Counting perfect matchings via random matrices

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based on joint works with
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Perfect matchings

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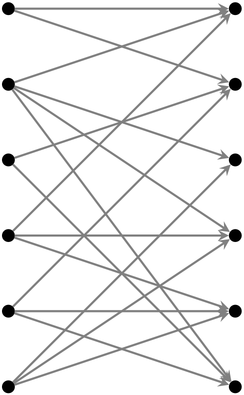

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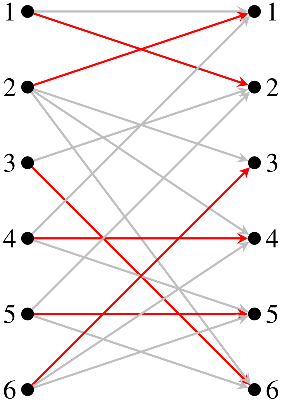

$\pi = (2, 1, 6, 4, 5, 3)$

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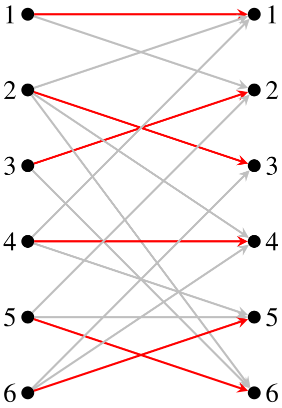

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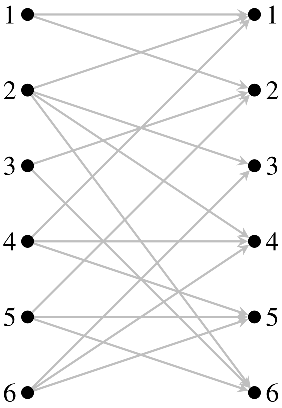
$$\#(\text{perfect matchings}) = \text{per}(A),$$

where A is the adjacency matrix of the graph.



Permanent of a matrix

Let A be an $n \times n$ matrix with $a_{i,j} \geq 0$.

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Running time: $O(n^{2.376})$.

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Evaluation of permanents is $\#P$-complete (Valiant 1979)

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Deterministic bounds

- Linial–Samorodnitsky–Wigderson algoritm: if $\operatorname{per}(A) > 0$, then one can find in polynomial time diagonal matrices D, D' such that the renormalized matrix $A' = D'AD$ is almost doubly stochastic:

$$1 - \varepsilon < \sum_{i=1}^{n} a'_{i,j} < 1 + \varepsilon, \qquad \text{for all } j = 1, \dots, n$$

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- Linial–Samorodnitsky–Wigderson algoritm: reduces permanent estimates to almost doubly stochastic matrices
- Van der Waerden conjecture, proved by Falikman and Egorychev: if A is doubly stochastic, then

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- Gurvits-Samorodnitsky estimator (2014) + Linial–Samorodnitsky–Wigderson algorithm yields the multiplicative error 2^n.

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- Barvinok's estimator (improves Godsil–Gutman estimator).
  Let A be the adjacency matrix.
  Let Γ be an $n \times n$ random matrix having independent $N(0, a_{ij})$ entries.
  Then
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- How accurate is it?

Precision bounds for Barvinok's estimator

Theorem (Barvinok)

Let A be any $n \times n$ matrix with non-negative entries.
Then, with probability $1-\delta$,

$$((1-\varepsilon)\cdot\theta)^n \operatorname{per}(A) \le \det^2(\Gamma) \le C \operatorname{per}(A),$$

where C is an absolute constant and

- $\theta = 0.28$ *for real Gaussian matrices;*
- $\theta = 0.56$ *for complex Gaussian matrices;*

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- Identity matrix: multiplicative error at least $\exp(cn)$ w.h.p.
- Matrix of all ones: multiplicative error at most $\exp(C\sqrt{\log n})$ (Goodman, 1963).

Question:

for which graphs would Barvinok's estimator yield a small error?

Concentration for Barvinok's estimator: how strong?

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Theorem (R'–Zeitouni, 2013)

Let A be the adjacency matrix A of an $n \times n$ expander-type bipartite graph of the minimal degree at least δn.
Let Γ be a random Gaussian matrix with variance matrix A.
Then for any $\tau \geq 1$

$$\mathbb{P}\left[\qquad \leq \frac{\det^2(\Gamma)}{M} \leq \qquad \right]$$

$\geq 1 -$ *small*

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- det is a highly non-linear function $\Rightarrow$ $\det(\Gamma)$ has heavy tails.

Counting perfect matchings in general graphs

Counting perfect matchings: permanent $\Rightarrow$ hafnian

Let G be a graph with $n = 2m$ vertices.

Bipartite graphs:

Let A be the $n \times n$ *non-trivial block* of the adjacency matrix of the graph.

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General graph:

Let A be the the adjacency matrix of the graph.

$$\mathrm{haf}(A) = \frac{1}{m!2^m} \sum_{\pi \in \Pi_n} \prod_{j=1}^{m} a_{\pi(2j-1),\pi(2j)}.$$

Bipartite graph algorithms

Let $G(V,E)$ be a graph with an even number of vertices.

- MCMC algorithm works in polynomial time whenever the minimal degree is greater than $n/2$ (Jerrum-Sinclair, 1989).

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- There is no lower bound for the hafnian of a doubly stochastic matrix.
- A deterministic algorithm of sub-exponential (but not polynomial) complexity (Bayati, Gamarnik, Katz, Nair, and Tetali, 2007)

Barvinok's estimator

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Let Γ be an $n \times n$ Gaussian matrix with $\gamma_{ij} \sim N(0,1)$ whenever $i \neq j$, $A_{ij} = 1$ and 0 otherwise.

$$\#\text{perf. match.} = \mathrm{per}(A') = \mathbb{E}\det^2(\Gamma)$$

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Let W be an $n \times n$ skew-symmetric Gaussian matrix with $\tilde{w}_{ij} \sim N(0,1)$ whenever $i < j$, $A_{ij} = 1$ and 0 otherwise.

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Barvinok's estimator

Theorem (Barvinok)

Let A be a symmetric matrix with non-negative entries.
Let W be a skew-symmetric Gaussian matrix with $\mathbb{E}w_{ij}^2 = a_{ij}$.
If $\mathrm{haf}(A) > 0$, *then*

$$\mathbb{P}\left(\exp(-\gamma n) \le \frac{\det(W)}{\mathrm{haf}(A)} \le C\right) = 1 - o(1),$$

where $\gamma = 0.577$ is the Euler constant.

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Problem

For which graphs (matrices) would Barvinok's estimator yield subexponential precision?

Strong expansion condition

Definition

A graph $G = (V, E)$ is called a vertex expander if for any set $J \subset V$ with $|J| \leq n/2$

$$|\partial(J)| \geq \kappa \cdot |J|$$

Key parameter: $\kappa > 0$.

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Definition

A graph $G = (V, E)$ is called a strong expander if for any set $J \subset V$ with $|J| \leq m$

$$|\partial(J)| - |\text{Con}(J)| \geq \kappa \cdot |J|$$

Con(J) – the set of connected components of J.
Key parameters: $\kappa > 0, m < n$.

Main difference

Highly disconnected sets should grow faster.

Spectral gap

Let A be the adjacency matrix. Let $B = D_1 A D_2$ be its doubly stochastic scaling.

- Bregman's theorem: the scaling exists iff any edge is a part of some perfect matching.
  (easy to check).
- A doubly stochastic scaling of a symmetric matrix is symmetric.

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A symmetric stochastic matrix B is said to possess a spectral gap δ if there there do not exist eigenvalues of A in $(-1, -1+\delta) \cup (1-\delta, 1)$.

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- Spectral gap $\Leftrightarrow$ edge expansion.
- Spectral gap $\approx$ vertex expansion.

Regular graphs

Theorem (R'-Samorodnitsky-Zeitouni, 2014)

Fix $\alpha, \kappa > 0$. Let A be the adjacency matrix of a d-regular graph G with $d \geq \alpha n$. Assume that

$$G \text{ is } \kappa \text{ strongly expanding up to level } \frac{1-\alpha}{1+\kappa/4} \cdot n.$$

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1. *Then for any $\epsilon > 0$ and $D > 0$,*

$$\mathbb{P}\left(\exp(-Cn^{4/5+\epsilon}) \leq \frac{\det(W)}{\operatorname{haf}(A)} \leq \exp(Cn^{4/5+\epsilon})\right) \geq 1 - n^{-D}.$$

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2. *If, in addition to the above assumptions, the matrix A/d possesses a spectral gap $\delta > 0$, then for any $D > 0$,*

$$\mathbb{P}\left(\exp(-Cn^{1/2}\log^{1/2} n) \leq \frac{\det(W)}{\operatorname{haf}(A)} \leq \exp(Cn^{1/2}\log^{1/2} n)\right) \geq 1 - n^{-D}.$$

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Fix $\alpha, \kappa > 0$*. Let A be the adjacency matrix of a graph G with minimal degree* $d \geq \alpha n$*. Assume that*

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and the doubly stochastic scaling B of A satisfies $\max_{i,j} B_{ij} \leq n^{-\vartheta}$*.*

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2. *Fix $\delta > 0$. If, in addition to the above assumptions, B possesses a spectral gap δ, then for any $D > 0$,*

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Let $\delta > 0$.

$$\forall J \subset [M] \quad |J| \le M/2 \;\Rightarrow\; |\partial(J)| - (1-\delta)|Con(J)| \ge \kappa |J|$$

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Let $\delta > 0$. For any $N \in \mathbb{N}$, there exists a graph G with $M > N$ vertices such that

$$\forall J \subset [M] \quad |J| \leq M/2 \Rightarrow |\partial(J)| - (1-\delta)|Con(J)| \geq \kappa |J|$$

and

$$\mathbb{P}\left(\frac{\det(W)}{\operatorname{haf}(A)} \leq e^{-cM}\right) \geq 1 - e^{-c'M}.$$

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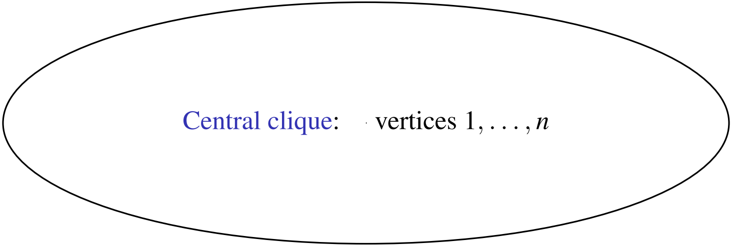

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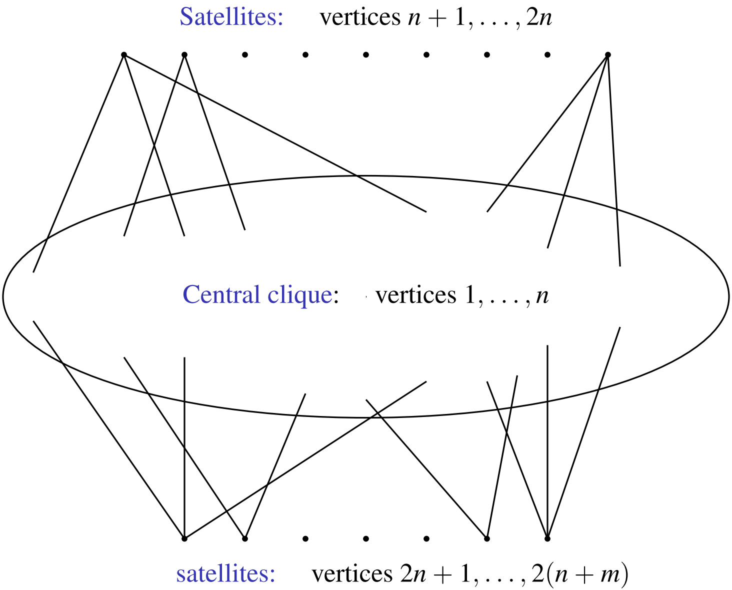

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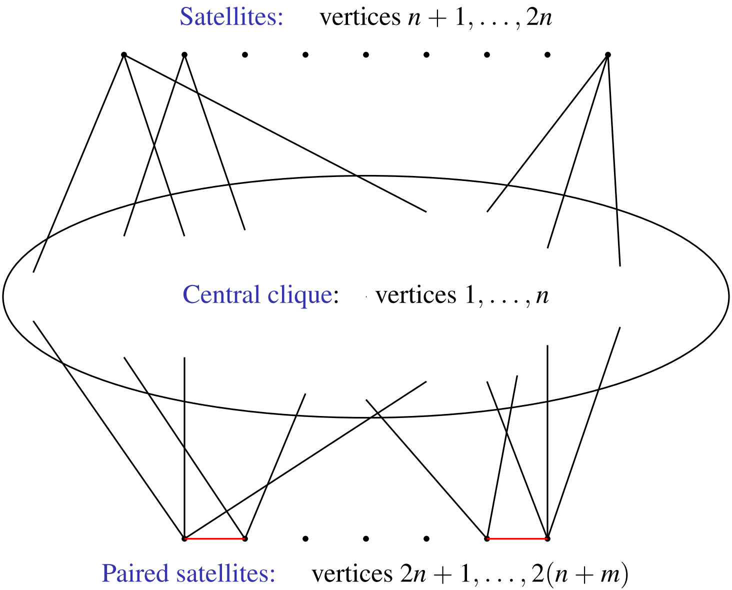

Approach to concentration

$W = A \odot G^s$ – the entry-wise product of A and a skew-symmetric Gaussian matrix.

- Aim: $X(G) := \det(A \odot G^s)$ is concentrated.
- $\det(A \odot G^s)$ is highly non-linear $\Rightarrow$ $\log(\det(A \odot G^s))$ is easier to control.
- Modified aim : $Y(G) = \log\det(A \odot G^s)$ is concentrated around its expectation. We will have to compare the concentration for $X(G)$ and $Y(G)$ at the end.

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- $\log\det(A \odot G^s)$ is not Lipschitz.

What is non-Lipschitz in log determinant?

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How to use Lipschitz concentration for a non-Lipschitz function?

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   This establishes concentration of $\log\det(A \odot G^s)$ around its median.
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3. Use measure concentration to show that the median of $\log\det(A \odot G^s)$ is close to the expectation.
4. Use measure concentration another time to swap the expectation and the logarithm:
   $$\mathbb{E}\det(A \odot G^s) \approx \exp\left(\mathbb{E}\log\det(A \odot G^s)\right)$$
   This is the main source of (**unavoidable**) errors.