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On log-concave functions

Analytic Tools in Probability and Applications
IMA – University of Minnesota

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Outline of the talk

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- Log-concave functions and their addition

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- Functional forms of Blaschke-Santaló and Rogers-Shephard inequalities
- Valuations on log-concave functions

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$$f : \mathbb{R}^n \to \mathbb{R}_+$$

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$$\lim_{|x|\to+\infty} u(x) = +\infty \quad \Rightarrow \quad \lim_{|x|\to+\infty} f(x) = 0,$$

and set

$$\mathcal{C}^n = \{f \text{ log-concave on } \mathbb{R}^n : \lim_{|x|\to+\infty} f(x) = 0\}.$$

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We will focus on the second aspect.

Addition of log-concave functions

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(so that, *e.g.*, $f \oplus f = 2 \cdot f$). $\mathcal{C}^n$ is closed with respect to these operations.

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In other words, $\oplus$ extends the Minkowski addition at the level of log-concave functions.

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$$\text{epi}(u \Box v) = \text{epi}(u) + \text{epi}(v).$$

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Another property of inf-convolution:

$$(u \Box v)^* = u^* + v^*$$

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Summarizing,

$$f \oplus g = e^{-u} \oplus e^{-v} = e^{-(u \Box v)} = e^{-(u^* + v^*)^*},$$

i.e. $\oplus$ coincides with the **usual addition** applied to the **conjugate** of the **exponent** (with sign changed) of log-concave functions.

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$I(f)$ is the counterpart of the *volume* (Lebesgue measure) of a convex body K, in convex geometry, denoted by V_n:

$$V_n(K) = \text{volume of } K.$$

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For every f, g log-concave and for every $t \in [0,1]$

$$\int_{\mathbb{R}^n} [(1-t)\cdot f \oplus t \cdot g] dz \geq \left(\int_{\mathbb{R}^n} f dx\right)^{1-t} \left(\int_{\mathbb{R}^n} g dy\right)^{t} \qquad \text{(PL)}$$

(Prékopa, 1971; Leindler 1972).

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See the exhausitve survey by Gardner: *The Brunn-Minkowski inequality*, 2002.

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- It can be viewed as a reverse form of the Hölder inequality, recalling that

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- It is a special case of a family of inequalities proved by Barthe, which reverse the Borell-Brascamp-Lieb inequalities.
- It is also a limit case of the reverse Young inequalities for convolutions.

From (PL) to Poincaré and Sobolev inequalities

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Bobkov and Ledoux (2000) showed how to obtain a Poincaré type inequality due to Brascamp and Lieb (see next slide), and the log-Sobolev inequality starting from the (PL) inequality. In 2007 they also provided a proof of the Sobolev inequality via the Brunn-Minkowski inequality.

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In general it is now well understood that concavity inequalities like (PL) and (BM) can be used to prove integral inequalities involving derivatives.

More ambitiously, we would like to see that (PL) is in fact *equivalent* to a class of integral inequalities of Poincaré type (precisely those of Brascamp and Lieb).

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Then for every $\phi \in C^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (\phi - \int_{\mathbb{R}^n} \phi d\mu)^2 d\mu \leq \int_{\mathbb{R}^n} ((D^2 u)^{-1} \nabla\phi, \nabla\phi) d\mu. \qquad \text{(BL)}$$

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Remark. This result applies to the Gaussian measure γ_n and provides the standard Poincaré inequality for this measure.

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$J''(f)$ can be thought as a bilinear quadratic form acting on test functions h, defined as follows

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The condition

$$(J''(f)h, h) \leq 0 \quad \forall h$$

turns out to be exactly the Poincaré inequality of Brascamp and Lieb.

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$$\nabla u = (\nabla u^*)^{-1}, \quad D^2 u(x) = [D^2 u^*(\nabla u(x))]^{-1}.$$

Moreover

$$u(x) = x \cdot \nabla u(x) - u^*(\nabla u(x)).$$

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$$\begin{aligned} I(f) = \int_{\mathbb{R}^n} f dx &= \int_{\mathbb{R}^n} e^{-u} dx \quad (y = \nabla u(x)) \\ &= \int_{\mathbb{R}^n} e^{u^*(y) - y \cdot \nabla u^*(y)} \det(D^2u^*(y)) dy. \end{aligned}$$

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for an arbitrary test function ψ. This condition plus the reverse change of variable $y = \nabla u(x)$ lead to the inequality by Brascamp and Lieb.

Remarks

- A similar procedure shows that the Brunn-Minkowski inequality is equivalent to a family of weighted Poincaré inequalities on the unit sphere ([C. '08]).

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- Many inequalities of (BM) type (proved or conjectured) admit an infinitesimal version, typically consisting of a family of functional inequalities of Poincaré type, but in general it is difficult to use them **to prove** (BM). At this regard see the recent result by Livshyts, Marsiglietti, Nayar and Zvavitch about the conjectured dimensional version of (BM) in Gauss space, for symmetric convex bodies.

The functional version of Blaschke-Santaló inequality

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Thm. *Let* $f = e^{-u}$ *be a log-concave function and assume that* f *is* **even**. *Then*

$$\left(\int_{\mathbb{R}^n} e^{-u} dx\right) \cdot \left(\int_{\mathbb{R}^n} e^{-u^*} dx\right) \leq (2\pi)^n.$$

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Thm. (Blaschke-Santaló inequality). *Let K be a centrally symmetric convex body, and let K° be its polar:*

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- For both theorems, to find an optimal lower bound for the corresponding products (of integrals or of volumes) is an open problem, for $n \geq 3$.

The difference body inequality

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Thm. (Rogers-Shephard inequality).

$$V_n(DK) \leq \binom{2n}{n} V_n(K).$$

Equality holds iff K is a symplex.

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This is an even function. By the (PL) inequality:

$$\int_{\mathbb{R}^n} Df \, dz \geq \int_{\mathbb{R}^n} f \, dx.$$

Thm. (C. '06).

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Equality holds iff (up to linear transformations)

$$f(x_1, \ldots, x_n) = \begin{cases} e^{-(x_1+\cdots+x_n)} & \text{if } x_i \geq 0 \text{ for every } i, \\ 0 & \text{elsewhere.} \end{cases}$$

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Extensions and other functional versions of the Rogers-Shephard inequality have been recently found by Artstein-Avidan, Einhorn, Florentin, Ostrover (2015), and Alonso-Gutiérrez, González, Jiménez, Villa (2015).

Valuations on convex bodies

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Let $\mathcal{K}^n = \{\text{convex bodies in } \mathbb{R}^n\}$. A mapping $\mu : \mathcal{K}^n \to \mathbb{R}$ is a *valuation* if

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The study of valuations, especially with continuity and invariance properties, is one of the most active and prolific branches of convex geometry, after the seminal contributions by Hadwiger, McMullen and Alesker.

Thm. (Volume theorem). *Let μ be a rigid motion invariant and simple valuation on $\mathcal{K}^n$, which is also either monotone or continuous. Then it is a multiple of the volume.*

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Let X be a space of functions. A mapping $\sigma : X \to \mathbb{R}$ is a valuation if

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- $X = W^{1,p}(\mathbb{R}^n)$ or $X = BV(\mathbb{R}^n)$; Ludwig ('11, '12, '13), Ma ('15), Wang ('13).

A result for log-concave functions

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Thm. (Cavallina, C., '15) *Let σ be a valuation defined on the space of log-concave functions, which is:*

- *rigid motion invariant, i.e.*

$$\mu(f) = \mu(f \circ T) \quad \forall\, T \text{ rigid motion of } \mathbb{R}^n;$$

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Then μ can be written in the form

$$\mu(f) = \int_{\mathbb{R}^n} G(f)dx \quad \forall\, f$$

for a suitable function G (continuous and monotone).