = K$	$d\kappa(k) \geq 0$ $d\alpha(a) \geq 0$	exogenous endogenous: $\alpha = z^\theta_\# \epsilon$

$$z^\theta(k, a) := (1 - \theta)k + \theta a$$

Sector	Exogenous parameters	Endogenous matching	Direct (exogenous) payoff	Indirect (endogenous) payoff
Education Labor	(N, θ) (N', θ')	$d\epsilon(k, a) \geq 0$ $d\lambda(a, a') \geq 0$	$cb_E(z)$ $b_L(z)$	$u(k)$ $v(a)$

MOTIVATING EXAMPLE: $N = N'$, $\theta = \frac{1}{2} = \theta'$ and $b_L(a) = e^a = b_E(a)$, $c \geq 0$, with $c = 0$ being a case of primary interest

Competitive equilbrium

STEADY-STATE

$$\epsilon^1 = \kappa \qquad \text{and} \tag{1a}$$

$$\lambda^1 + \frac{1}{N'}\lambda^2 + \frac{1}{N}\epsilon^2 = z^\theta_\# \epsilon, \tag{1b}$$

i.e. worker + manager + teacher skills = output of educational match

STABLE

$$u(k) + \frac{1}{N} v(a) \geq c b_E(z^\theta(k,a)) + v(z^\theta(k,a)) \quad \text{and} \tag{2a}$$

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$$u(k) + \frac{1}{N} v(a) \geq c b_E(z^\theta(k,a)) + v(z^\theta(k,a)) \quad \text{and} \tag{2a}$$

$$v(a) + \frac{1}{N'} v(a') \geq b_L((1-\theta')a + \theta' a') \qquad \text{on } \bar{K} \times \bar{A}, \tag{2b}$$

BUDGET FEASIBLE

equality holds ϵ-a.e. in (2a) and λ-a.e. in (2b) (3)

A variational approach...

But how can we find and analyze such equilibria?

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Assume a marriage of man k to woman a generates surplus $s(k, a)$, to be divided between them as they see fit. Given probability measures $d\kappa(k)$ and $d\alpha(a)$ representing the frequency of different types of men and women in a given population, can we pair each man to a woman STABLY, meaning that, when the pairing is done, no man and woman would both prefer to leave their assigned partners and marry each other?

e.g. M men and M women:

$$\kappa = \frac{1}{M}\sum_{i=1}^{M}\delta_{k_i} \text{ and } \alpha = \frac{1}{M}\sum_{j=1}^{M}\delta_{a_j}, \text{ payoff matrix } (s_{ij}) = s(k_i, a_j)$$

Shapley and Shubik's (1972) solution:

The solutions are precisely those pairings $d\epsilon(a,k)$ of men to women which attain the maximum

$$\max_{\{\epsilon \geq 0 \mid \epsilon^1 = \kappa,\ \epsilon^2 = \alpha\}} \int_{\bar{K} \times \bar{A}} s(k,a) d\epsilon(k,a).$$

This is a LINEAR PROGRAM; in fact an *optimal transportation* problem.

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The solution $(u,v) \in C(\bar{A})^2$ to its DUAL PROGRAM,

$$\inf_{u(k)+v(a) \geq s(k,a)} \int_{\bar{K}} u(k) d\kappa(k) + \int_{\bar{A}} v(a) d\alpha(a)$$

shows how the surplus $s(k,a)$ will be split between the husband and wife in each couple at equilibrium, *provided the infimum is attained*;
it satisfies $u(k) + v(a) \geq s(k,a)$ on $\bar{K} \times \bar{A}$, with equality ϵ-a.e.

The analogous linear programs for our steady-state match

PLANNER'S PROBLEM: a maximization over steady-state matches (ϵ, λ)

$$LP^* := \max_{\{\epsilon,\lambda \geq 0 | (1a)-(1b)\}} c \int_{\bar{K}\times\bar{A}} b_E \circ z^\theta d\epsilon + \int_{\bar{A}\times\bar{A}} b_L \circ z^{\theta'} d\lambda$$

(recall $z^\theta = (1-\theta)k + \theta a$)

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DUAL LINEAR PROGRAM: a minimization over stable payoffs (u, v)

$$LP_* := \inf_{\{u,v \in F | (2a)-(2b)\}} \int_{\bar{K}} u(k) d\kappa(k)$$

$$F = \{u_0 + u_1 = u \in L^1(d\kappa) \mid u_0 \in C(\bar{K}) \text{ and } u_1 > 0 \text{ non-decreasing}\}$$

Proof of duality ($LP_* = LP^*$): $\geq$ 'easy'; $\leq$ standard

Rockafellar-Fenchel duality in $(C(K), \|\cdot\|_\infty)^2$ implies $\boxed{LP_* \leq LP^*.}$

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The reverse inequality is formally clear: integrating educational stability

$$u(k) - cb_E(z^\theta(k,a)) \geq v(z^\theta(k,a)) - \frac{1}{N}v(a) \tag{2a}$$

against $d\epsilon(k,a)$ yields

$$\int_{\bar{K}} ud\kappa - c\int_{\bar{A}} b_E d(z^\theta_\# \epsilon) \quad \geq \quad \int_{\bar{A}} vd(z^\theta_\# \epsilon) - \frac{1}{N}\int_{\bar{A}} vd\epsilon^2$$

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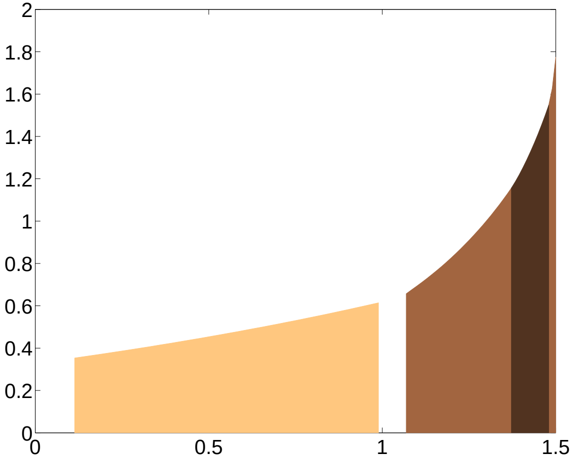
against $d\epsilon(k,a)$ yields

$$\begin{aligned} \int_{\bar K} u d\kappa - c \int_{\bar A} b_E d(z^\theta_\# \epsilon) &\geq \int_{\bar A} v d(z^\theta_\# \epsilon) - \frac{1}{N} \int_{\bar A} v d\epsilon^2 \\ &= \int_{\bar A} v d(\lambda^1 + \frac{1}{N'} \lambda^2) \\ &\geq \int_{\bar A \times \bar A} b_L((1-\theta')a + \theta' a') d\lambda(a,a') \end{aligned}$$

for all (ϵ, λ) & $u, v \in F \subset L^1(d\kappa)$ satisfying (1)-(2)
provided the integrals converge.

Strict inequalities would violate the budget constraint.

Numerics: Equilibrium wage $v(a)$ as a function of adult skill $a \in [0, \bar{a}[$



$\kappa = \mathcal{L}^1$ uniform, $c = 0$, $b_L(a) = e^a$, and $(N, \theta) = (N', \theta') = (10, \frac{1}{2})$,
Note segregation: workers=yellow, managers=brown, and teachers=beige

Doubling condition

To guarantee this convergence, we henceforth assume a doubling condition on the student skill distribution at its upper endpoint: for some $C < \infty$ and all $D > 0$:

$$\int_{[\bar{k}-2D,\bar{k}]} d\kappa \leq C \int_{[\bar{k}-D,\bar{k}]} d\kappa. \qquad (D.C.)$$

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Proposition (Variational characterization of competitive equilibria)

(D.C.) implies $LP_* = LP^*$. If LP_* is attained, then for (u, v) and (ϵ, λ) to optimize LP_* and LP^* is equivalent to forming a competitive equilibrium.

Thus it is crucial to know the infimum is attained — if not in $C(\bar{A})^2$ — then at least in the larger class $u, v \in F$.

Moreover, we want to analyze the optimal (ϵ, λ) and (u, v).

Theorem (Existence of equilibrium wages)

Fix $c \geq 0$ and positive constants $\bar{k} = \bar{a}$ and $\max\{\theta, \theta'\} < 1 < \min\{N, N'\}$. If $b_{E/L} \in C^1(\bar{A})$ are positive, uniformly convex and increasing and κ satisfies (D.C.) on $K = [0, \bar{k}[$, then LP_* is attained by convex non-decreasing functions $u, v \in F$, uniformly convex and increasing if either $c > 0$ or $N\theta^2 \geq 1$.

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Moreover, $v = \max\{v_w, v_m, v_t\}$ and

$$u(k) = \sup_{a \in \bar{A}} cb_E(z^\theta(k,a)) + v(z^\theta(k,a)) - \frac{1}{N} v(a)$$

where the worker / manager / teacher wages for an adult of skill $a \in K$ are

$$v_w(a) := \sup_{a' \in \bar{A}} b_L((1-\theta')a + \theta' a') - \frac{1}{N'} v(a')$$

$$v_m(a') := N' \sup_{a \in \bar{A}} b_L(z^{\theta'}(a, a')) - v_w(a)$$

$$v_t(a) := N \sup_{k \in \bar{K}} cb_E(z^\theta(k,a)) + v(z^\theta(k,a)) - u(k).$$

Idea of proof:

1) the conclusions becomes true if we restrict the infimum by replacing F with the compact set $F_0 = \{v \in F \mid v \text{ convex, non-decreasing}\}$

2) we then need to check that these artificially imposed constraints do not bind for the minimizing pair $u, v \in F_0$ under this restriction;

3) this requires positive lower bounds for first two derivatives of $v_{w/m/t}$

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A few technical issues:

5) get $v = \max\{v_w, v_m, v_t\}$ for a.e. adult, but need it $\mathcal{L}^1$-a.e. in K

7) need to perturb the problem to ensure $\mathcal{L}^1 \ll \kappa$ and $\mathcal{L}^1 \ll \alpha = z^\theta_\# \epsilon$

8) finally, let this perturbation (and $c > 0$ if desired) tend to zero, using convexity and monotonicity of (u_k, v_k) to extract a convex monotone limit

9) more work shows uniform convexity/monotonicity survives if $N\theta^2 \geq 1$

Let

$$\underline{b}'_{E/L} := b'_{E/L}(0)$$
$$\bar{b}'_{E/L} := b'_{E/L}(\bar{a}).$$

Lemma (Specialization by type; the educational pyramid)

In any equilibrium:

a) $N'\theta' \geq \bar{b}'_L / \underline{b}'_L \implies$ *least manager skill* $\geq$ *highest worker skill*

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c) if $N\theta > 1$ *education strictly improves everyone's skill and*

d) in this case the academic descendents of a skill $a \in A$ *teacher display at most finitely many skill types unless differentiability of* v *fails at* a*.*

i.e. finitely many academic descendents, yet INFINITELY many ancestors

Corollary (Uniqueness; positive assortativity)

a) *If* (ϵ, λ) *maximize the planner's problem, spt* $\lambda \subset \mathbf{R}^2$ *is non-decreasing; i.e. managerial skill varies directly with worker skill.*

b) *Moreover, there exist maximizers* (ϵ, λ) *with spt* ϵ *non-decreasing also, i.e. with teacher skill varying directly with student skill.*

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d) If also κ is free from atoms, the equilibrium match (ϵ, λ) is unique.

e) If also $N\theta > 1$, then $u'(k)$ and $v'(a)$ are uniquely determined for κ-a.e. student k and $\alpha = z^\theta_\# \epsilon$-a.e. adult $a \in K$

f) If also κ dominates some a.c. measure whose support fills $\bar{K}$, then u is unique (among locally Lipschitz functions on $K = [0, \bar{k}[$).

Theorem (Transition to unbounded wage gradients)

If κ given by an L^∞ probability density, continuous and positive at $\bar{k}$, and
i) all sufficiently skilled adults become teachers (as when $N\theta \geq \bar{b}'_L / \underline{b}'_L$)
ii) spt $\epsilon \subset$ Graph(a_t) for $a_t : \bar{K} \longrightarrow \bar{A}$ non-decreasing (as when $N\theta^2 > 1$)
iii) the student-to-teacher skill map $a = a_t(k)$ is differentiable at $\bar{k}$
iv) $v(a)$ is differentiable on $]\bar{a} - \delta, \bar{a}[$ for some $\delta > 0$, then for $N\theta \neq 1$

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$$\frac{dv}{da}(\bar{a} - \Delta a) = \frac{const}{|\Delta a|^{\frac{\log N\theta}{\log N}}} + \frac{c\bar{b}'_E}{\frac{1}{N\theta} - 1} + o(1) \qquad \text{as } \Delta a \downarrow 0.$$

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Corollary

If $N\theta > 1$ then (i)&(ii) $\implies$ a singularity must occur in u or in v near $\bar{a}$.

If also (iii)-(iv) hold, then $\lim_{a \to \bar{a}} v(a) < +\infty$

Idea of proof (theorem and corollary):

A student-teacher match produces equality in the stability constraint

$$u(k) + \frac{1}{N} v(a) - [cb_E + v]((1-\theta)k + \theta a) \geq 0 \tag{2a}$$

Assuming differentiability, the first-order conditions for equality

$$\frac{u'(k)}{1-\theta} = [cb_E' + v']_{(1-\theta)k+\theta a} =$$

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$$\frac{u'(k)}{1-\theta} = [cb'_E + v']_{(1-\theta)k+\theta a} = \frac{v'(a)}{N\theta}$$

$$\implies a = a_t(k) = \frac{1}{\theta}[cb'_E + v']^{-1}\left(\frac{u'(k)}{1-\theta}\right) + (1-\theta^{-1})k$$

and

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$$u(k) + \frac{1}{N} v(a) - [cb_E + v]((1-\theta)k + \theta a) \geq 0 \tag{2a}$$

Assuming differentiability, the first-order conditions for equality

$$\frac{u'(k)}{1-\theta} = [cb'_E + v']_{(1-\theta)k+\theta a} = \frac{v'(a)}{N\theta}$$

$$\implies a = a_t(k) = \frac{1}{\theta}[cb'_E + v']^{-1}\left(\frac{u'(k)}{1-\theta}\right) + (1-\theta^{-1})k$$

and

$$v'(a) = N\theta[cb'_E + v']_{(1-\theta)k+\theta a}$$

This last formula shows $N\theta$ acts as a multiplier relating the marginal wage $v'(z)$ of an adult with skill $z = (1-\theta)k + \theta a$ to the marginal wage $v'(a)$ of his or her teacher. If $N\theta > 1$ and we know to first-order how a and hence z relate to k, we can compute the rate at which $v'(a)$ diverges as $a \to \bar{a}$.

QED

Returing to point (9) of our earlier proof:

uniform convexity of v_c for $N\theta^2 \geq 1$ survives $c \downarrow 0$

is derived from the analogous second-order condition for a minimum of

$$v(a) + \frac{1}{N'} v(a') - b_L((1-\theta')a + \theta' a') \geq 0 : \qquad (2b)$$

$$v_c''(a) \geq \begin{cases} (1-\theta')^2 b_L''|_{(1-\theta')a+\theta' a'} & \geq \delta \quad \text{if } a \text{ works} \\ N'(\theta')^2 b_L''|_{(1-\theta')a'+\theta' a} & \geq \delta \quad \text{if } a \text{ manages} \\ N\theta^2 [c b_E'' + v_c'']_{(1-\theta)k+\theta a} & \geq 0 \quad \text{if } a \text{ teaches.} \end{cases}$$

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Thus

$$\underline{v}_c'' := \inf_{a \in A} v_c''(a) \geq \begin{cases} \delta & \text{independent of } c > 0 \text{ or} \\ N\theta^2 (c \underline{b}_E'' + \underline{v}_c'') & \end{cases}$$

Since $\underline{v}_c'' \geq 0$, we get a $c > 0$ independent bound $\bar{v}_c'' > \delta > 0$ if $N\theta^2 \geq 1$.

QED

Conclusions and open questions:

1) Hidden recursion in education market can generate wage singularities with universal exponents;

2) but only if a teacher's impact $N\theta \geq 1$ does not decrease from one generation of students to the next.

3) Such singularities lead to subtle questions, some remaining open.

4) In this model, they occur at the level of gradients rather than wages.

5) Do singularities persist with discounting? (ie. interest on student loans)

6) What about models with a countable number of management layers?

7) Does competition allow a tiny fraction of the population to extract a positive fraction of the total wealth?

8) Can one analyze the limiting behaviour of finite population models?

9) Parallels to statistical physics?

10) Economic theory remains largely ripe for mathematization...

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Thank you