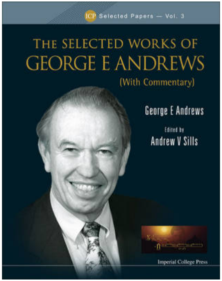
Geometric and Enumerative Combinatorics, IMA
University of Minnesota, Nov 10–14, 2014

Combinatorics, Modular Forms, and Discrete Geometry

Peter Paule
(joint work with: G.E. Andrews, S. Radu)

Johannes Kepler University Linz
Research Institute for Symbolic Computation (RISC)

Partition Analysis

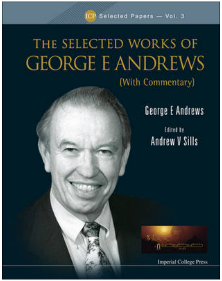


"The no. of partitions of N of the form $N = b_1 + \cdots + b_n$ satisfying

$$\frac{b_n}{n} \geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0$$

equals the no. of partitions of N into odd parts each $\leq 2n-1$.

This problem cried out for MacMahon's Partition Analysis, ...



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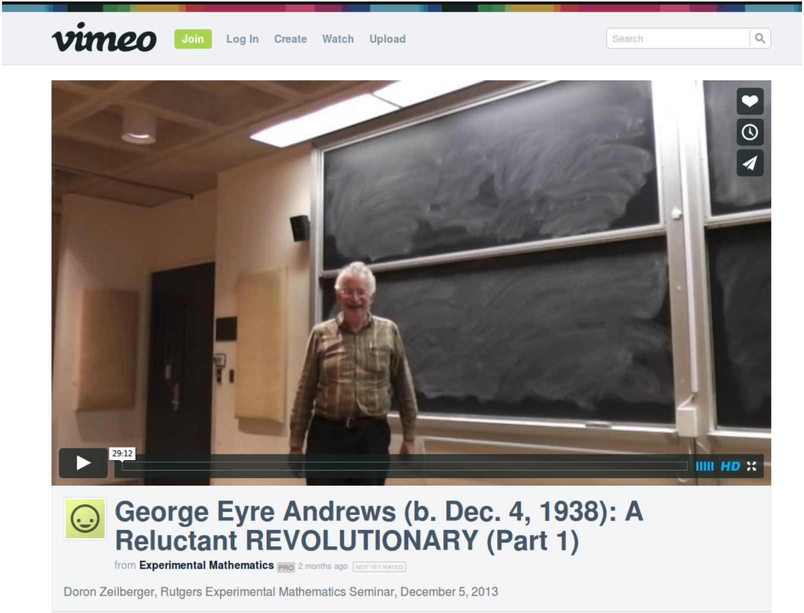
Given that Partition Analysis is an algorithm for producing partition generating functions, I was able to convince Peter Paule and Axel Riese to join an effort to automate this algorithm."

How Zeilberger tells the story of partition analysis (and more):



Example (PA and the Omega package)

Find a suitable closed form of

$$L(x_1,x_2,x_3) \quad := \sum_{b_1,b_2,b_3\in\mathbb{N} \text{ s.t. } 2b_3-3b_2\geq 0,\, b_2-2b_1\geq 0} x_1^{b_1}x_2^{b_2}x_3^{b_3}$$

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$$= \Omega_{\geqq} \sum_{b_1, b_2, b_3 \geq 0} \lambda_1^{2b_3 - 3b_2} \lambda_2^{b_2 - 2b_1} x_1^{b_1} x_2^{b_2} x_3^{b_3}$$

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$$= \underset{\geqq}{\Omega} \frac{1}{1 - \frac{x_1}{\lambda_2^2}} \frac{1}{1 - \frac{\lambda_2 x_2}{\lambda_1^3}} \frac{1}{1 - \lambda_1^2 x_3}$$

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```
In[1]:= << Omega2.m
```

Omega Package by Axel Riese (in cooperation with George E. Andrews and Peter Paule) - ©RISC, JKU Linz - V 2.47

```
In[2]:= LCrude = OSum[ x1^b1 x2^b2 x3^b3,
          {2 b3 - 3 b2 ≥ 0, b2 - 2 b1 ≥ 0 , b1 ≥ 0}, λ]
```

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         {2 b3 - 3 b2 ≥ 0, b2 - 2 b1 ≥ 0 , b1 ≥ 0}, λ]
```

Out[2]= $$\underset{\lambda_1,\lambda_2}{\underset{\geq}{\Omega}} \frac{1}{\left(1-\frac{x1}{\lambda_2^2}\right)\left(1-\frac{\lambda_2\,x2}{\lambda_1^3}\right)\left(1-\lambda_1^2\,x3\right)}$$

```
In[3]:= L=OR[LCrude]
```

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```
In[3]:= L=OR[LCrude]
```

Out[3]= $\frac{1+x2\, x3^2}{(1-x3)\left(1-x2^2\, x3^3\right)\left(1-x1\, x2^2\, x3^3\right)}$

```
In[4]:= L /.  {x1->q, x2->q, x3->q}
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```
In[4]:= L /.  {x1->q, x2->q, x3->q}
```

Out[4]= $\frac{1+q^3}{(1-q)(1-q^5)(1-q^6)}$

GENERAL THEME: linear Diophantine constraints

- Find $b_1, \dots b_n \in \mathbb{N}$ such that

$$\begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \\ \vdots & \ddots & \vdots \\ c_{m,1} & \cdots & c_{m,n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

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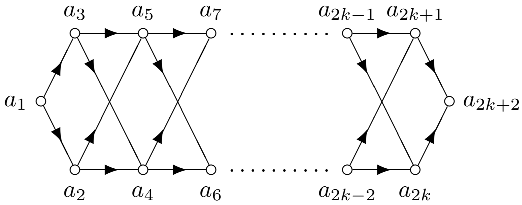
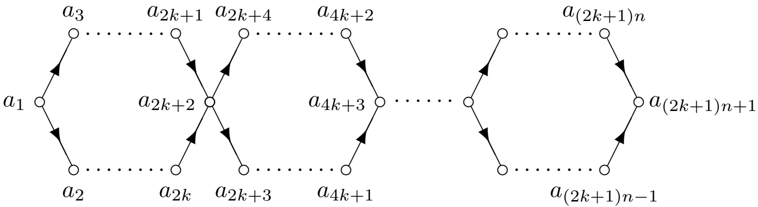
- New algorithm by F. Breuer & Z. Zafeirakopolous [poster: "A Linear Diophantine System Solver", Lind Hall 400, 4 pm]

Polyhedral Omega is a new algorithm for solving linear Diophantine systems (LDS), i.e., for computing a multivariate rational function representation of the set of all non-negative integer solutions to a system of linear equations and inequalities. Polyhedral Omega combines methods from partition analysis with methods from polyhedral geometry. In particular, we combine MacMahon's iterative approach based on the Omega operator and explicit formulas for its evaluation with geometric tools such as Brion decompositions and Barvinok's short rational function representations. In this way, we connect two recent branches of research that have so far remained separate, unified by the concept of symbolic cones which we introduce.

Omega and Mathematical Discovery



A k-elongated partition diamond of length 1



A k-elongated partition diamond of length n

Generating function for k-elongated diamonds of length n:

$$h_{n,k}(q) = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+2})(1+q^{(2k+1)j+4})\cdots(1+q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1}(1-q^j)}$$

Andrews' great idea: delete the source:

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$$h^*_{n,k}(q) = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+1})(1+q^{(2k+1)j+3})\cdots(1+q^{(2k+1)j+2k-1})}{\prod_{j=1}^{(2k+1)n}(1-q^j)}$$

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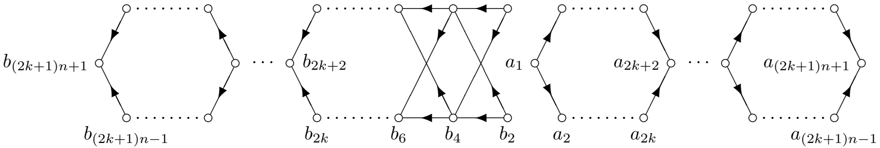


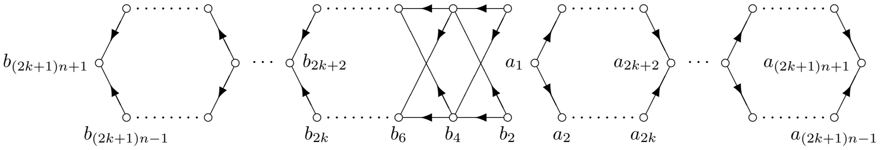
A broken k-diamond of length $2n$

$$\sum_{m=0}^{\infty} \Delta_k(m) q^m := \lim_{n\to\infty} h_{n,k}(q)\, h^*_{n,k}(q)$$

$$= \frac{\prod_{j=1}^{\infty}(1+q^j)}{\prod_{j=1}^{\infty}(1-q^j)^2 \prod_{j=1}^{\infty}(1+q^{(2k+1)j})}$$

$$= \frac{\prod_{j=1}^{\infty}(1+q^j)(1-q^j)}{\prod_{j=1}^{\infty}(1-q^j)^3 \prod_{j=1}^{\infty}(1+q^{(2k+1)j})}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{(2k+1)j})}{(1-q^j)^3(1-q^{(4k+2)j})}$$



A broken k-diamond of length $2n$

Consequently,

$$\sum_{m=0}^{\infty} \Delta_k(m) q^m = \lim_{n\to\infty} h_{n,k}(q)\, h^*_{n,k}(q)$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{(2k+1)j})}{(1-q^j)^3(1-q^{(4k+2)j})}$$

$$= q^{(k+1)/12} \frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}$$

with η the Dedekind eta function:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \quad (q = e^{2\pi i\tau})$$

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NOTE. η^{24} is a modular form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$, because of

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\sqrt{-i(c\tau+d)}\,\eta(\tau)$$

where $a\,d - b\,c = 1$ and $c > 0$.

Recall

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\epsilon(a,b,c,d)\sqrt{-i(c\tau+d)}\eta(\tau).$$

Hence, for $\tau \in \mathbb{H}$ (upper half complex plane):

$$\begin{aligned}\eta(\tau+1)^{24} &= \eta\left(\frac{1\,\tau+1}{0\,\tau+1}\right)^{24}\\ &= \epsilon(1,1,0,1)^{24}\sqrt{-i(0\,\tau+1)}^{24}\eta(\tau)^{24}\\ &= \eta(\tau)^{24}\end{aligned}$$

The Fourier series expansion ("q-series expansion", $q=e^{2\pi i\tau}$) is

$$\eta(\tau)=q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

WHY η-QUOTIENTS?

In[1]:= bd[n_, k_] := $\prod_{j=1}^{n} \frac{(1-q^{2j})(1-q^{(2k+1)j})}{(1-q^j)^3 (1-q^{(4k+2)j})}$

In[6]:= bd1 = Normal[Series[bd[30, 1], {q, 0, 30}]]

$1 + 3q + 8q^2 + 18q^3 + 38q^4 + 75q^5 + 142q^6 + 258q^7 + 455q^8 + 780q^9 + 1308q^{10} + 2148q^{11} + 3467q^{12} + 5505q^{13} + 8618q^{14} + 13\,314q^{15} + 20\,327q^{16} + 30\,693q^{17} + 45\,882q^{18} + 67\,944q^{19} + 99\,745q^{20} + 145\,239q^{21} + 209\,882q^{22} + 301\,128q^{23} + 429\,148q^{24} + 607\,710q^{25} + 855\,414q^{26} + 1\,197\,228q^{27} + 1\,666\,585q^{28} + 2\,308\,014q^{29} + 3\,180\,668q^{30}$

In[7]:= Mod[CoefficientList[bd1, q], 2]

{1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0}

In[8]:= Mod[CoefficientList[bd1, q], 3]

{1, 0, 2, 0, 2, 0, 1, 0, 2, 0, 0, 0, 2, 0, 2, 0, 2, 0, 0, 0, 1, 0, 2, 0, 1, 0, 0, 0, 1, 0, 2}

In[9]:= Mod[CoefficientList[bd1, q], 4]

{1, 3, 0, 2, 2, 3, 2, 2, 3, 0, 0, 0, 3, 1, 2, 2, 3, 1, 2, 0, 1, 3,

Congruences for $\Delta_k(n)$

Theorem [Andrews & P, Partition Analysis XI]. For all $n \in \mathbb{N}$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3}.$$

Proof.

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Proof. Because of $(1-q^j)^3 \equiv 1-q^{3j} \pmod{3}$,

$$\sum_{m=0}^{\infty} \Delta_1(m) q^m = \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{3j})}{(1-q^j)^3(1-q^{6j})}$$

$$\equiv \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{3j})}{(1-q^{3j})(1-q^{6j})} \pmod{3}.$$

Hence the coefficients of odd powers of q have to be zero.

Recall:

Theorem. For all $n \in \mathbb{N}$,

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Algorithmic Proof [Radu 2014]:

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$$\sum_{n=0}^{\infty} \Delta_1(2n+1) q^n = 3 \prod_{j=1}^{\infty} \frac{(1-q^{2j})^2(1-q^{6j})^2}{(1-q^j)^6}$$

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NOTE. Human proof [Hirschhorn & Sellers, 2007]

Some conjectures [Andrews & P, PA XI]: For all $n \in \mathbb{N}$,

$$\Delta_2(10n+2) \equiv 0 \pmod 2$$

and

$$\Delta_2(25n+14) \equiv 0 \pmod 5.$$

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NOTE. First proof of the 10-case [Hirschhorn & Sellers, 2007]

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Algorithmic Proof [Radu 2014]. Human preprocessing: since $(1-q^j)^5 \equiv 1-q^{5j} \pmod{5}$,

$$\Delta_2(n) \equiv d(n) \pmod{5},$$

where

$$\sum_{m=0}^{\infty} \Delta_2(n) q^n = \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{5j})}{(1-q^j)^3(1-q^{10j})}$$

and

$$\sum_{m=0}^{\infty} d(n) q^n := \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^j)^2}{(1-q^{10j})}$$

Radu's program "Ramanujan-Kolberg" delivers:

$$q^{\frac{3}{2}}\frac{\eta(2\tau)\eta(5\tau)^{10}}{\eta(\tau)^6\eta(10\tau)^{20}}\left(\sum_{m=0}^{\infty} d(25n+14)q^n\right)\left(\sum_{m=0}^{\infty} d(25n+24)q^n\right)$$
$$= 25(2t^4+28t^3+155t^2+400t+400)$$

where

$$t=\frac{\eta(\tau)^3\eta(5\tau)}{\eta(2\tau)\eta(10\tau)^3}\in M(\Gamma_0(10)).$$

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NOTE 2. There are numerous other congruences for broken diamonds and generalizations.

Radu's Ramanujan-Kolberg Package

Back to Euler (= limit of Lecture Hall)

Define

$$\sum_{n=0}^{\infty} Q(n) q^n := \prod_{j=1}^{\infty} \frac{1}{1-q^{2j-1}} :$$

Radu's "Ramanujan-Kolberg" package delivers (computing over $E^{\infty}(14)$):

$$\sum_{n=0}^{\infty} Q(7n+3)q^n \cdot \sum_{n=0}^{\infty} Q(7n+4)q^n \cdot \sum_{n=0}^{\infty} Q(7n+6)q^n$$

$$= 8\, q^5 \prod_{j=1}^{\infty} \frac{(1-q^{2j})^5(1-q^{14j})^{16}}{(1-q^j)^{13}(1-q^{7j})^8}(-16\, E_1 + 9\, E_1^2 + 2\, E_1 E_4)$$

NOTE. This implies:

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NOTE. This implies:

$$Q(7n+3) \equiv Q(7n+4) \equiv Q(7n+6) \equiv 0 \pmod{2}.$$

Radu's "Ramanujan-Kolberg" package also delivers:

$$\sum_{n=0}^{\infty} Q(7n)q^n \cdot \sum_{n=0}^{\infty} Q(7n+1)q^n \cdot \sum_{n=0}^{\infty} Q(7n+5)q^n$$
$$= q^6 \prod_{j=1}^{\infty} \frac{(1-q^{2j})^5(1-q^{14j})^{16}}{(1-q^j)^{13}(1-q^{7j})^8}(3\,E_1^3 + 24\,E_1^2 + 64\,E_1)$$

and

$$\sum_{n=0}^{\infty} Q(7n+2)q^n$$
$$= q^3 \prod_{j=1}^{\infty} \frac{(1-q^{14j})^8}{(1-q^j)^3(1-q^{2j})(1-q^{7j})^4}(8\,E_1 + E_4 - 8)$$

- STEP 1. Find generators of the multiplicative monoid $E^{\infty}(14)$:

Solving a problem for nonnegative integers with linear Diophantine constraints, we obtain the generators

$$E_1 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{1} \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^{7} \left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-7} \in E^{\infty}(14),$$

$$E_2 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{8} \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^{4} \left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-8} \in E^{\infty}(14),$$

$$E_3 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-5} \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^{5} \left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-13} \in E^{\infty}(14),$$

and

$$E_4 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^1 \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^3 \left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-7} \in E^\infty(14),$$

$$E_5 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^5 \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^7 \left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-11} \in E^\infty(14),$$

and

$$E_6 = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-2} \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^6 \left(\frac{\eta(14\tau)}{\eta(\tau)}\right)^{-10} \in E^\infty(14).$$

Summary: STEP 1 computes generators $E_1, \ldots, E_6$ of the multiplicative monoid $E^\infty(14)$ consisting of eta quotients which are modular functions with poles only at infinity.

A crucial FINITE representation

- GOAL: We want to represent our object as an element in the infinite dimensional vectorspace

$$\begin{aligned}\langle E^{\infty}(14)\rangle_{\mathbb{Q}} &= \{c_1\, e_1 + \cdots + c_k\, e_k : c_i \in \mathbb{Q}, e_j \in E^{\infty}(14)\}\\ &= \mathbb{Q}[E_1, \ldots, E_6].\end{aligned}$$

- STEP 2. Represent $E^{\infty}(14)$ as a $\mathbb{Q}[E_1]$-module which is freely generated by 1 and E_4; i.e.,

$$\langle E^{\infty}(14)\rangle_{\mathbb{Q}} = \langle 1, E_4\rangle_{\mathbb{Q}[E_1]}\,.$$

NOTE 1. Ramanujan [1919] proved for

$$\sum_{n=0}^{\infty} p(n)q^n := \prod_{j=1}^{\infty} \frac{1}{1-q^j} :$$

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n$$
$$= 7 \prod_{j=1}^{\infty} \frac{(1-q^{7j})^3}{(1-q^j)^4} + 49q \prod_{j=1}^{\infty} \frac{(1-q^{7j})^7}{(1-q^j)^8}.$$

NOTE 2. An alternative formulation in terms of

$$z_5 := q \prod_{j=1}^{\infty} \frac{(1-q^{5j})^6}{(1-q^j)^6} = \left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6$$

and

$$z_7 := q \prod_{j=1}^{\infty} \frac{(1-q^{7j})^4}{(1-q^j)^4} = \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^4 :$$

$$q \prod_{j=1}^{\infty}(1-q^{5j}) \sum_{n=0}^{\infty} p(5n+4)q^n = 5\, z_5$$

and

$$q \prod_{j=1}^{\infty}(1-q^{7j}) \sum_{n=0}^{\infty} p(7n+5)q^n = 7\, z_7 + 49\, q\, z_7^2.$$

NOTE 3.

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}$$

"It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities . . . ; but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting . . ." [G.H. Hardy]

NOTE. The "Ramanujan-Kolberg" package computes in $E^{\infty}(22)$:

$$\sum_{n=0}^{\infty} p(11n+6)q^n = q^{14}\prod_{j=1}^{\infty}\frac{(1-q^{22j})^{22}}{(1-q^j)^{10}(1-q^{2j})^2(1-q^{11j})^{11}}$$

$$\times(1078t^4+13893t^3+31647t^2+11209t-21967$$

$$+z_1(187t^3+5390t^2+594t-9581)$$

$$+z_2(11t^3+2761t^2+5368t-6754)$$

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with

$$t:=\frac{3}{88}w_1+\frac{1}{11}w_2-\frac{1}{8}w_3,\ z_1:=-\frac{5}{88}w_1+\frac{2}{11}w_2-\frac{1}{8}w_3-3,$$
$$z_2:=\frac{1}{44}w_1-\frac{3}{11}w_2+\frac{5}{4}w_3,$$

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and

$$[r_2,r_{11},r_{22}]:=\prod_{\delta|22}\left(\frac{\eta(\delta\tau)}{\eta(\tau)}\right)^{r_\delta}\in E^{\infty}(22).$$

Reference

- Cristian-Silviu Radu: An Algorithmic Approach to Ramanujan-Kolberg Identities. Journal of Symbolic Computation, 2014.