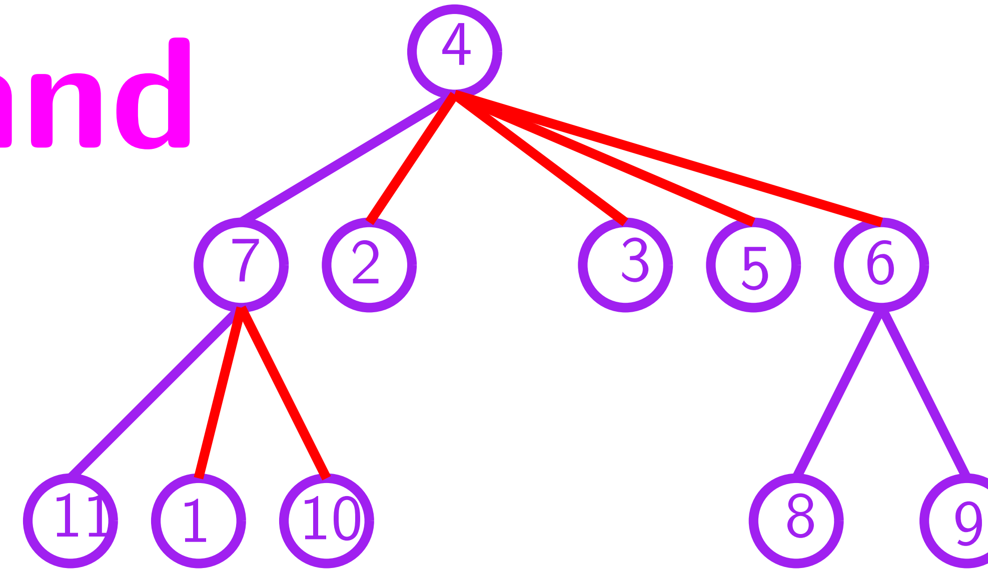
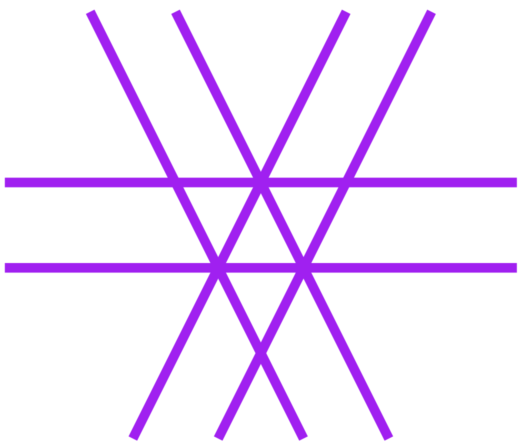
Hyperplane arrangements and trees



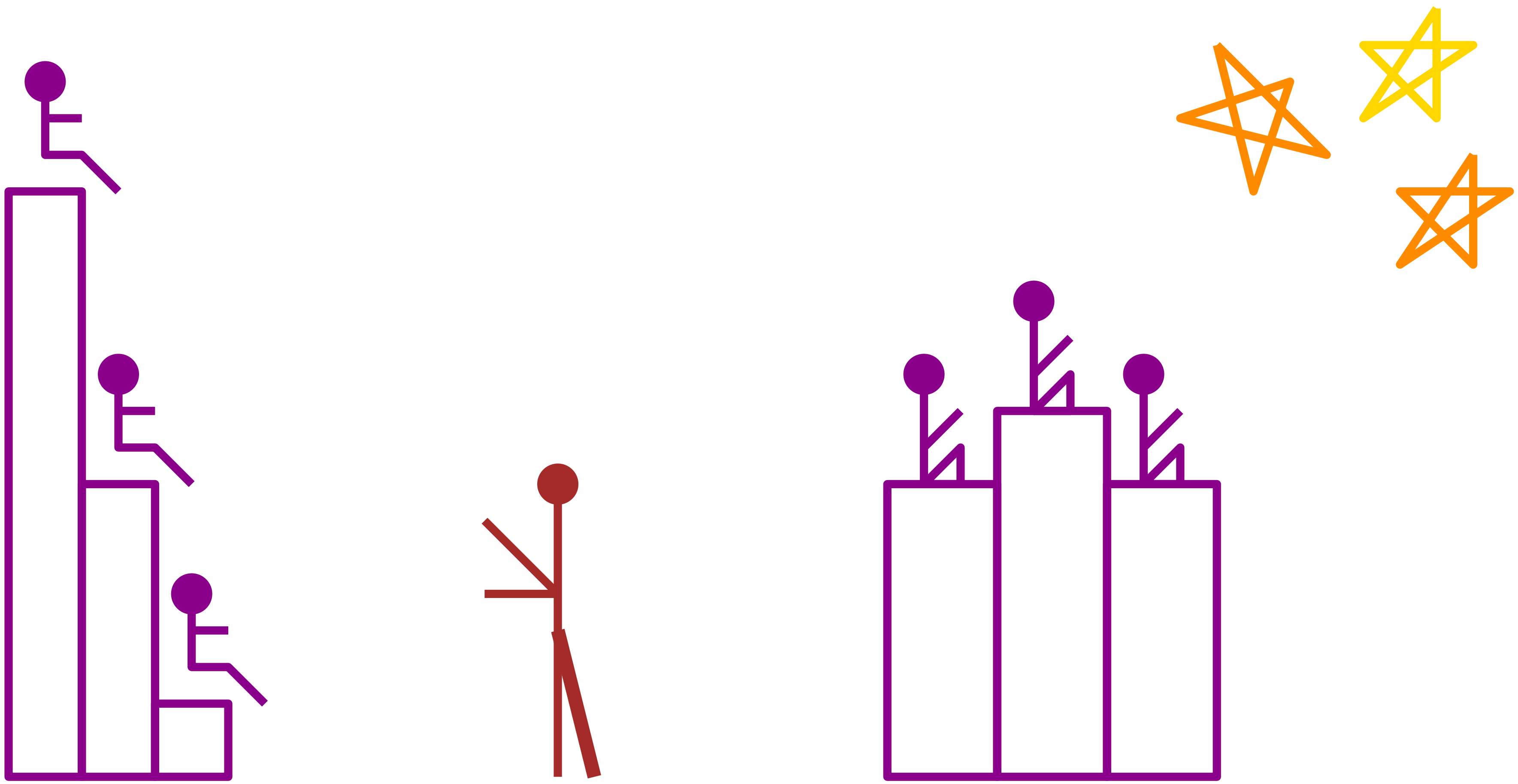


Sylvie Corteel (CNRS Paris 7)

D. Forge (Paris Sud) and A. Micheli (Paris 7)

IMA, Enumerative and Geometric Combinatorics, November 2014

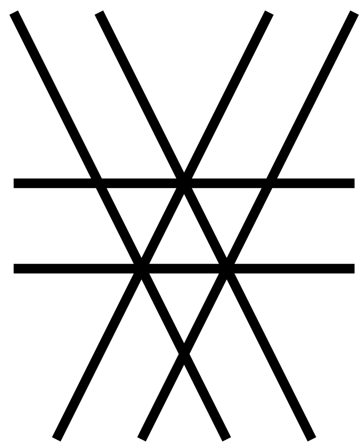
Happy birthday!



Introduction

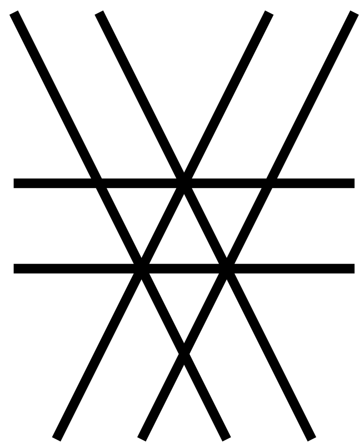
Hyperplane arrangements

$\mathcal{A}$: finite set of affine hyperplanes in $\mathbb{R}^n$



Hyperplane arrangements

$\mathcal{A}$: finite set of affine hyperplanes in $\mathbb{R}^n$



$x_i - x_j = 0$ or 1

$1 \le i < j \le 3$

Hyperplane arrangements

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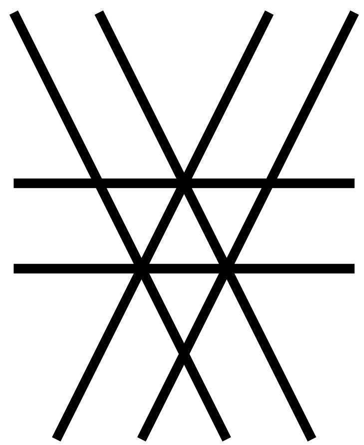

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Regions: number of connected components obtained from $\mathbb{R}^n$ by removing $\mathcal{A}$

Hyperplane arrangements

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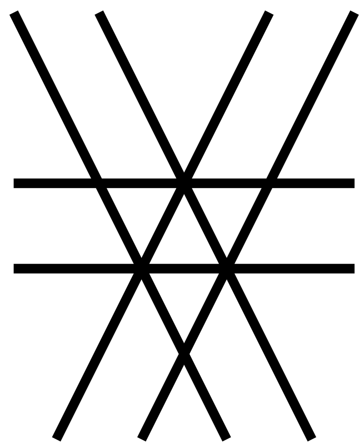

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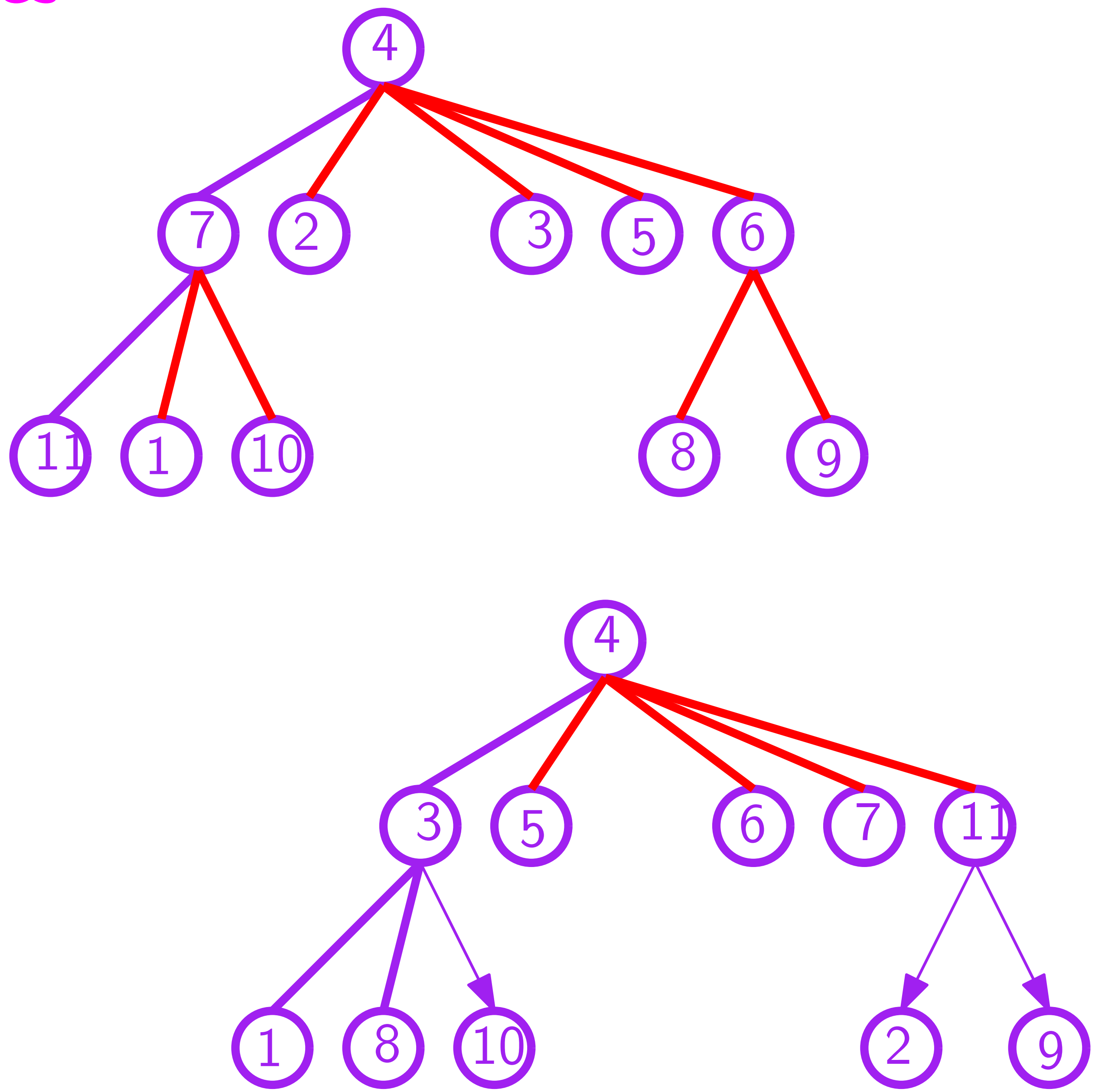
16 regions

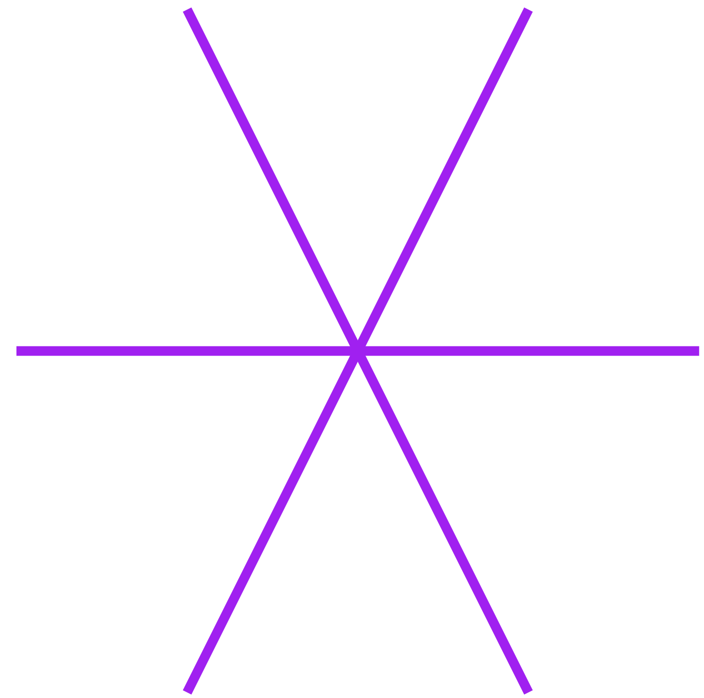
Labelled trees



Example of arrangements

Braid arrangement: $x_i - x_j = 0$

6 regions



Shi arrangement: $x_i - x_j = 0, 1$

Catalan arrangement: $x_i - x_j = -1, 0, 1$

Linial arrangement: $x_i - x_j = 1$

Example of arrangements

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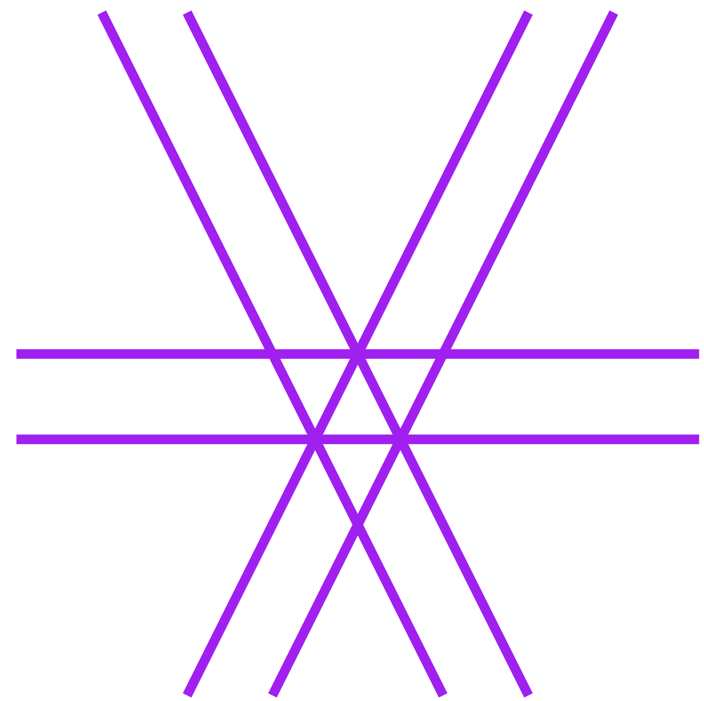


Example of arrangements

Braid arrangement: $x_i - x_j = 0$

6 regions

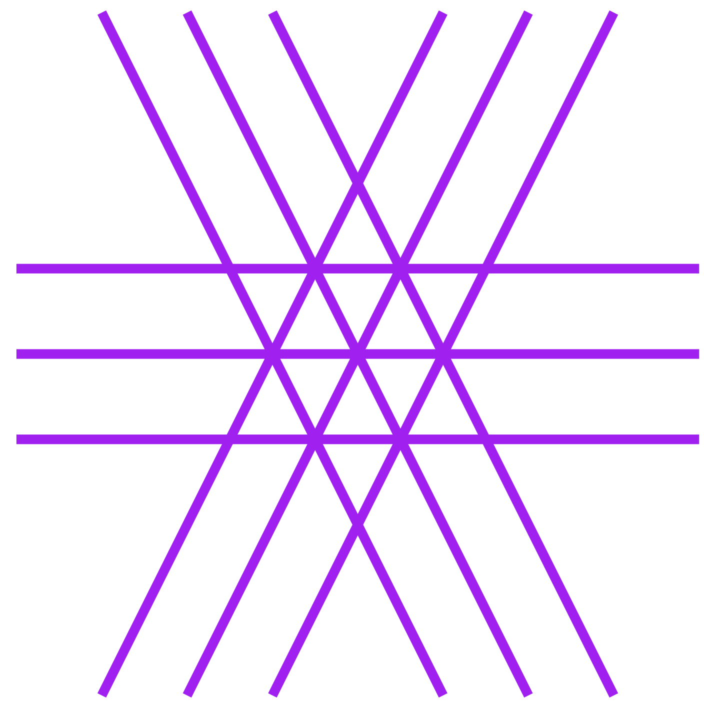
Shi arrangement: $x_i - x_j = 0, 1$

16 regions

Catalan arrangement: $x_i - x_j = -1, 0, 1$

30 regions

Linial arrangement: $x_i - x_j = 1$



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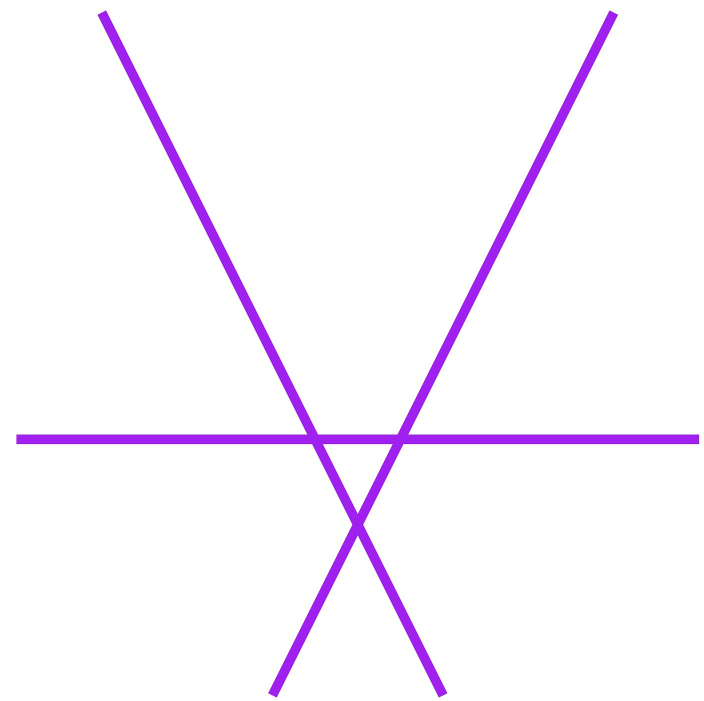
30 regions

Linial arrangement: $x_i - x_j = 1$

7 regions



Example of arrangements

Braid arrangement: $x_i - x_j = 0$ $n!$ regions

Shi arrangement: $x_i - x_j = 0, 1$ $(n+1)^{n-1}$ regions

Catalan arrangement: $x_i - x_j = -1, 0, 1$ $n! C_n$ regions

Linial arrangement: $x_i - x_j = 1$ $2^{-n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}$ regions

Lots of names, 90s Postnikov, Stanley, Athanasiadis, Linusson...

Example of arrangements

Braid arrangement: $x_i - x_j = 0$

Shi arrangement: $x_i - x_j = 0, 1$

Catalan arrangement: $x_i - x_j = -1, 0, 1$

Linial arrangement: $x_i - x_j = 1$

[Gessel, Oberwolfach 14] $\frac{(1+u_1B)(1+v_2B)}{(1+v_1B)(1+u_2B)} = \exp(x(u_1 - u_2 - v_1 + v_2 + u_1v_2 - u_2v_1)B)$

Example of arrangements

Braid arrangement: $x_i - x_j = 0$ $u_1 = v_1 = 1,\ u_2 = v_2 = 0$

Shi arrangement: $x_i - x_j = 0, 1$ $u_1 = v_1 = u_2 = 1,\ v_2 = 0$

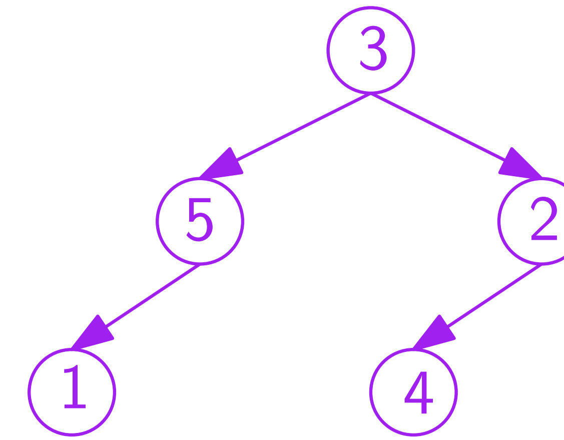
Catalan arrangement: $x_i - x_j = -1, 0, 1$ $u_1 = v_1 = u_2 = v_2 = 1$

Linial arrangement: $x_i - x_j = 1$ $u_2 = v_1 = 1,\ u_1 = v_2 = 0$

[Gessel, Oberwolfach 14] $\frac{(1+u_1B)(1+v_2B)}{(1+v_1B)(1+u_2B)} = \exp(x(u_1 - u_2 - v_1 + v_2 + u_1v_2 - u_2v_1)B)$

Arrangements and trees [Gessel]

labelled binary trees



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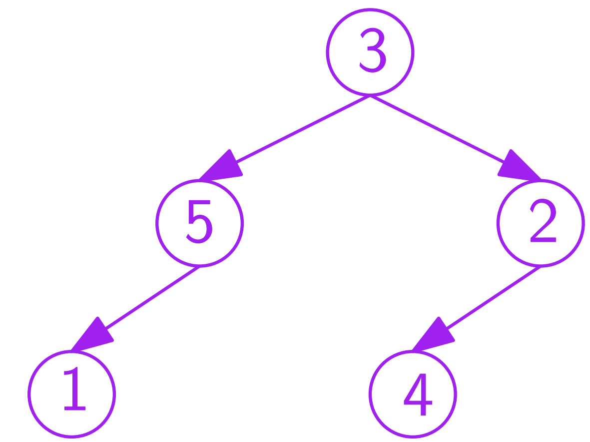

Braid arrangement: $x_i - x_j = 0$

No left descent, no right descent

Shi arrangement: $x_i - x_j = 0, 1$

No right descent

Catalan arrangement: $x_i - x_j = -1, 0, 1$

All possible

Linial arrangement: $x_i - x_j = 1$

No left ascent, no right descent

Arrangements and combinatorial objects

Braid arrangement: $x_i - x_j = 0$

Permutations

Shi arrangement: $x_i - x_j = 0, 1$

Parking functions [Athanasiadis and Linusson 99], Cayley forests

Catalan arrangement: $x_i - x_j = -1, 0, 1$

Sequences of Cayley trees

Linial arrangement: $x_i - x_j = 1$

Alternating trees and local binary search trees [Postnikov 97]

Arrangements and combinatorial objects

Braid arrangement: $x_i - x_j = 0$

13524 $\leftrightarrow x_1 > x_3 > x_5 > x_2 > x_4$

Permutations

Shi arrangement: $x_i - x_j = 0, 1$

13524 $\leftrightarrow x_1 - x_2 > 1,\ x_3 - x_4 > 1$

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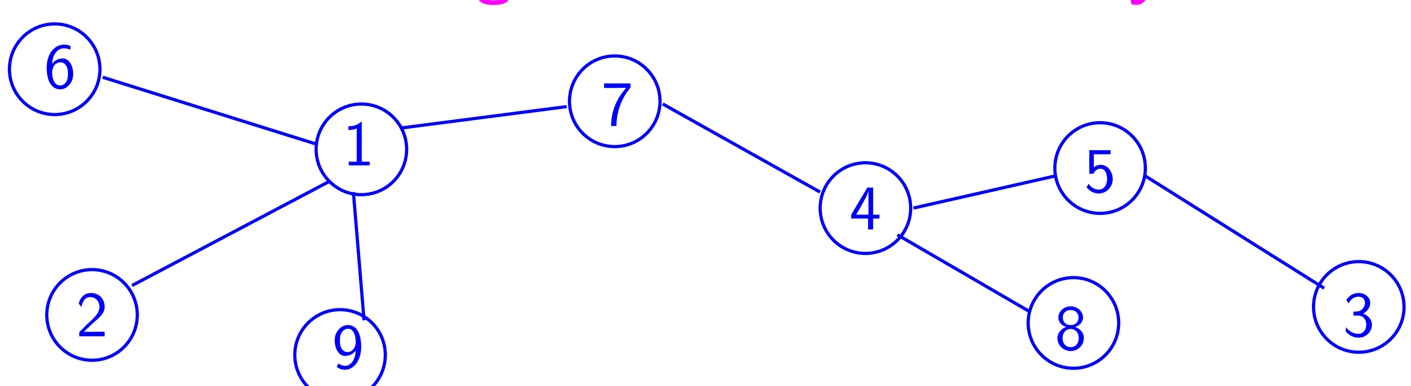

Deformation of Coxeter arrangements

[Postnikov and Stanley, 00]

$$x_i - x_j = g$$

$g \in [a, b]$ and 0 or 1 $\in [a, b]$

$f_n^{ab} = f_n$ number of regions

$$f(x) = \sum_n f_n \frac{x^n}{n!}$$

Theorem

- If $a + b = 0$ then $f = 1 + xf^{b+1}$
- Otherwise $f = \exp\left(xf^{1-a}\left(\frac{1-f^{a+b}}{1-f}\right)\right)$

Deformation of Coxeter arrangements

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Tools: NBC theorem and generating functions

No tree interpretation

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Theorem

- If $a + b = 0$ then $f = 1 + xf^{b+1}$
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Our goal

Combinatorial interpretation!

Strategy:

- Gain graphs [Zavlavsky]
- No broken circuit (NBC)-trees
- Bijections with colored (non)-increasing trees
- Generating functions [Gessel et al]
- Bijections 2-Increasing $\leftrightarrow$ Rooted Cayley trees

Main result

Theorem [CFM 14]. There is a bijection between the regions of the arrangements $\mathcal{A}_n^{ab}$ and

- If $a+b \leq 1$ the $(1-a-b)$-increasing and b- free forests with n vertices
- If $a+b > 0$ the $(a+b-1)$-non-increasing and $(1-a)$-free forests with n vertices

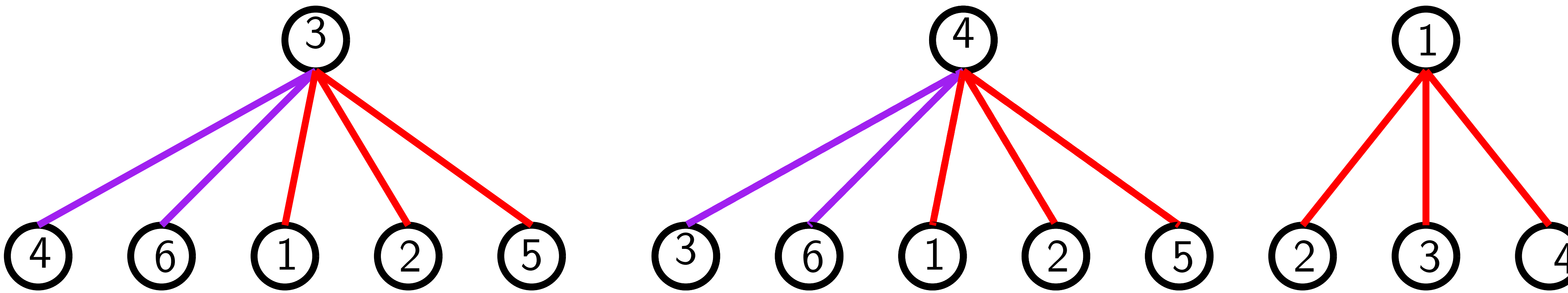
Generalization. Poincaré polynomial

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_{\text{forests}} q^{\#\text{edges(forests)}}$$

Labelled trees

k-increasing/non-decreasing and j-free trees

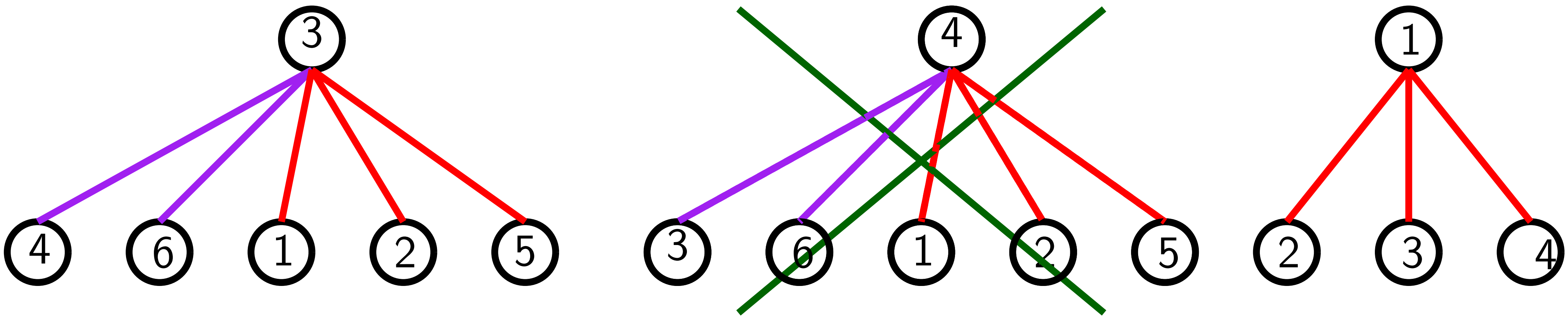
k-increasing



ALL edges of the smallest appearing color MUST be increasing

k-increasing/non-decreasing and j-free trees

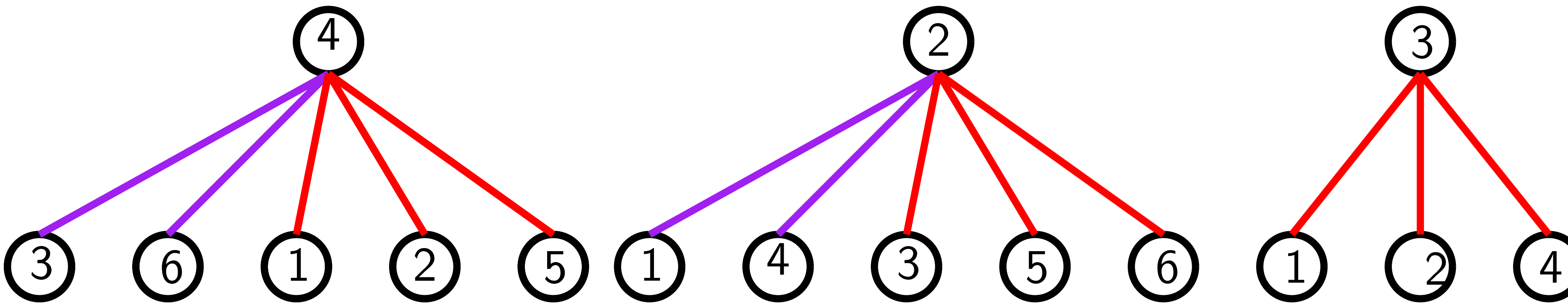
k-increasing



ALL edges of the smallest appearing color MUST be increasing

k-increasing/non-decreasing and j-free trees

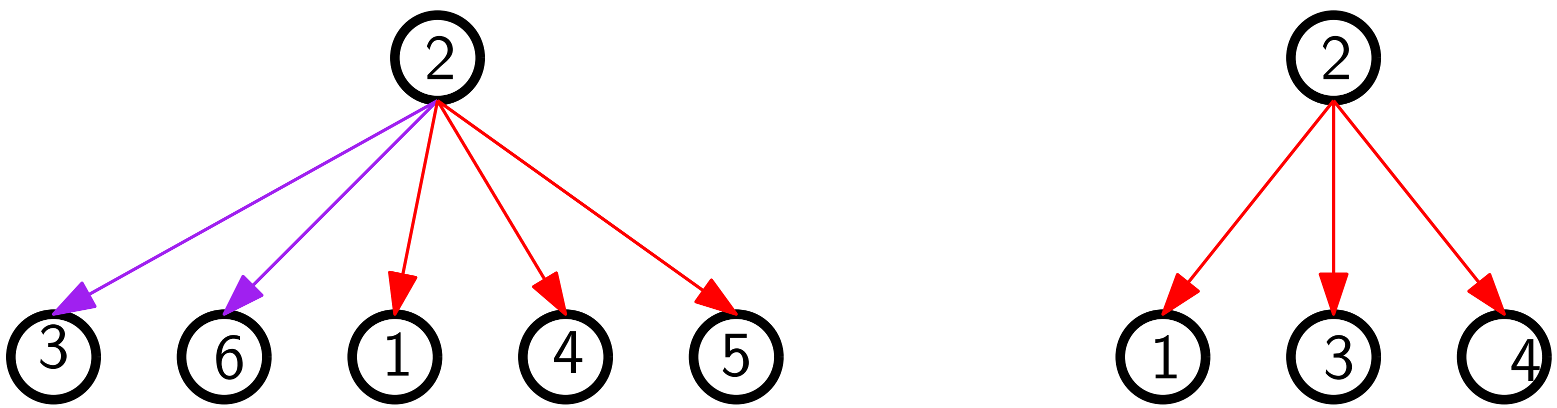
k-non-increasing



AT LEAST one edge of the smallest appearing color is decreasing

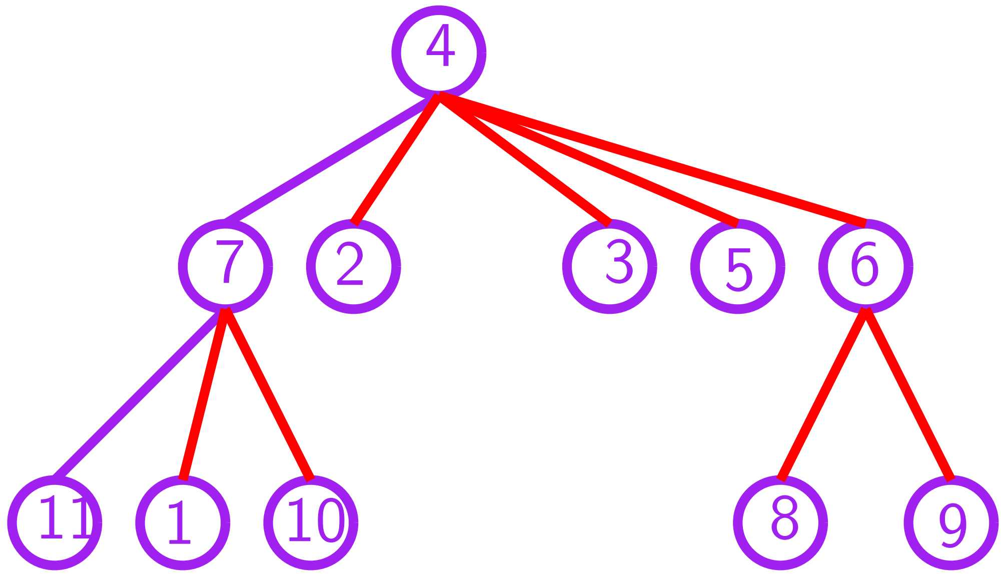
k-increasing/non-decreasing and j-free trees

2-free

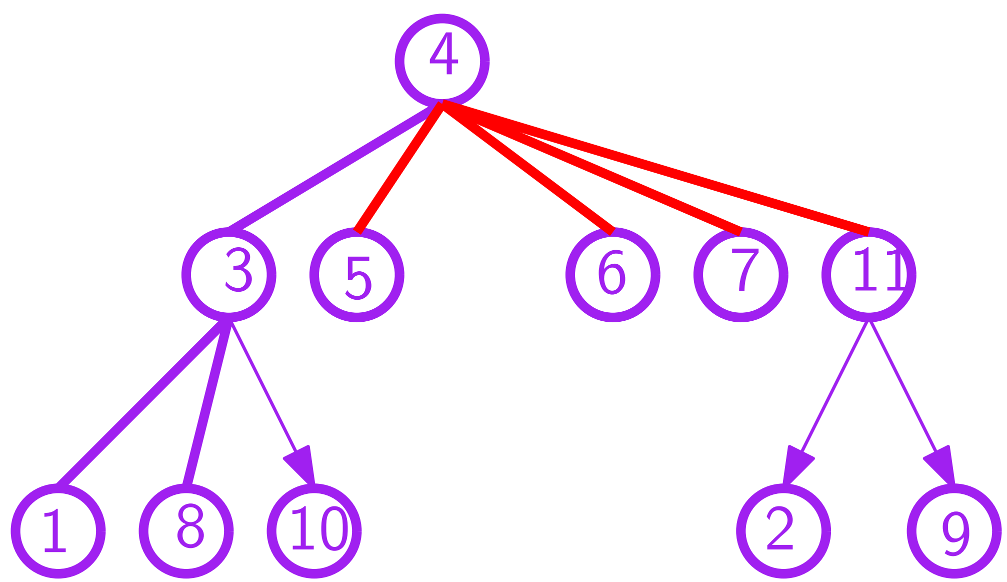


No conditions

free color color 1 color 2



2-increasing



2-non-increasing

1-free

Generating functions

Enumeration of k-increasing trees

Increasing forests $F_{n,k}$: n vertices, k internal vertices, $n-k$ leaves

$$F(x,y,t) = \sum_{n,k} F_{n,k} \frac{x^n}{n!} y^k t^{n-k}$$

$$F(x,y,t) = \frac{y-t}{y-t\exp(x(y-t))}$$

Increasing trees $A(x,y,t) = \ln F(x,y,t)$

k-increasing trees

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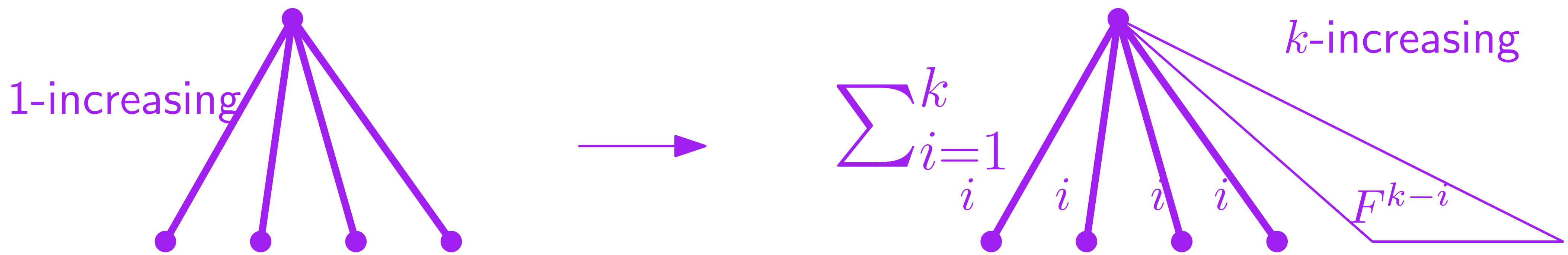

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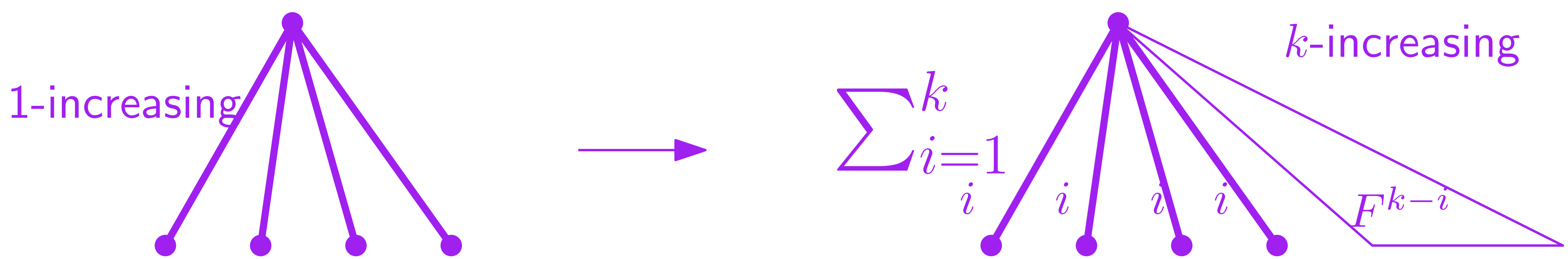
k-increasing trees



Lemma $\qquad F_k(x) = F(x, 1 + F_k + F_k^2 + \ldots + F_k^{k-1}, 1)$

Enumeration of k-increasing trees

Increasing forests $F_{n,k}$: n vertices, k internal vertices, $n-k$ leaves

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Increasing trees $A(x,y,t) = \ln F(x,y,t)$

k-increasing trees $A_k(x) = \ln F_k(x)$

Proposition [CFM, 14] For $k \geq 2$, $F_k^{k-1} = \exp\left(xF_k \frac{1-F_k^{k-1}}{1-F_k}\right)$

$$(k-1)A_k = x\exp(A_k)(1+\exp(A_k)+\ldots+\exp((k-2)A_k))$$

Enumeration of k-increasing trees

Increasing forests $F_{n,k}$: n vertices, k internal vertices, $n-k$ leaves

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$$(k-1)A_k = x\exp(A_k)(1+\exp(A_k)+\ldots+\exp((k-2)A_k))$$

$k = 2 : A_2 = x\exp(A_2)$

Rooted Cayley trees!

Enumeration of k-non-increasing trees

Increasing forests

$$F(x, y, t) = \frac{y - t}{y - t\exp(x(y - t))}$$

Enumeration of k-non-increasing trees

Increasing forests

$$F(x,y,t) = \frac{y-t}{y-t\exp(x(y-t))}$$

k-non-increasing trees $\quad \tilde{A}_k(x) = \ln \tilde{F}_k(x)$

Lemma $\quad \tilde{F}_k(x) = F(x, -1-\tilde{F}_k - \ldots - \tilde{F}_k^{k-1}, \tilde{F}_k^k)$

[Marked forests, Gessel 96]

Proposition [CFM, 14] For $k \geq 2$, $\tilde{F}_k^{k+1} = \exp\left(x\frac{1-\tilde{F}_k^{k+1}}{1-\tilde{F}_k}\right)$

$$(k+1)\tilde{A}_k = x(1+\exp(A_k)+\ldots+\exp(kA_k))$$

Link with the regions of the arrangements

Using [Postnikov and Stanley, 00], we get again

Theorem [CFM 14]. There is a bijection between the regions of the arrangements $\mathcal{A}_n^{ab}$ and

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$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_{\mathrm{forests}} q^{\#\mathrm{edges(forests)}}$$

Gain graphs and NBC-trees

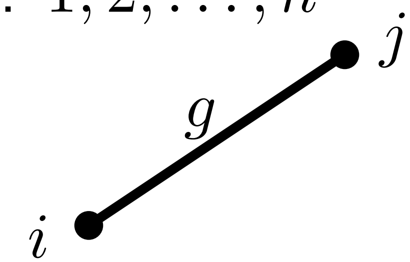
Gain graphs

[Zavlavsky]

Vertices: $1, 2, \ldots, n$

$$x_i - x_j = g$$

$$i < j$$



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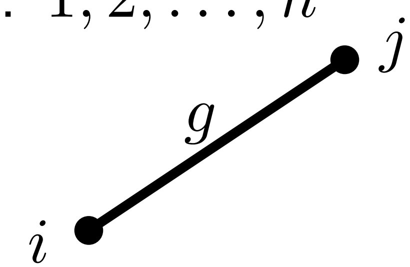

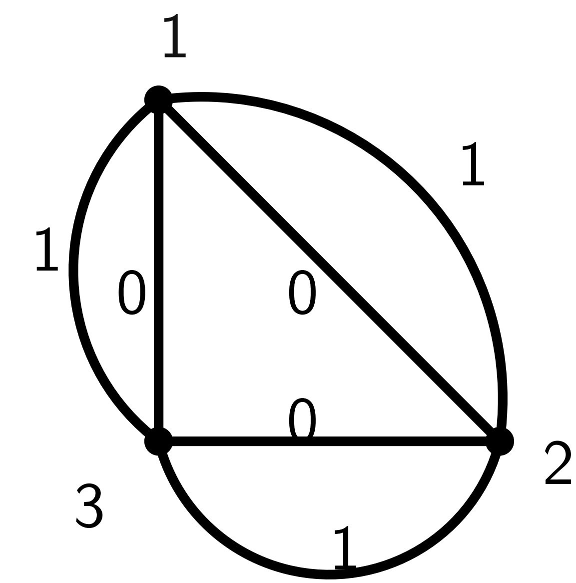

Gain graphs

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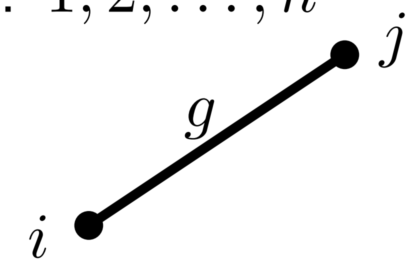
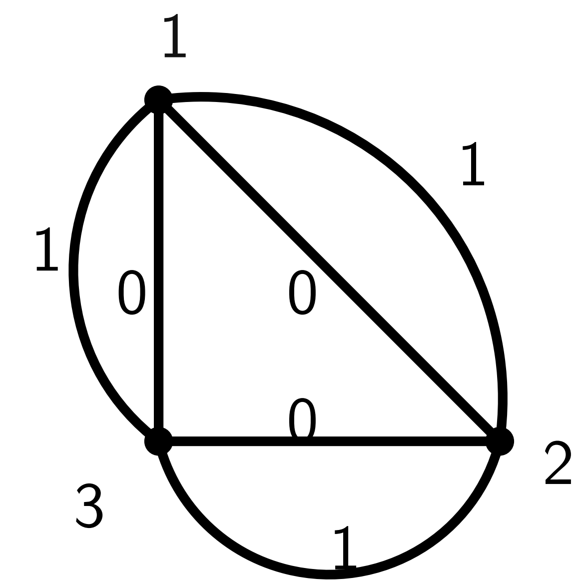
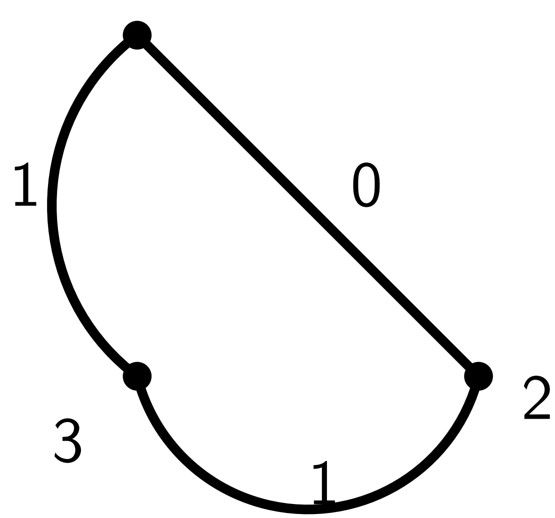
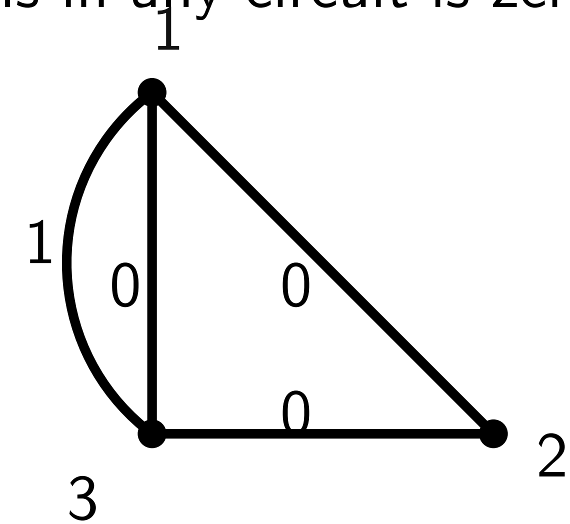


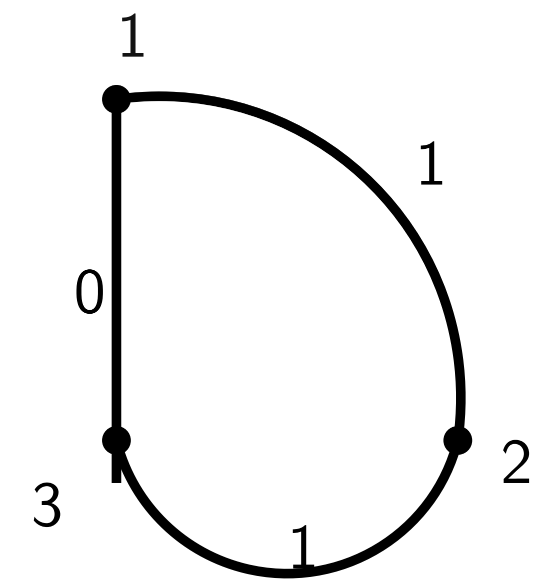


Balanced gain graph

sum of the gains in any circuit is zero







Gain graphs

[Zavlavsky]

Vertices: $1, 2, \ldots, n$

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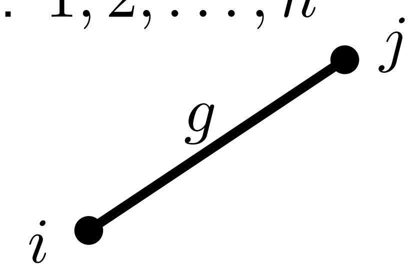

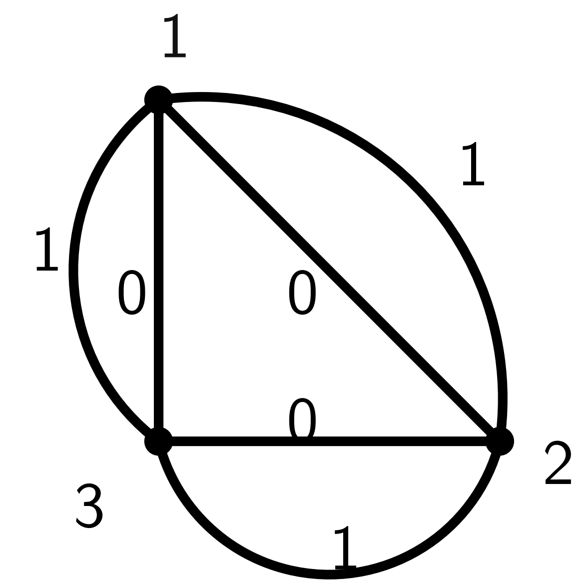

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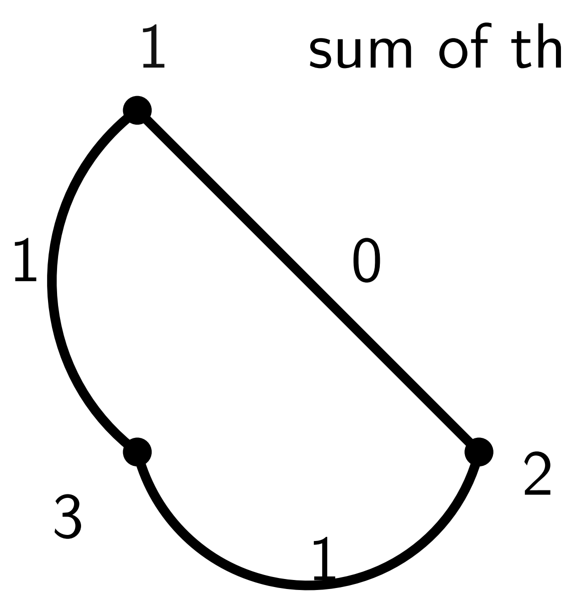
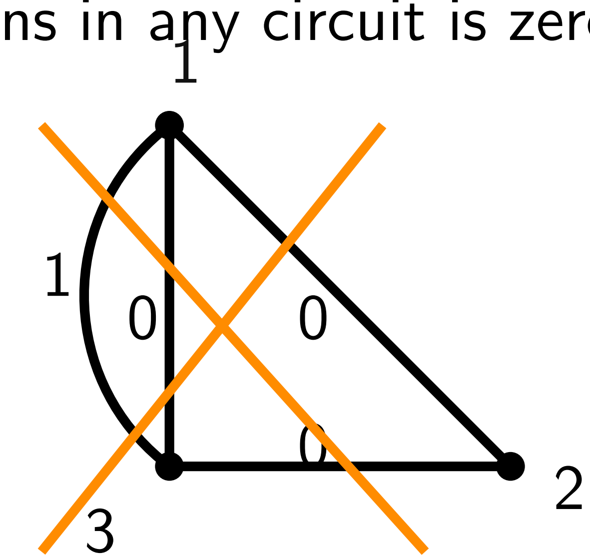
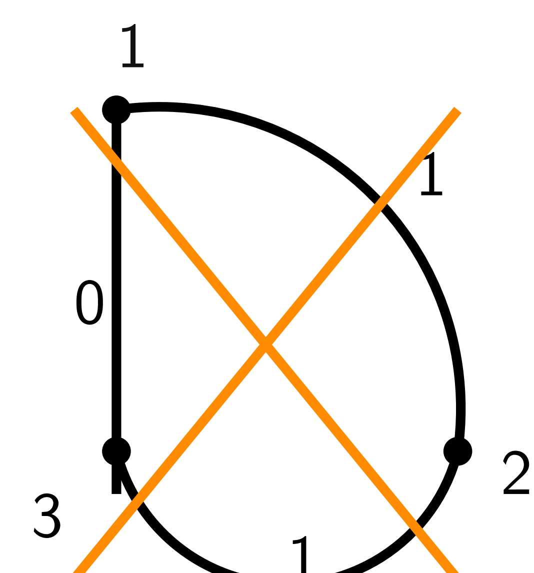


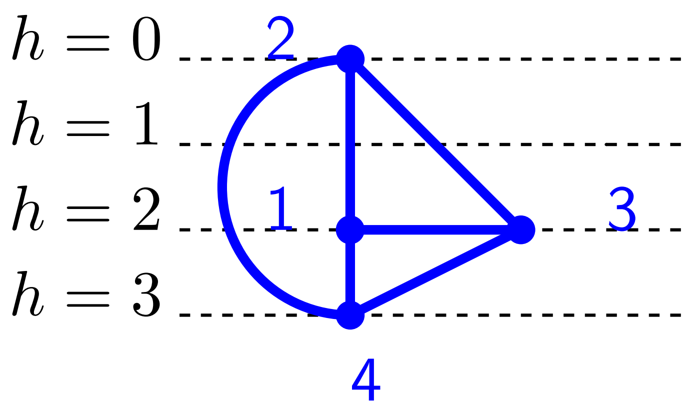




Height function

Height function ⇔ Balanced gain graph

Height function



$$h(2)=0,\ h(1)=h(3)=2,\ h(4)=1$$

$$x_i - x_j = g \quad \Leftrightarrow \quad h(i)-h(j)=g$$

$$g \in [-3, 2]$$

No broken circuit

Order the edges of the gain graph

Broken circuit: balanced circuit without its minimal edge

Lemma. The NBC are forests

Theorem [Whitney] There exists a bijection between

- the NBC-forests with n vertices and
- the regions of $\mathcal{A}_n$

No broken circuit

Order the edges of the gain graph

Broken circuit: balanced circuit without its minimal edge

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- the regions of $\mathcal{A}_n$

Our order: O_h Choose a height function

$i <_{O_h} j$ if $h(i) < h(j)$ or $h(i) = h(j)$ and $i < j$

No broken circuit

Order the edges of the gain graph

Broken circuit: balanced circuit without its minimal edge

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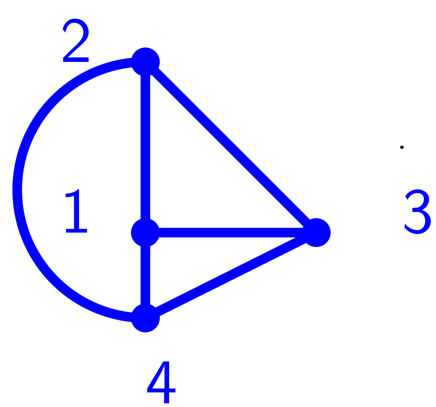

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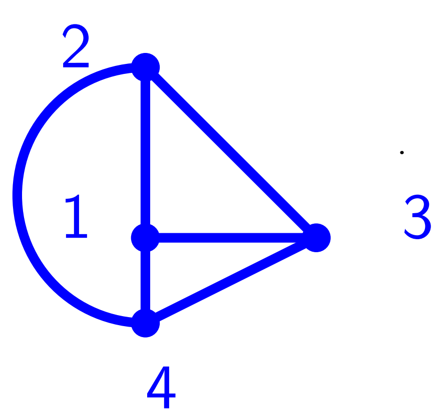

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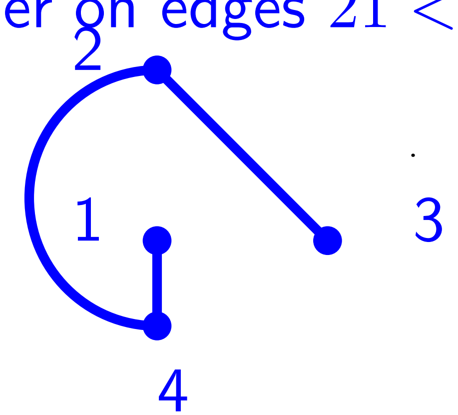

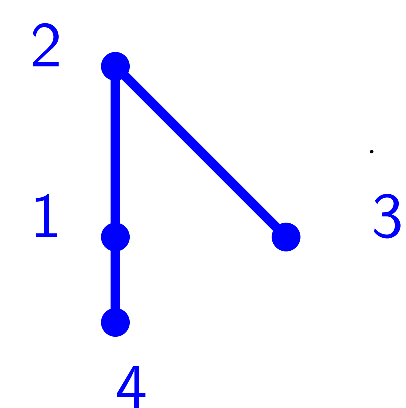

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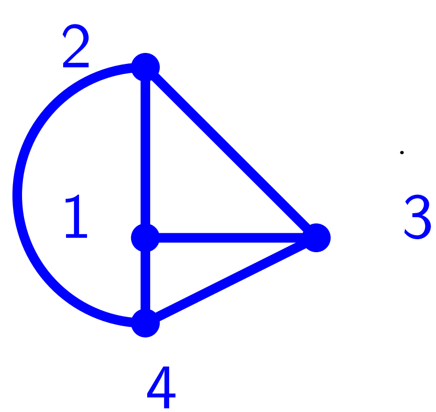

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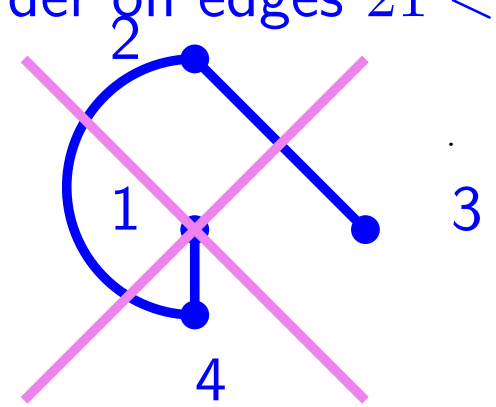
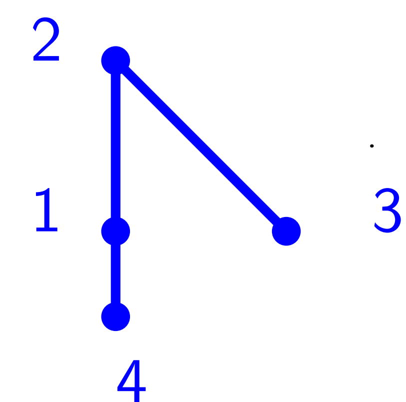
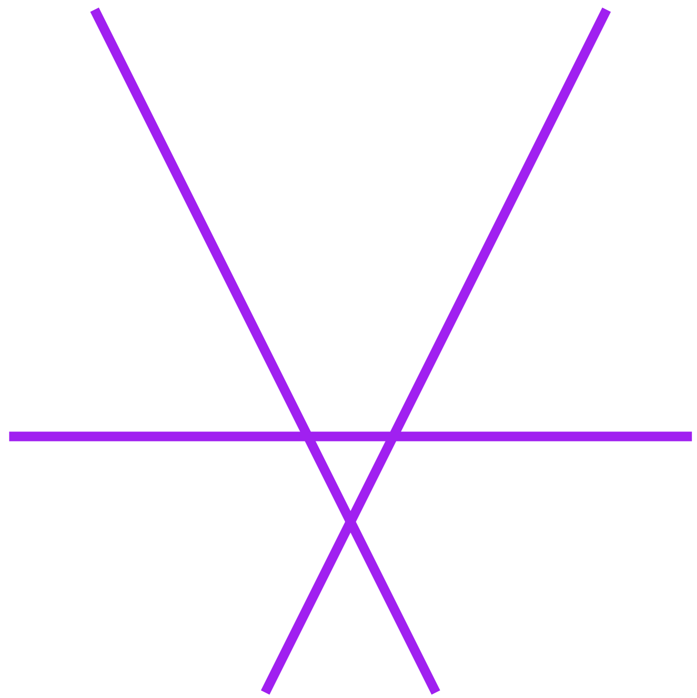
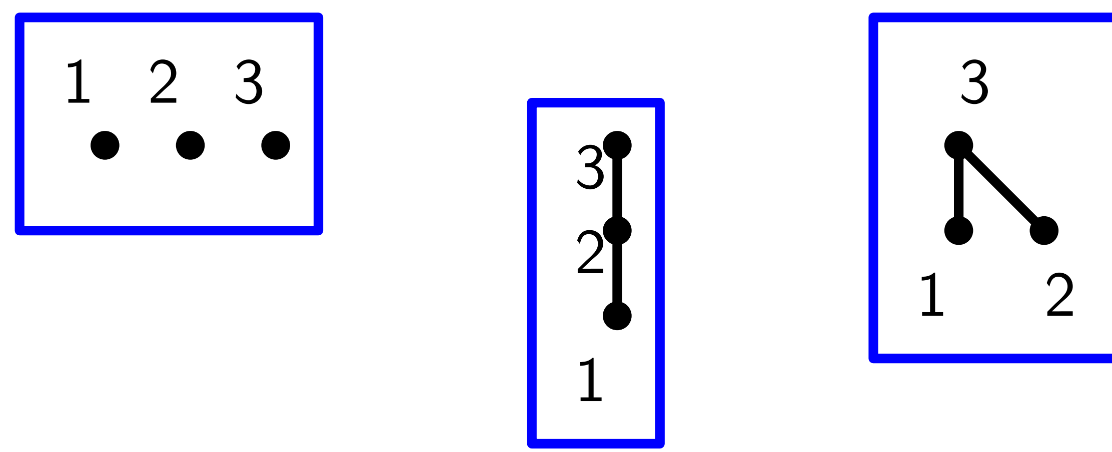


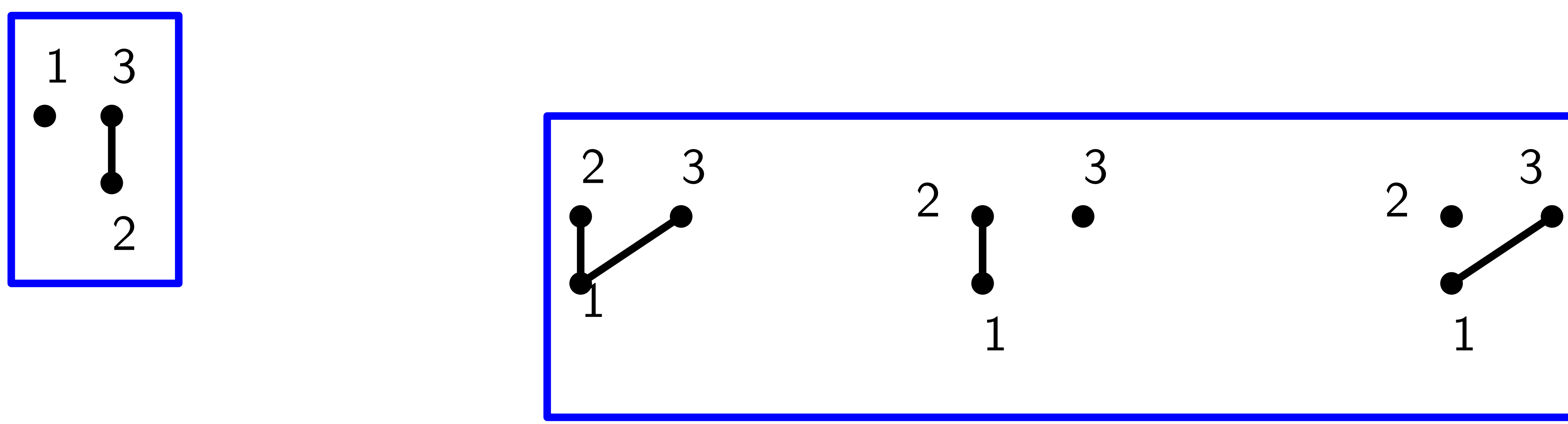


Example: linial

$$x_i - x_j = 1$$



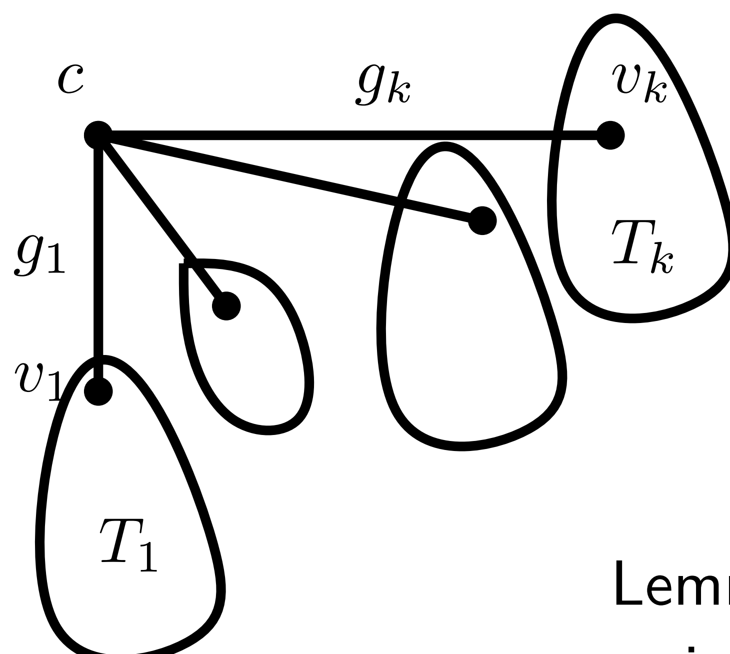




Bijection
NBC
trees
$\leftrightarrow$
Colored
trees

Bijection NBC-trees and increasing/non-increasing trees

Corner: smallest for O_h



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v_i is the smallest vertex such that (c, v_i) exists

Bijection NBC-trees and increasing/non-increasing trees

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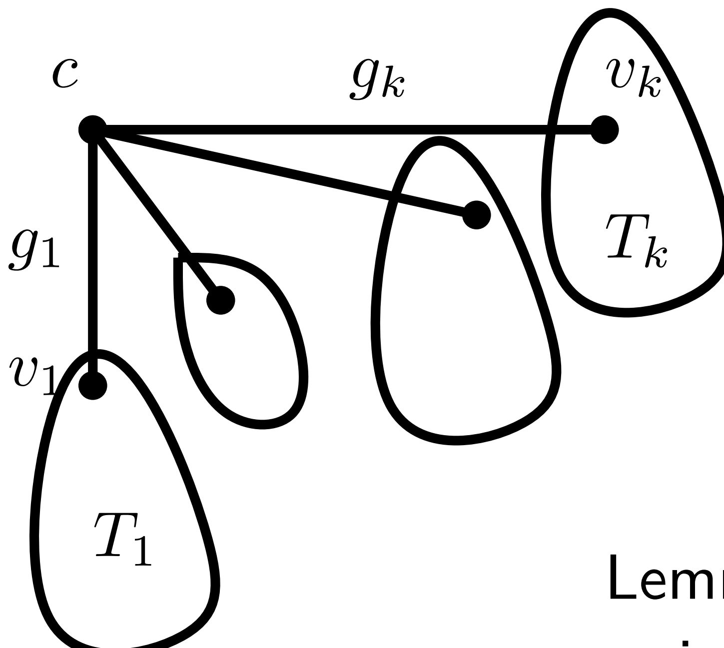

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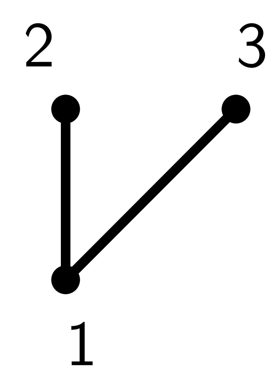

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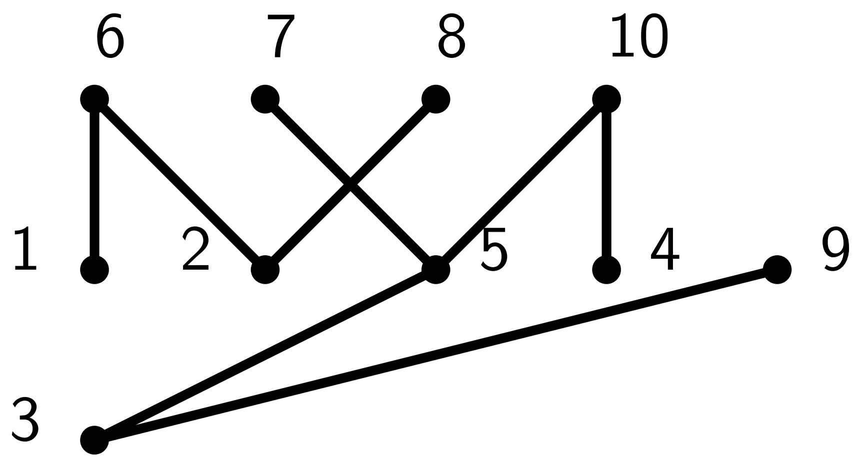

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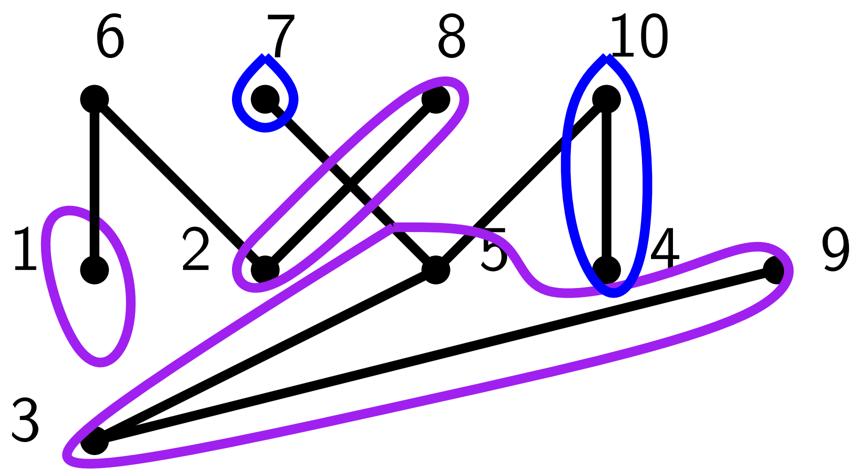

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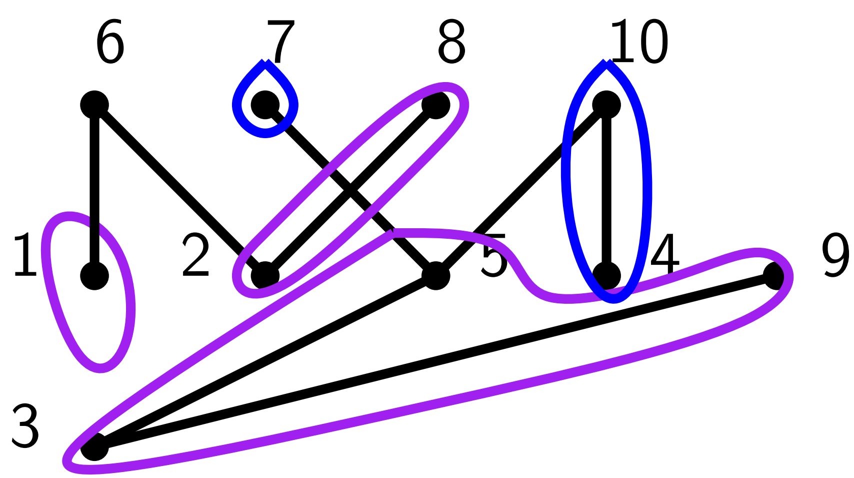
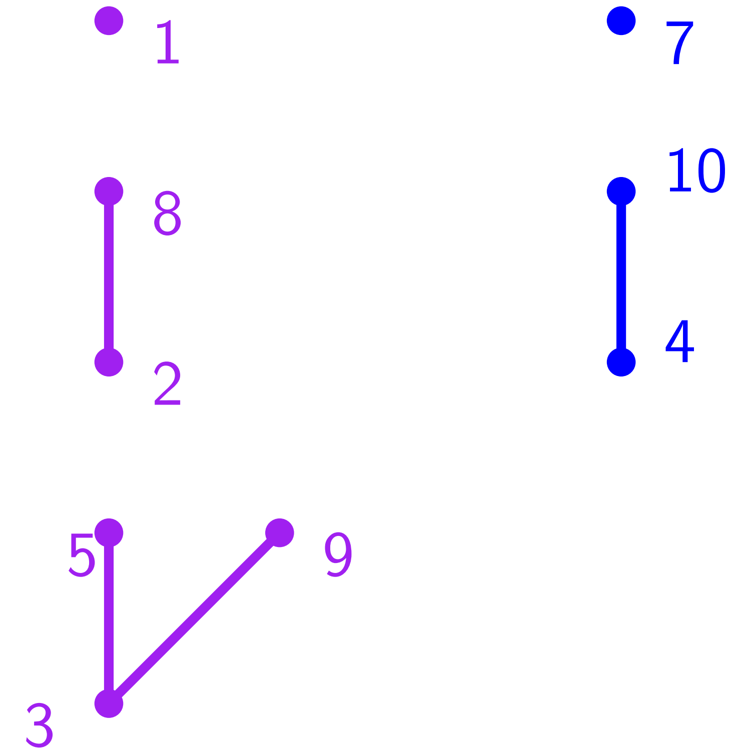


Apply the bijection on each tree and attach the root



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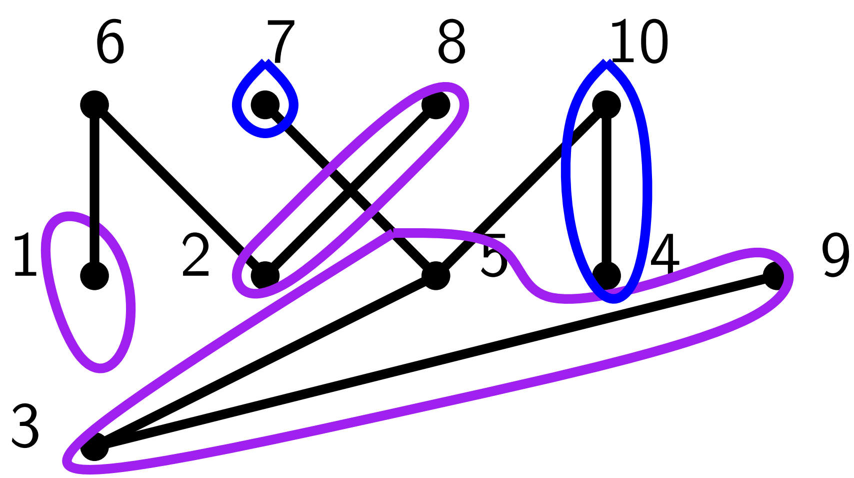

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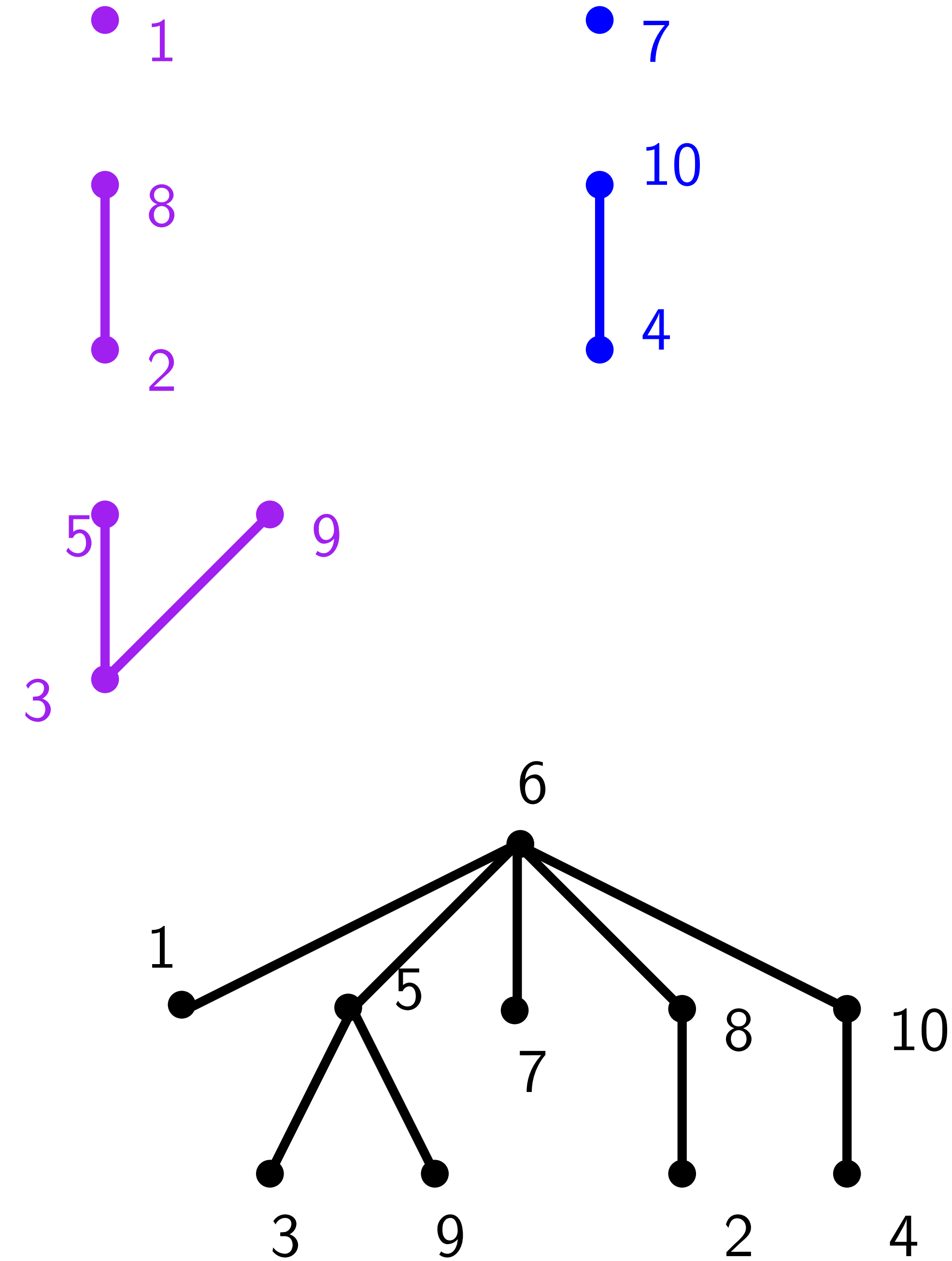

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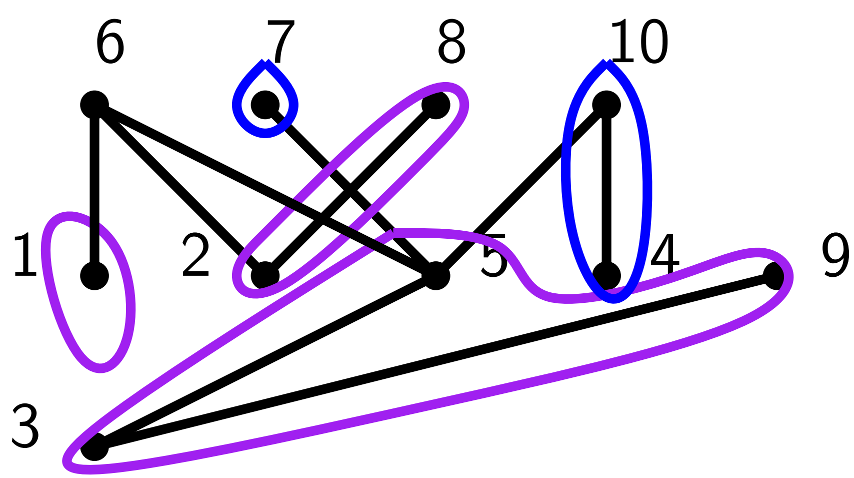

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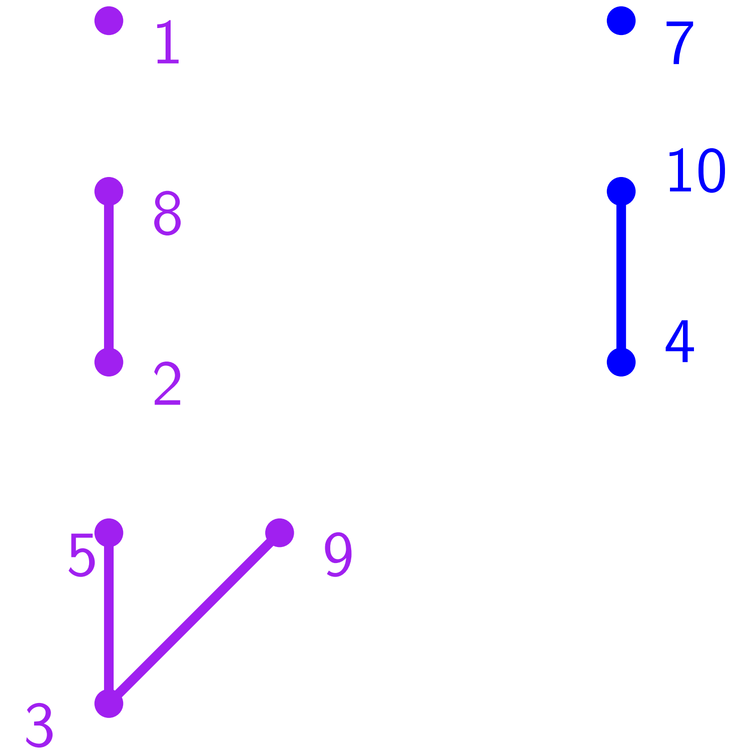

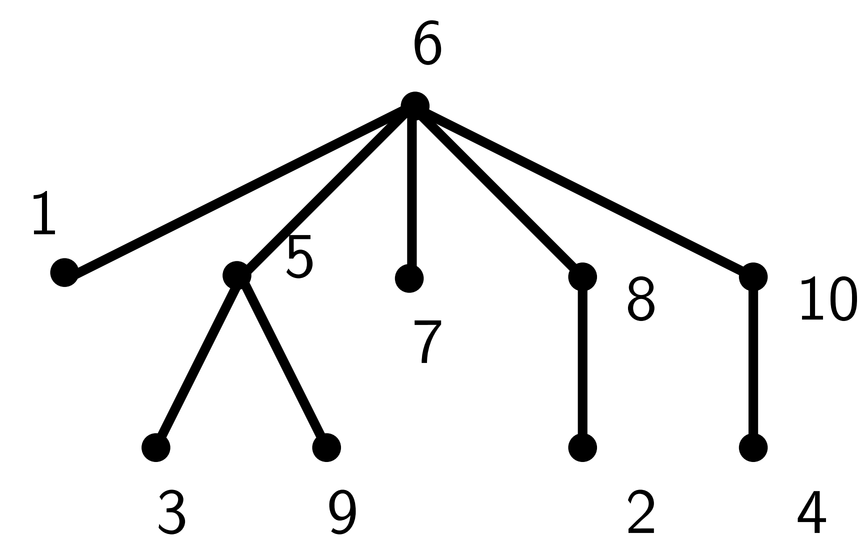

Non-increasing tree!

corner $\leftrightarrow$ root

General results

Theorem [CFM 14]. There is a bijection between the NBC-trees of the gain graph $\mathcal{K}_n^{ab}$ and

- If $a+b>0$ the $(a+b-1)$-non-increasing and $(1-a)$- free forests with n vertices

- If $a+b\leq 0$ the $(1-a-b)$-increasing and b- free trees with n vertices

General results

Regions of the arrangements $\mathcal{A}_n^{ab}$

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Other order O'_h

$i <_{O'_h} j$ if $h(i) < h(j)$ or $h(i) = h(j)$ and $i > j$

Enumerative consequences

Theorem [CFM 14] There exists a bijection between k-increasing and j-free trees with n vertices and $(k-2)$-non-increasing and $(j+1)$-free trees with n vertices

k-decreasing vertices $\leftrightarrow$
decreasing vertices on the first free color

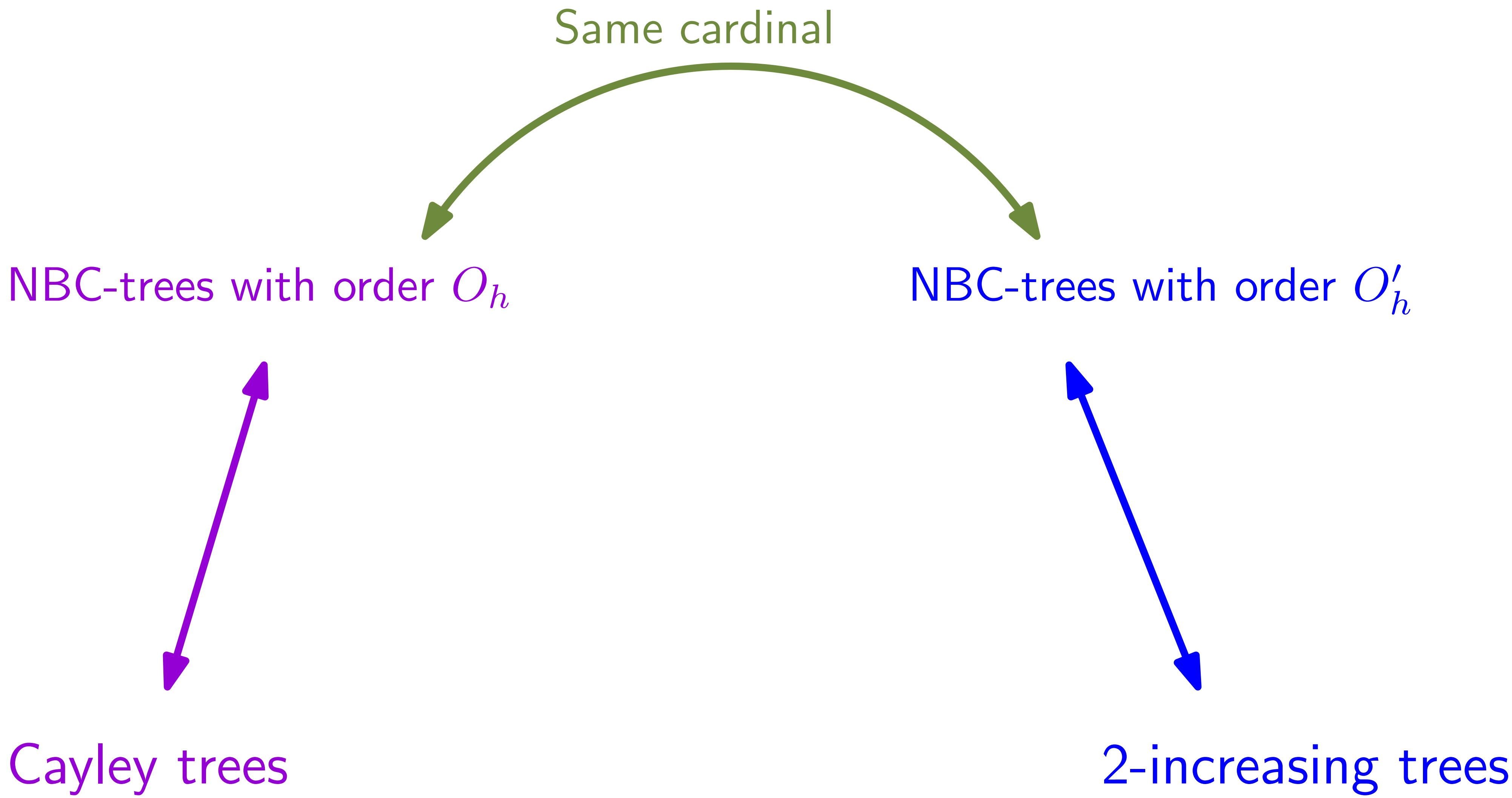
[Conjectured first thanks to Sage and F. Chapoton]

The case of Shi

NBC-trees with order O_h ↔ Cayley trees

NBC-trees with order O'_h ↔ 2-increasing trees

The case of Shi



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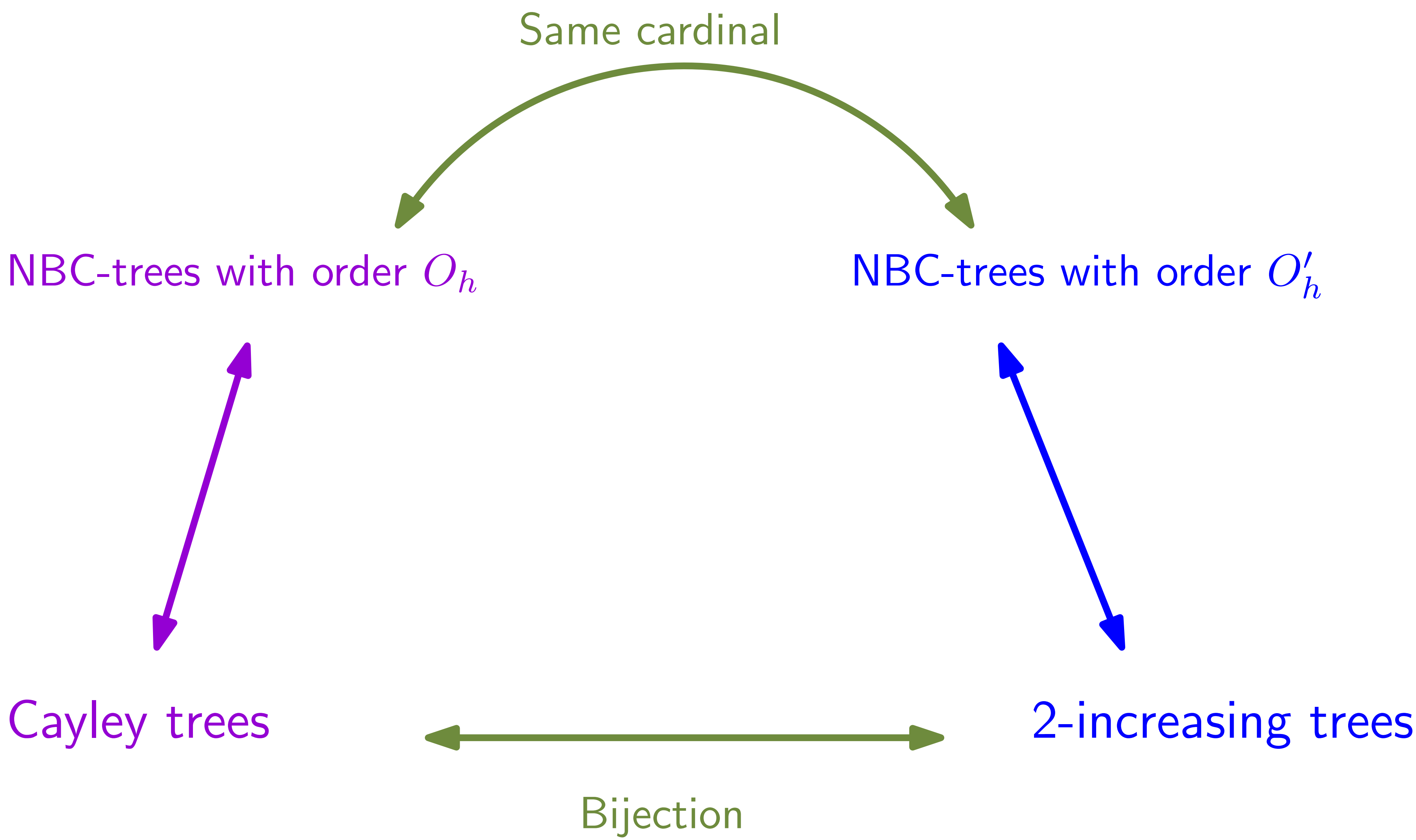
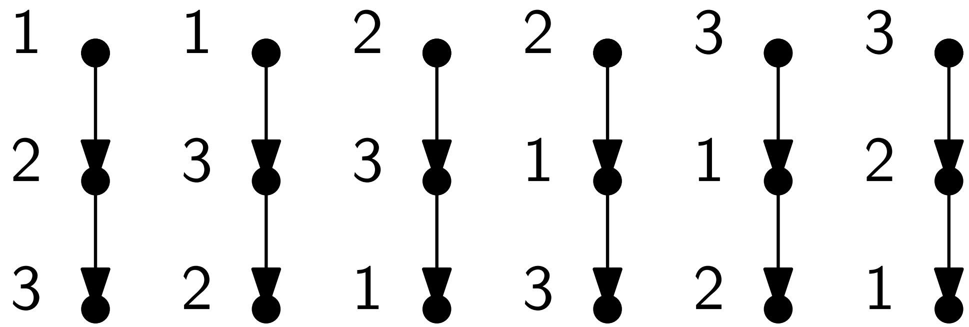
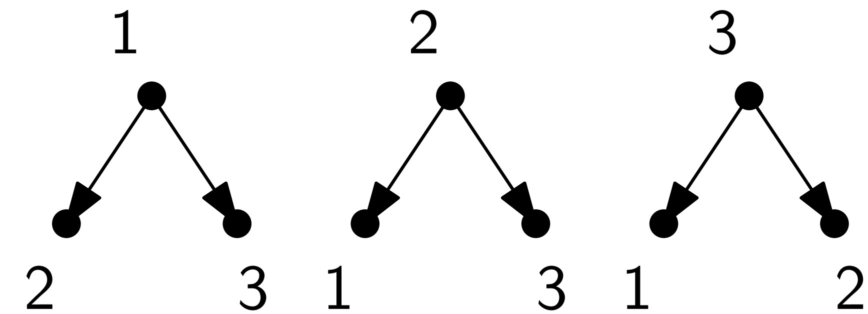






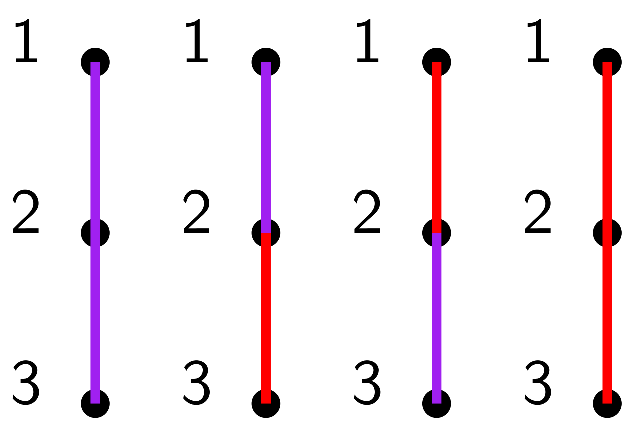
Bijection

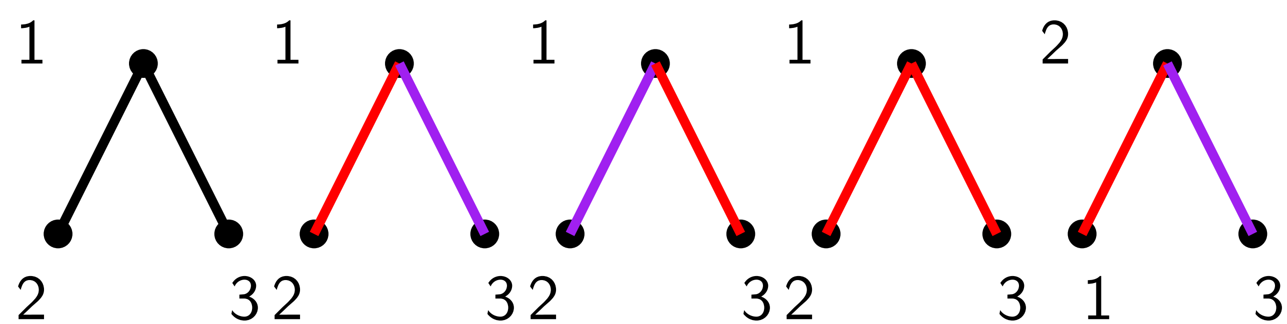
$\leftrightarrow$

Cayley

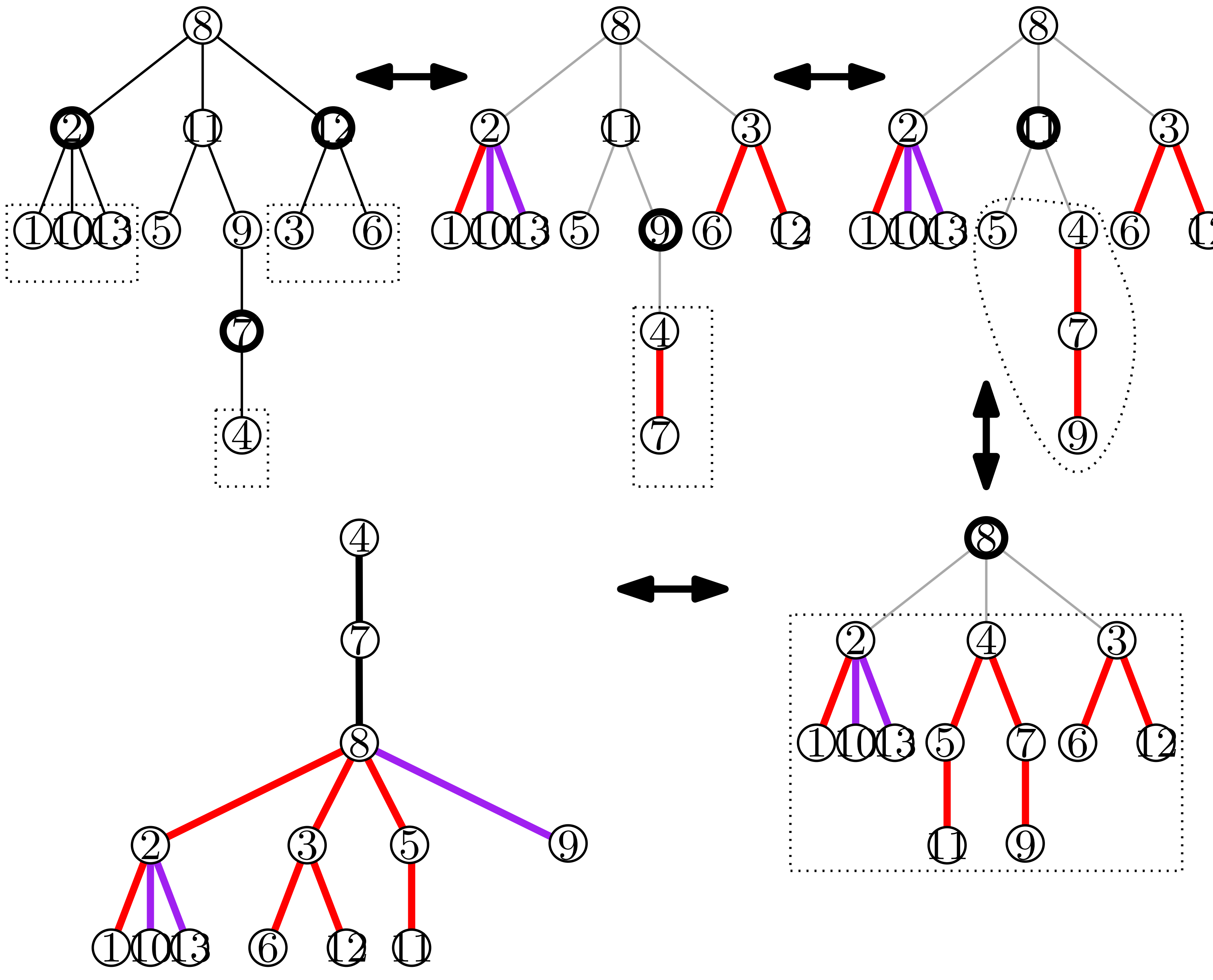
2-

increasing





Cayley trees ↔ 2-increasing trees



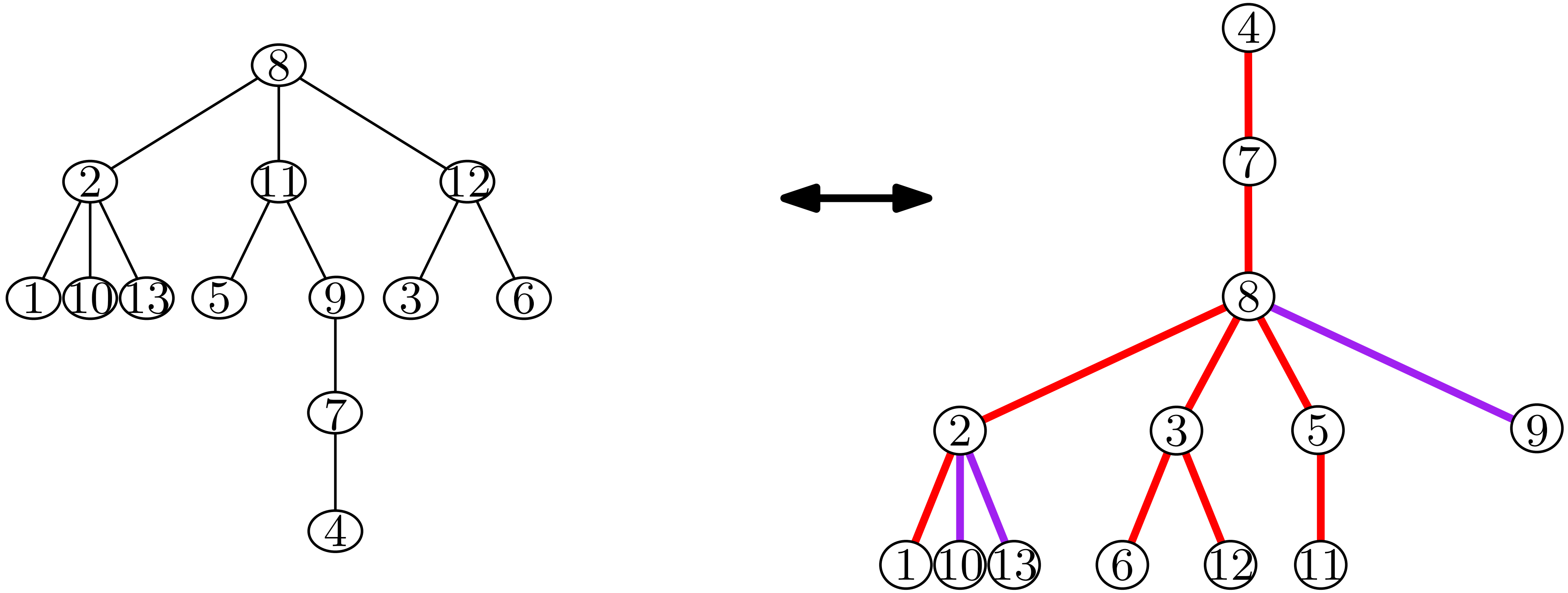
Cayley trees $\leftrightarrow$ 2-increasing trees

Lemma. Root of the Cayley tree

$\leftrightarrow$ vertex at the end of the 2-increasing path of the 2-increasing tree.

Lemma. Decreasing vertices in the Cayley trees

$\leftrightarrow$ vertices who have an increasing 2-edge in the 2-increasing tree.



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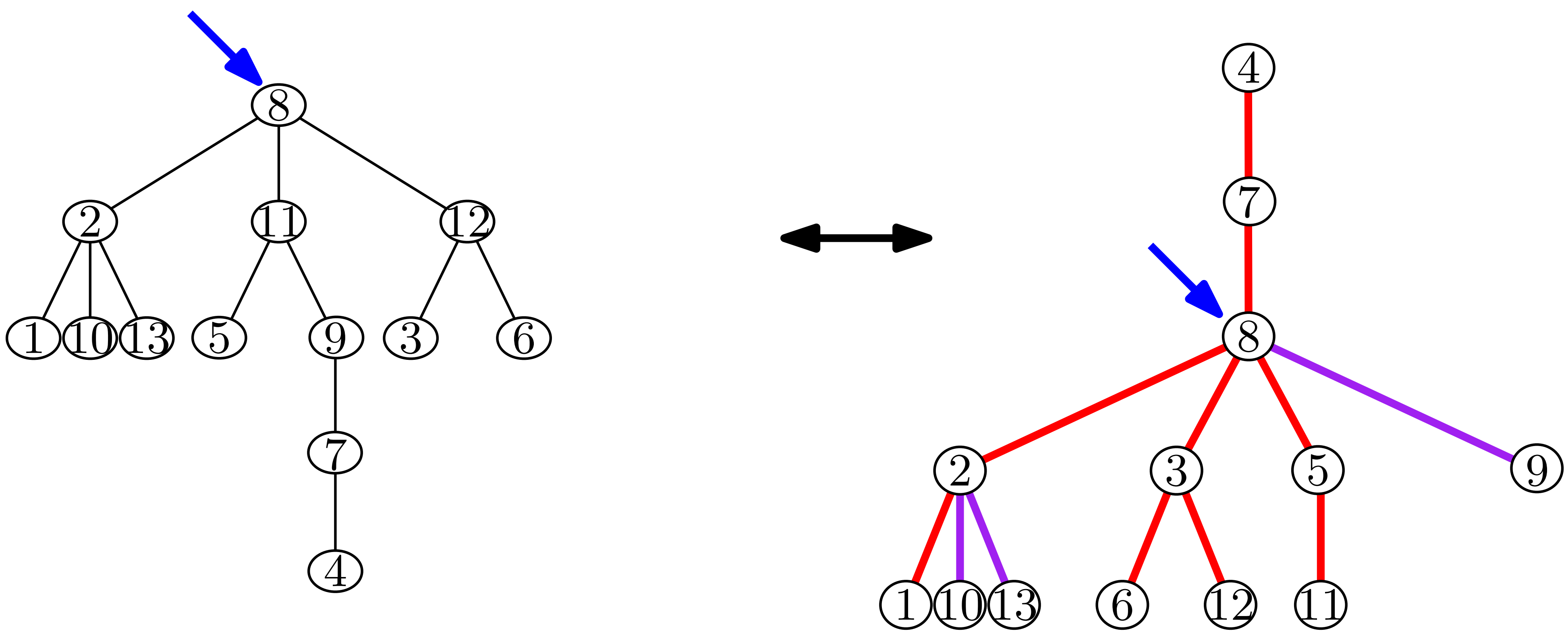

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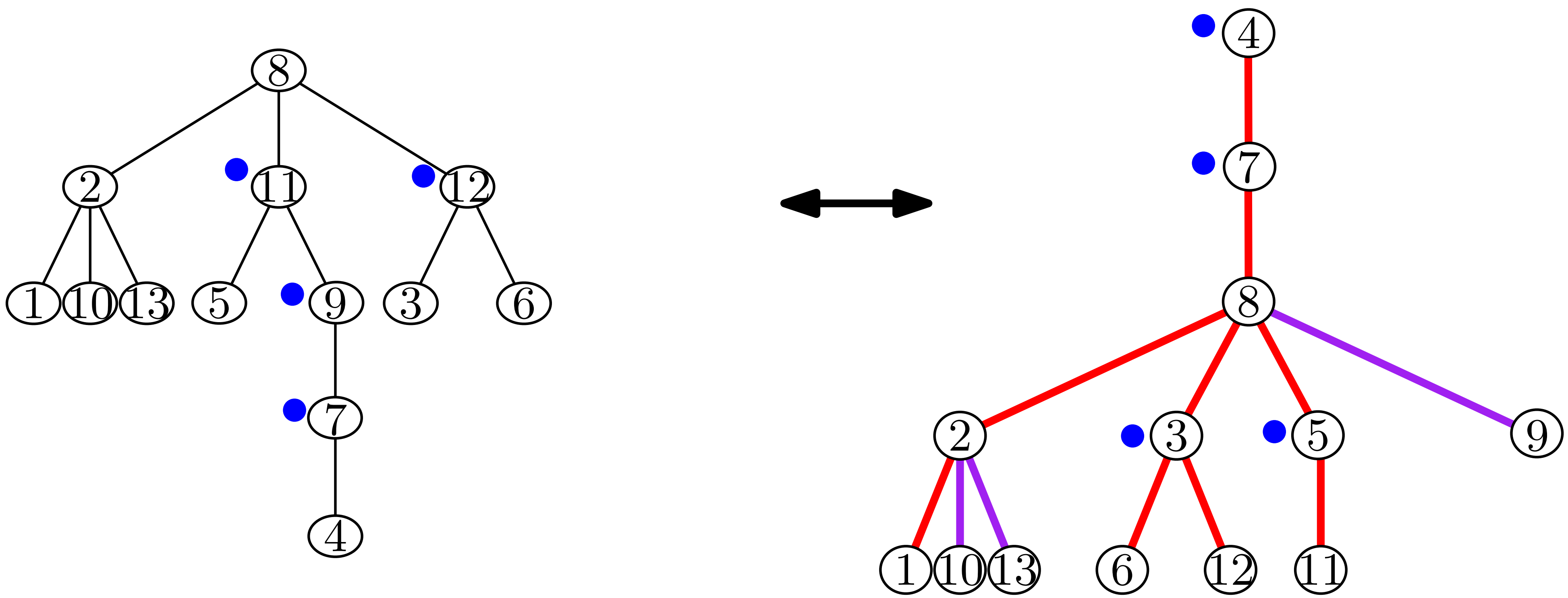

Generalization

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vertices with an increasing k-edge $\leftrightarrow$
decreasing vertices on the first free color

Ongoing projects

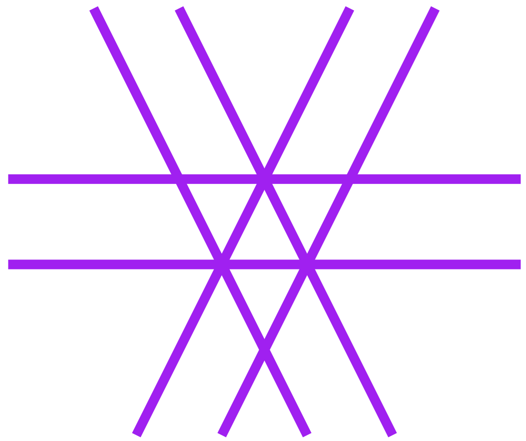
- Other Coxeter arrangements $x_i \pm x_j = g$ [Athanasiadis 99]

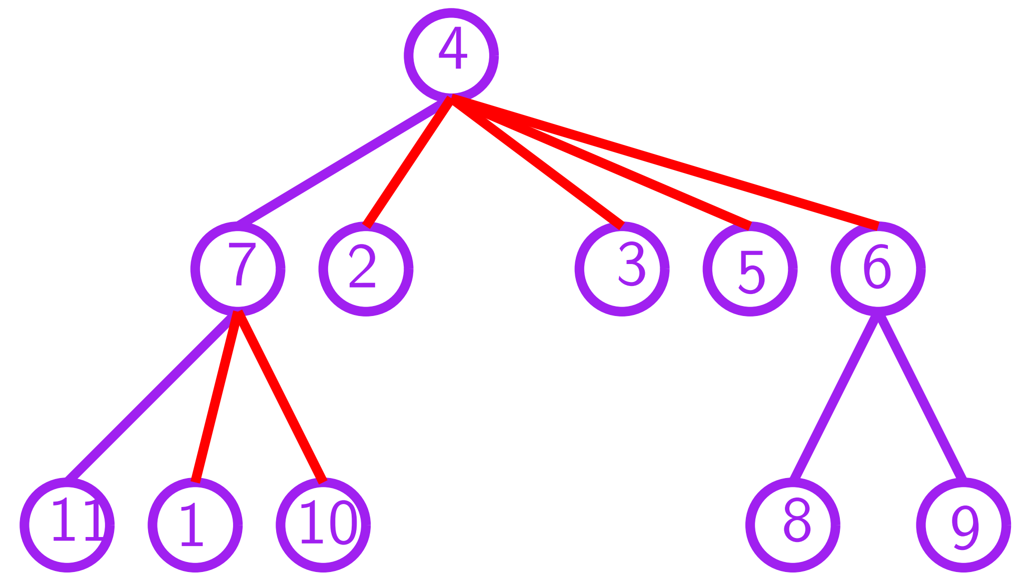
- Ish arrangement [Armstrong et al, 10]

- Counting bounded regions, activities...

- Simple bijections? 2-increasing trees $\leftrightarrow$ rooted Cayley trees

  1-increasing, 1-free forests $\leftrightarrow$ labelled binary trees....





Merci!