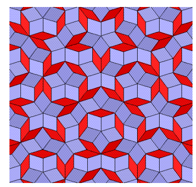
Counting with Wang tiles

Igor Pak, UCLA

Joint work with Scott Garrabrant

IMA (UMN, Minneapolis)

November 14, 2014



HIS, EIS and OEIS

OEIS now has over 250,000 sequences!

> *Our policy has been to include all interesting sequences, no matter how obscure the reference.* [N.J.A. Sloane and S. Plouffe, EIS, 1995]

> [The EIS contains] *the unrelenting cascade of numbers,* [..] *lists Hard, Disallowed and Silly sequences.* [Richard Guy, 1997]

Question 1: What makes an integer sequence ***combinatorial***?

Question 2: What makes a combinatorial sequence ***nice***?

Selected integer sequences (from OEIS)

`A000001`: 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, ... $\leftarrow$ finite groups

`A000029`: 1, 2, 3, 4, 6, 8, 13, 18, 30, 46, 78, 126, 224, 380, 687, ... $\leftarrow$ necklaces

`A000037`: 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, ... $\leftarrow$ non-squares

`A000040`: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, ... $\leftarrow$ primes

`A000041`: 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, ... $\leftarrow p(n)$

`A000042`: 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, ... $\leftarrow n$ in unary

`A000045`: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 232, 375, 610, 987, ... $\leftarrow F_n$

`A000050`: 1, 2, 3, 5, 9, 16, 29, 54, 97, 180, 337, ... $\leftarrow \#k \leq 2^n$ s.t. $k = a^2 + b^2$

`A000052`: 8, 5, 4, 9, 1, 7, 6, 3, 2, 0, 18, 80, 88, 85, 84, ... $\leftarrow$ alphabetical ordering

`A000054`: 4, 14, 23, 34, 42, 50, 59, 72, 81, 86, 96, 103, 110, 116, ... $\leftarrow$ NYC A line

`A000085`: 1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, ... $\leftarrow$ involutions in S_n

`A000088`: 1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, ... $\leftarrow$ graphs

Traditional Answers:

(1) A sequence is *combinatorial* if it counts combinatorial objects.

(2) Combinatorial sequence is *nice* if it is given by a nice formula.

(2′) The nicer the formula the nicer the sequence.

(2″) Nice formulas can be efficiently computed.

What is a formula?

(A) *The most satisfactory form of $f(n)$ is a completely explicit closed formula involving only well-known functions, and free from summation symbols. Only in rare cases will such a formula exist. As formulas for $f(n)$ become more complicated, our willingness to accept them as "determinations" of $f(n)$ decreases.*

We will be concerned almost exclusively with enumerative problems that admit solutions that are more concrete than an algorithm.

Richard Stanley, *Enumerative Combinatorics,* Vol. 1 (1986)

(B) Formula $=$ Algorithm working in time $o(f(n))$.

Herb Wilf, *What is an answer?* (1982)

Cayley's Formula

Let $f(n)$ denote the number of *rooted* labeled trees. Then:

$$(*) \qquad f(n) \;=\; n^{n-1}$$

$$(**) \qquad f(n) \;=\; n \cdot \sum_{T \subset K_n} 1$$

Observe: These two are both are formulas according to Wilf! Indeed,

$$\log n \cdot n^{n-2} \;=\; o\big(n^{n-1}\big)$$

Moral: Time complexity gives a *quantitative*, not a qualitative difference!

Fibonacci Numbers:

$$(\dagger) \qquad F_n \;=\; F_{n-1} \;+\; F_{n-2}$$

$$(\dagger\dagger) \qquad F_n \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}$$

$$(\dagger\dagger\dagger) \qquad F_n \;=\; \frac{1}{\sqrt{5}} \cdot \left(\phi^n + \phi^{-n}\right) \qquad \text{where} \qquad \phi = \frac{\sqrt{5}+1}{2}$$

Observe: "Closed formula" (†††) is not useful for the exact computation. Summations can be very helpful.

Moral: What's a "nice" formula is complicated!

The number of derangements:

Let $D(n)$ denote the number of $\sigma \in S_n$ such that $\sigma(i) \neq i$ for all $1 \leq i \leq n$.

$$(\diamond) \qquad D(n) \, = \, \left[n!/e \right]$$

$$(\diamond\diamond) \qquad D(n) \, = \, \sum_{k=0}^{n} (-1)^k \, \frac{n!}{k!}$$

$$(\diamond \diamond \diamond) \qquad D(n) \, = \, n D(n-1) \, + \, (-1)^n$$

Observation: Formula $(\diamond)$ is neither combinatorial nor useful for the exact computation. Summation formula $(\diamond\diamond)$ explains $(\diamond)$, but the recursive formula $(\diamond \diamond \diamond)$ is most useful for computation.

Note: Formulas $(\diamond\diamond)$ and $(\diamond \diamond \diamond)$ are *non-positive* and thus *non-combinatorial*!

Ménage Problem

From `Wikipedia`:

A_n = number of different ways in which it is possible to seat a set of male-female couples at a dining table so that men and women alternate and nobody sits next to his or her partner.

$$A_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!$$

$$A_n = nA_{n-1} + 2A_{n-2} - (n-4)A_{n-3} - A_{n-4}$$

(cf. Zeilberger's "The Past and Future of Combinatorics" rant on YouTube; you must be 18+)

Our Answers:

(1) A sequence is *combinatorial* if it counts combinatorial objects.

(1′) Objects are *combinatorial* if they can be verified by an algorithm.

(2) Combinatorial sequence is *nice* if the corresponding algorithm is efficient.

(2′) The algorithm *efficient* if it requires $Const$ memory space.

More Precisely:

(3) A sequence $\{a_n\}$ is *combinatorial* and *nice* if there exists a finite set T of Wang tiles, so that $a_n = \#$ tilings of an n-rectangle.

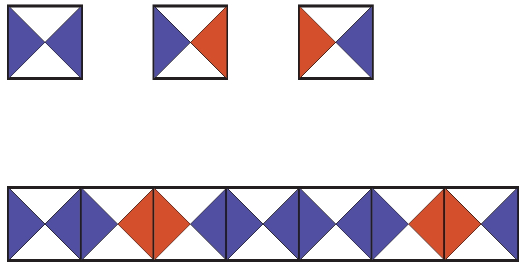
Note: Here *nice* = algorithmically efficient.

Efficient means restrictions on the model of computation.

Motivation: Think of this as a special combinatorial interpretation. When such an interpretation is found, it in itself can lead to better understanding AND new algorithmic solutions.

Counting with Wang tiles

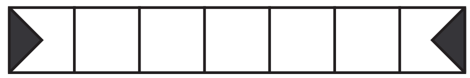
Fibonacci numbers:



12112

Wang tilings of a rectangle

T



n

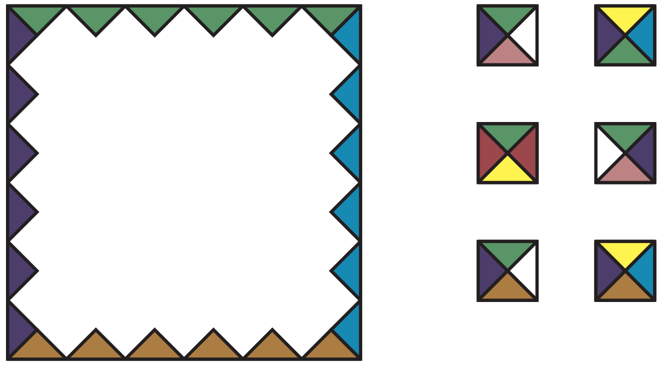
Let $a_n(T) =$ the number of tilings of $[1 \times n]$ with T.

Transfer matrix method:

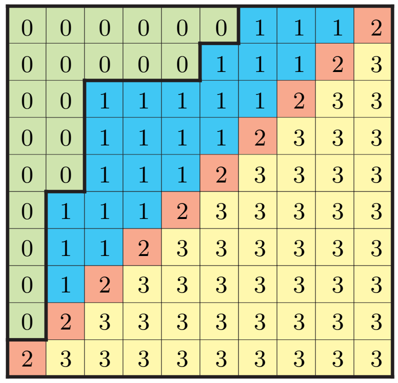
$$\mathcal{A}(t) \, = \, \sum_{n=0}^{\infty} a_n t^n \, = \, \frac{P(t)}{Q(t)}$$

Note: Complete characterization via $\mathbb{N}$-rational functions (the Berstel–Soittola Thm).

Wang tilings of a square



Catalan numbers



An example Catalan number matrix, and the corresponding lattice path.

Note: Can be implemented with 169 Wang tiles.

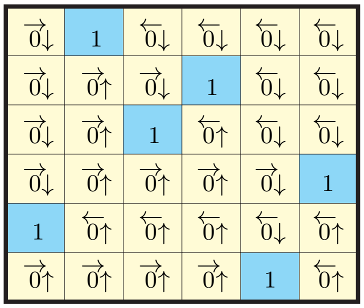
Main Theorem (Garrabrant, P.)

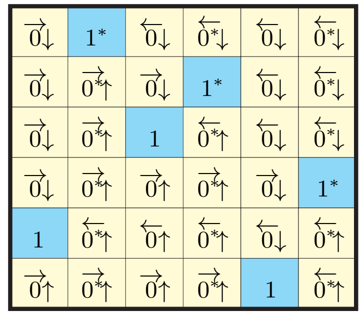
The following functions count Wang Tilings of a square:

(1) The number of integer partitions of n,
(2) The number of set partitions of an n element set (ordered Bell numbers),
(3) The Catalan number C_n,
(4) The Motzkin number M_n.
(5) The number of Gessel walks of length n,
(6) $n!$,
(7) The number of alternating permutations $Alt(n)$ of length n,
(8) The number of permutations of length n whose assents and descents follow a given periodic sequence,
(9) The number $D(n)$ of derangements of length n,
(10) The ménage numbers A_n,
(11) The Menger number $L(k, n)$ of n by k Latin squares for any fixed k,
(12) The number $Pat_k(n)$ of permutations of length n with no increasing subsequence of length k,
(13) The number $B(n)$ of Baxter permutations of length n,
(14) The number $Alt(n)$ of alternating sign matrices of size n,
(15) The number $G(n)$ of labeled connected graphs on n vertices.

Permutations and Alternating Permutations:

Permutation $\sigma \in S_n$ is *alternating* if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$





Note: Can be implemented with 405 and 146410 Wang tiles, respectively.

Baxter Permutations:

Baxter permutations are permutations $\sigma \in S_n$ such that there are no indices $i < j < k$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j+1) > \sigma(i) > \sigma(k) > \sigma(j)$.

Observation: a given permutation matrix is a Baxter permutation is equivalent to ensuring that the two given 2×2 submatrices do not appear.

0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	0	0	0	1	0	0
0	0	0	1	0	0	0
0	0	0	0	0	1	0
0	0	1	0	0	0	0
0	0	0	0	0	0	1

$\overrightarrow{0}\downarrow$	1	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$
1	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\downarrow$
$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\downarrow$	$\overrightarrow{0}\downarrow$	$\overrightarrow{0}\downarrow$	1	$\overleftarrow{0}\downarrow$
$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\downarrow$	1	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\downarrow$
$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\downarrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\downarrow$	$\overrightarrow{0}\uparrow$	1
$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	1	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\downarrow$	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\uparrow$
$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	$\overrightarrow{0}\uparrow$	1	$\overleftarrow{0}\uparrow$	$\overleftarrow{0}\uparrow$

Set Partitions:

1	← 1	← 0↓	← 0↓	← 1	← 0↓
0↑	0↑	1	← 0↓	← 0↑	← 1
0↑	0↑	0↑	1	← 0↑	← 0↑
0↑	0↑	0↑	0↑	0↑	0↑
0↑	0↑	0↑	0↑	0↑	0↑
0↑	0↑	0↑	0↑	0↑	0↑

The set partition $\big\{\{1,2,5\},\{3,6\},\{4\}\big\}$.

Integer Partitions:

1^c	2^a	2^a	3^a	0^a	0^a	0^a	0^a	0^a	0^a
4^b	5^c	5^a	6^a	0^a	0^a	0^a	0^a	0^a	0^a
4^b	5^b	5^c	6^a	0^a	0^a	0^a	0^a	0^a	0^a
7^b_d	8^b	8^b	9^c	0^a	0^a	0^a	0^a	0^a	0^a
0^b_e	0^b_d	0^b	0^b	1^c	3^a	0^a	0^a	0^a	0^a
0^b	0^b_e	0^b_d	0^b	7^b_d	9^c	0^a	0^a	0^a	0^a
0^b	0^b	0^b_e	0^b_d	0^b_e	0^b_d	1^c	3^a	0^a	0^a
0^b	0^b	0^b	0^b_e	0^b_d	0^b_e	7^b_d	9^c	0^a	0^a
0^b	0^b	0^b	0^b	0^b_e	0^b_d	0^b_e	0^b_d	10^c_d	0^a
0^b	0^b	0^b	0^b	0^b	0^b_e	0^b_d	0^b_e	0^b_{de}	10^c_d

The matrix corresponding to the partition 42211.

Number of connected graphs $g(n)$ on $n+1$ vertices

Note the asymptotics: $g(n) \sim 2^{n(n+1)/2}$ (so, it barely fits).

Lemma:

$$g(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} (2^k - 1)\, g(k-1)\, g(n-k).$$

There is a way to realize this recurrence relation with Wang tiles.

This is used to prove part (15). Our construction requires over 10^7 tiles.

Thank you!